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REMARKS ON THE QUENCHING ESTIMATE FOR A NONLOCAL DIFFUSION PROBLEM WITH A REACTION TERM

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Abstract. In this paper, we address the following initial value problem

$$u_t = \int_\Omega J(x-y)(u(y,t)-u(x,t))\mathrm{d}y + f(u(x,t)) \quad \text{in} \quad \overline{\Omega}\times(0,T),$$
$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $f:(-\infty,b)\to(0,\infty)$ is a C^1 convex nondecreasing function, $\lim_{s\to b^-} f(s)=\infty$, $\int^\infty \frac{\mathrm{d}\sigma}{f(\sigma)}<\infty$, with b a positive constant, $J:\mathbb{R}^N\to\mathbb{R}$ is a kernel which is measurable, nonnegative and bounded in $\mathbb{R}^N$. Under some conditions, we show that the solution of a perturbed form of the above problem quenches in a finite time and estimate its quenching time. We also prove the continuity of the quenching time as a function of the initial datum. Finally, we give some numerical results to illustrate our analysis.

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1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial value problem

$$(1)\quad u_t = \int_\Omega J(x-y)(u(y,t)-u(x,t))\mathrm{d}y + f(u(x,t)) \quad \text{in} \quad \overline{\Omega}\times(0,T),$$

$$(2)\qquad u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where $f:(-\infty,b)\to(0,\infty)$ is a C^1 convex nondecreasing function, $\int^\infty \frac{\mathrm{d}\sigma}{f(\sigma)}<\infty$, $\lim_{s\to b^-} f(s)=\infty$, with b a positive constant, $J:\mathbb{R}^N\to\mathbb{R}$ is a kernel which is measurable, nonnegative and bounded in $\mathbb{R}^N$. In addition, J is symmetric ($J(z)=J(-z)$) and $\int_{\mathbb{R}^N} J(z)\mathrm{d}z=1$. The initial datum $u_0\in C^0(\overline{\Omega})$, $0\leq$

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$u_0(x) < b$ in $\overline{\Omega}$. Let us notice that, if $f(s) = (b-s)^{-p}$ with p a positive constant, then f satisfies the above conditions. Here, $(0,T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T}\|u(\cdot,t)\|_\infty = b,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega}|u(x,t)|$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u. Recently, nonlocal diffusion has been the subject of investigation of many authors (see, [2], [8], [11], [13], [15], [17], [19], [20], [25], [29], [32], and the references cited therein). Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\mathrm{d}y,$$

and variations of it, have been used by several authors to model diffusion processes (see, [4], [5], [11], [19], [20]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at the time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then the convolution $(J*u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\mathrm{d}y$ is the rate at which individuals are arriving to position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(y,t)\mathrm{d}y$ is the rate at which they are leaving location x to travel to any other site (see, [19]). For the problem described in (1)–(2), the integral is taken over Ω. Consequently, there is no individuals that arrive or leave the domain Ω. It is the reason why in the title of the paper, we have added Neumann boundary condition. On the other hand, the term of the source $f(u)$ can be rewritten as follows

$$f(u(x,t)) = \int_{\mathbb{R}^N} J(x-y)f(u(x,t))\mathrm{d}y.$$

Therefore, in view of the above equality, the term $f(u)$ can be interpreted as a force that decreases the rate of individuals which are leaving location x to travel to any other site, provoking as we shall see later, the phenomenon of quenching of the solution u. For local diffusions, solutions which quench in a finite time has been the subject of investigation of many authors (see, for instance [10], [18], [23], [24], [26], [28], [30], and the reference cited therein).

Similar results have been obtained in [32], where the authors considered analogous problems within the framework of the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time).

The first paper which deals with blow-up of (1)–(2) that we are aware as that of Perez-LLanos and Rossi in [32], where they considered the problem (1)-(2) in the case where $f(u) = u^p$ with $p = \text{const} > 1$. They proved that the solution u of (1)–(2) blows up in a finite time and localized the blow-up set. Some results about blow-up rate are also given. In the same way, in [29], Nabongo and Boni examined the initial value problem

$$u_t = \varepsilon \int_\Omega J(x-y)(u(y,t)-u(x,t))\mathrm{d}y + f(u) \quad \text{in} \quad \overline{\Omega}\times(0,T),$$

$$u = 0 \quad \text{on} \quad (\mathbb{R}-\Omega)\times(0,T),$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where ε is a positive parameter. They showed that, if ε is small enough, then the solution u of the above problem blows up in a finite time, and its blow-up time goes to that of the solution of the following ODE

