REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 41 (2012) no. 2, pp. 125–129
ictp.acad.ro/jnaat

A SEPARATION OF SOME SEIFFERT-TYPE MEANS BY POWER MEANS

IULIA COSTIN* and GHEORGHE TOADER†

Abstract. Consider the identric mean $\mathcal{I}$, the logarithmic mean $\mathcal{L}$, two trigonometric means defined by H. J. Seiffert and denoted by $\mathcal{P}$ and $\mathcal{T}$, and the hyperbolic mean $\mathcal{M}$ defined by E. Neuman and J. Sándor. There are a number of known inequalities between these means and some power means $\mathcal{A}_p$. We add to these inequalities some new results obtaining the following chain of inequalities

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/3} < \mathcal{P} < \mathcal{A}_{2/3} < \mathcal{I} < \mathcal{A}_{3/3} < \mathcal{M} < \mathcal{A}_{4/3} < \mathcal{T} < \mathcal{A}_{5/3}.$$

MSC 2000. 26E60.
Keywords. Seiffert type means; power means; logarithmic mean; identric mean; inequalities of means.

1. INTRODUCTION

A **mean** is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a,b) \le M(a,b) \le \max(a,b), \quad \forall a,b>0.$$

Each mean is **reflexive**, that is

$$M(a,a) = a, \quad \forall a > 0.$$

This is also used as the definition of $M(a,a)$.

A mean is **symmetric** if

$$M(b,a) = M(a,b), \quad \forall a,b>0;$$

it is **homogeneous** (of degree 1) if

$$M(ta,tb) = t \cdot M(a,b), \quad \forall a,b,t>0.$$

We shall refer here to the following symmetric and homogeneous means:

- the power means $\mathcal{A}_p$, defined by

$$\mathcal{A}_p(a,b) = \left[\tfrac{a^p+b^p}{2}\right]^{\frac{1}{p}}, \quad p \ne 0;$$

*Department of Computer Science, Technical University of Cluj-Napoca, Baritiu st. no. 28, Cluj-Napoca, Romania, e-mail: `Iulia.Costin@cs.utcluj.ro`.

†Department of Mathematics, Technical University of Cluj-Napoca, Baritiu st. no. 25, Cluj-Napoca, Romania, e-mail: `Gheorghe.Toader@math.utcluj.ro`.

- the geometric mean $\mathcal{G}$, defined as $\mathcal{G}(a,b) = \sqrt{ab}$, but verifying also the property

$$\lim_{p\to 0} \mathcal{A}_p(a,b) = \mathcal{A}_0(a,b) = \mathcal{G}(a,b);$$

- the identric mean $\mathcal{I}$ defined by

$$\mathcal{I}(a,b) = \tfrac{1}{e}\left(\tfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \quad a \neq b;$$

- the Gini mean $\mathcal{S}$ defined by

$$\mathcal{S}(a,b) = \left(a^a b^b\right)^{\frac{1}{a+b}};$$

- the first Seiffert mean $\mathcal{P}$, defined in [9] by

$$\mathcal{P}(a,b) = \frac{a-b}{2\sin^{-1}\frac{a-b}{a+b}}, \quad a \neq b;$$

- the second Seiffert mean $\mathcal{T}$, defined in [10] by

$$\mathcal{T}(a,b) = \frac{a-b}{2\tan^{-1}\frac{a-b}{a+b}}, \quad a \neq b;$$

- the Neuman-Sándor mean $\mathcal{M}$, defined in [6] by

$$\mathcal{M}(a,b) = \frac{a-b}{2\sinh^{-1}\frac{a-b}{a+b}}, \quad a \neq b;$$

- the logarithmic mean $\mathcal{L}$ defined by

$$\mathcal{L}(a,b) = \frac{a-b}{\ln a - \ln b}, \quad a \neq b.$$

As remarked B.C. Carlson in [1], the logarithmic mean can be represented also by

$$\mathcal{L}(a,b) = \frac{a-b}{2\tanh^{-1}\frac{a-b}{a+b}}, \quad a \neq b,$$

thus the last four means are very similar.

Being rather complicated, these means were evaluated by simpler means, first of all by power means. For two means M and N we write $M < N$ if $M(a,b) < N(a,b)$ for $a \neq b$. It is known that the family of power means is an increasing family of means, thus

$$\mathcal{A}_p < \mathcal{A}_q \text{ if } p < q.$$

The **evaluation** of a given mean M by power means assumes the determination of some real indices p and q such that $\mathcal{A}_p < M < \mathcal{A}_q$. The evaluation is **optimal** if p is the the greatest and q is the smallest index with this property. This means that M cannot be compared with $\mathcal{A}_r$ if $p < r < q$.

Optimal evaluation were given for the logarithmic mean in [5]

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/3},$$

for the identric mean in [8]

$$\mathcal{A}_{2/3} < \mathcal{I} < \mathcal{A}_{\ln 2},$$

and for the first Seiffert mean in [3]

$$\mathcal{A}_{\ln 2/\ln \pi} < \mathcal{P} < \mathcal{A}_{2/3}.$$

Following evaluations are also known:

$$\mathcal{A}_{1/3} < \mathcal{P} < \mathcal{A}_{2/3},$$

proved in [4],

$$\mathcal{A}_1 < \mathcal{T} < \mathcal{A}_2,$$

given in [10],

$$\mathcal{A}_1 < \mathcal{M} < \mathcal{T},$$

as it was shown in [6] and

$$\mathcal{S} > \mathcal{A}_2$$

as it is proved in [7]. In [2] it is proven that

$$\mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} \tag{1}$$

and using some of the above results, it is obtained the following chain of inequalities

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/2} < \mathcal{P} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} < \mathcal{A}_2.$$

Here we retain another chain of inequalities

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/3} < \mathcal{P} < \mathcal{A}_{2/3} < \mathcal{I} < \mathcal{A}_1 < \mathcal{M} < \mathcal{T} < \mathcal{A}_2 < \mathcal{S}. \tag{2}$$

Our aim is to prove that $\mathcal{A}_{4/3}$ can be put between $\mathcal{M}$ and $\mathcal{T}$ and $\mathcal{A}_2$ can be replaced by $\mathcal{A}_{5/3}$. We obtain so another nice separation of these means by "equidistant" power means.

2. MAIN RESULTS

We add to the inequalities (2) the next results.

THEOREM 1. *The following inequalities*

$$\mathcal{M} < \mathcal{A}_{4/3} < \mathcal{T} < \mathcal{A}_{5/3}$$

hold.

Proof. As the means are symmetric and homogenous, for the first inequality

$$\frac{a-b}{2\sinh^{-1}\frac{a-b}{a+b}} < \left(\frac{a^{4/3}+b^{4/3}}{2}\right)^{\frac{3}{4}}, \quad a \neq b,$$

we can assume that $a > b$ and denote $a/b = t^3 > 1$. The inequality becomes

$$\frac{t^3-1}{2\sinh^{-1}\frac{t^3-1}{t^3+1}} < \left(\frac{t^4+1}{2}\right)^{\frac{3}{4}}, \quad t > 1,$$

or

$$\frac{2^{\frac{3}{4}}(t^3-1)}{2(t^4+1)^{\frac{3}{4}}} < \sinh^{-1}\frac{t^3-1}{t^3+1}, \quad t > 1.$$

Denoting

$$f(t) = \sinh^{-1} \tfrac{t^3-1}{t^3+1} - 2^{-\frac{1}{4}} \left(t^3 - 1\right) \left(t^4 + 1\right)^{-\frac{3}{4}}$$

we have to prove that $f(t) > 0$ for $t > 1$. As $f(1) = 0$, we want to prove that $f'(t) > 0$ for $t > 1$. We have

$$\begin{aligned} f'(t) &= \tfrac{6t^2}{(t^3+1)\sqrt{2(t^6+1)}} - 2^{-\frac{1}{4}} \tfrac{3t^2(t+1)}{(t^4+1)^{\frac{7}{4}}} \\ &= \frac{3t^2\left[2^{\frac{3}{4}}\left(t^4+1\right)^{\frac{7}{4}} - (t+1)\left(t^3+1\right)\sqrt{t^6+1}\right]}{2^{\frac{1}{4}}(t^3+1)\sqrt{t^6+1}(t^4+1)^{\frac{7}{4}}} \end{aligned}$$

and so it is positive if

$$g(t) = \left[2^{\frac{3}{4}} \left(t^4 + 1\right)^{\frac{7}{4}}\right]^4 - \left[(t + 1) \left(t^3 + 1\right) \sqrt{t^6 + 1}\right]^4$$

is positive. Or

$$\begin{aligned} g(t) = (t-1)^4 &(7t^{24} + 24t^{23} + 48t^{22} + 68t^{21} + 112t^{20} \\ &+ 184t^{19} + 264t^{18} + 296t^{17} + 344t^{16} + 428t^{15} \\ &+ 512t^{14} + 488t^{13} + 466t^{12} + 488t^{11} + 512t^{10} \\ &+ 428t^9 + 344t^8 + 296t^7 + 184t^5 + 112t^4 \\ &+ 68t^3 + 48t^2 + 24t + 7) \end{aligned}$$

so that the property is certainly true. The second inequality is a simple consequence of (1) because $\mathcal{A}_{4/3} < \mathcal{A}_{3/2}$. For the last inequality

$$\frac{a-b}{2\tan^{-1}\frac{a-b}{a+b}} < \left(\tfrac{a^{5/3}+b^{5/3}}{2}\right)^{\frac{3}{5}}, \quad a \neq b,$$

we can again consider $\frac{a}{b} = t^3 > 1$ and we have to prove that

$$\frac{t^3-1}{2\tan^{-1}\frac{t^3-1}{t^3+1}} < \left(\tfrac{t^5+1}{2}\right)^{\frac{3}{5}}, \quad t > 1.$$

This is equivalent with the condition that the function

$$h(t) = \tan^{-1} \tfrac{t^3-1}{t^3+1} - \tfrac{t^3-1}{2^{\frac{2}{5}}(t^5+1)^{\frac{3}{5}}}$$

is positive for $t > 1$. As $h(1) = 0$ and

$$\begin{aligned} h'(t) &= \tfrac{3t^2}{t^6+1} - \tfrac{3t^2\left(t^2+1\right)}{2^{\frac{2}{5}}(t^5+1)^{\frac{8}{5}}} \\ &= \frac{3t^2\left[2^{\frac{2}{5}}\left(t^5+1\right)^{\frac{8}{5}} - \left(t^2+1\right)\left(t^6+1\right)\right]}{2^{\frac{2}{5}}(t^5+1)^{\frac{8}{5}}(t^6+1)}, \end{aligned}$$

we have $h(t) > 0$ for $t > 1$ if $h'(t) > 0$ for $t > 1$, thus if the function

$$k(t) = \left[2^{\frac{2}{5}}\left(t^5+1\right)^{\frac{8}{5}}\right]^5 - \left[\left(t^2+1\right)\left(t^6+1\right)\right]^5$$

is positive for $t > 1$. Or this is obvious because

$$\begin{aligned} k(t) = (t-1)^4 & (185t^{28} + 200t^{27} + 221t^{26} + 365t^{24} \\ & + 410t^{22} + 520t^{19} + 580t^{18} + 520t^{17} + 430t^{16} \\ & + 400t^{15} + 410t^{14} + 440t^{13} + 365t^{12} + 284t^{11} \\ & + 221t^{10} + 200t^9 + 185t^8 + 140t^7 + 90t^6 \\ & + 60t^5 + 45t^4 + 25t^2 + 40t^3 + 12t + 3). \end{aligned}$$

$\square$

Remark 2. For the factorization of the polynomials g and k we have used the computer algebra Maple. $\square$

Remark 3. It is an open problem for us to find a mean N, related to the above mentioned means, with the property that

$$\mathcal{A}_{5/3} < N < \mathcal{A}_2.$$

For instance, the mean $\mathcal{S}$, which is similar to $\mathcal{I}$, is not convenient as follows from (2). $\square$

Corollary 4. *For each $x \in (0,1)$ we have the following evaluations*

$$1 < \frac{x}{\sinh^{-1} x} < \mathcal{A}_{4/3}(1-x, 1+x) < \frac{x}{\tan^{-1} x} < \mathcal{A}_{5/3}(1-x, 1+x).$$

REFERENCES

[1] B.C. Carlson, *The logarithmic mean*, Amer. Math. Monthly, **79** (1972), pp. 615–618.
[2] I. Costin and G. Toader, *A nice evaluation of some Seiffert type means by power means*, Int. J. Math. Math. Sc., **2012**, Article ID 430692, 6 pages, doi:10.1155/2012/430692.
[3] P.A. Hästö, *Optimal inequalities between Seiffert's means and power means*, Math. Inequal. Appl., **7** (2004) no. 1, pp. 47–53.
[4] A.A. Jagers, *Solution of problem 887*, Niew Arch. Wisk. (Ser. 4), **12** (1994), pp. 230–231.
[5] T.P. Lin, *The power and the logarithmic mean*, Amer. Math. Monthly, **81** (1974), pp. 879–883.
[6] E. Neuman and J. Sándor, *On the Schwab-Borchardt mean*, Math. Panon., **14** (2003) no. 2, pp. 253–266.
[7] E. Neuman and J. Sándor, *Comparison inequalities for certain bivariate means*, Appl. Anal. Discrete Math., **3** (2009), pp. 46–51.
[8] A.O. Pittenger, *Inequalities between arithmetic and logarithmic means*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **678-715** (1980), pp. 15–18.
[9] H.-J. Seiffert, *Problem 887*, Niew Arch. Wisk. (Ser. 4), **11** (1993), pp. 176–176.
[10] H.-J. Seiffert, *Aufgabe β16*, Die Wurzel, **29** (1995), pp. 221–222.

Received by the editors: June 12, 2012.