$$\begin{cases} \alpha'(t) = f(\alpha(t)), \quad t>0, \\ \alpha(0) = \max_{x\in\overline{\Omega}} u_0(x), \end{cases}$$

as ε goes to zero. In this paper, we are interested in the the phenomenon of quenching of the solution u, and the continuity of the quenching time for the problem describe in (1)-(2). More precisely, consider the following initial value problem

$$v_t = \int_\Omega J(x-y)(v(y,t)-v(x,t))\mathrm{d}y + f(v) \quad \text{in} \quad \overline{\Omega}\times(0,T_h), \tag{3}$$

$$v(x,0) = u_0^h(x) \quad \text{in} \quad \overline{\Omega}, \tag{4}$$

where $u_0^h \in C^0(\overline{\Omega})$, $0 \leq u_0^h(x) \leq u_0(x)$ in $\overline{\Omega}$, $\lim_{h\to 0} u_0^h = u_0$. Here $(0,T_h)$ is the maximal time interval of existence of the solution v. In the current paper, under some hypotheses, we show that the solution v of (3)–(4) quenches in a finite time and estimate its quenching time. We demonstrate in passing that, when the norm of the initial datum is large enough, then the solution v of (3)–(4) quenches in a finite time and its quenching time goes to that of the solution of a certain differential equation

$$\alpha'(t) = f(\alpha(t)), \quad t>0, \quad \alpha(0) = \|u_0^h\|_\infty,$$

as $\|u_0^h\|_\infty$ goes to b. Finally, under some hypotheses, we prove that the solution v of (3)–(4) quenches in a finite time and its quenching time goes to that of the solution u of (1)–(2) when h goes to zero. The remainder of the paper is organized in the following manner. In the next section, we prove the local existence and uniqueness of solutions. In the third section, under some conditions, we show that the solution v of (3)–(4) quenches in a finite time and estimate its quenching time. We also show that its quenching time goes to that of the solution u of (1)–(2) when h goes to zero, in the last section, we give some computational results to illustrate our analysis.

2. LOCAL EXISTENCE

In this section, we shall establish the existence and uniqueness of nonnegative solutions of (1)–(2) in $\Omega \times (0,T)$ for all small T. We shall also prove some results concerning the maximum principle within the framework of nonlocal diffusion problems for our subsequent use. Let us notice that results on local existence and uniqueness are known for our problem if one modifies slightly the proof given by Perez-LLanos and Rossi in [32]. However, for the sake of completeness, we outline them. Let t_0 be fixed, and define the function space $Y_{t_0} = \{u; u \in C([0,t_0], C(\overline{\Omega}))\}$ equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0\leq t\leq t_0} \|u(\cdot,t)\|_\infty$ for $u \in Y_{t_0}$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set

$$X_{t_0} = \{u; u \in Y_{t_0}, \|u\|_{Y_{t_0}} \leq b_0\},$$

where $b_0 = \frac{\|u_0\|_\infty + b}{2}$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows

$$R : X_{t_0} \to X_{t_0}$$

$$R(v)(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)(v(y,s) - v(x,s))\mathrm{d}y\mathrm{d}s + \int_0^t f(v(x,s))\mathrm{d}s.$$

THEOREM 1. *Assume that $u_0 \in C^0(\overline{\Omega})$, and $0 \leq u_0(x) < b$ in $\overline{\Omega}$. Then R maps X_{t_0} into X_{t_0}, and R is strictly contractive if t_0 is approximately small relative to $\|u_0\|_\infty$.*

Proof. Due to the fact that $\int_\Omega J(x-y)\mathrm{d}y \leq \int_{\mathbb{R}^N} J(x-y)\mathrm{d}y = 1$, a straightforward computation reveals that

$$|R(v)(x,t) - u_0(x)| \leq 2\|v\|_{Y_{t_0}} t + f(\|v\|_{Y_{t_0}})t,$$

which implies that $\|R(v)\|_{Y_{t_0}} \leq \|u_0\|_\infty + 2b_0t_0 + f(b_0)t_0$. If

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2b_0 + f(b_0)}, \tag{5}$$

then

$$\|R(v)\|_{Y_{t_0}} \leq b_0.$$

Therefore, if (5) holds, then R maps X_{t_0} into X_{t_0}. Now, we are going to prove that the map R is strictly contractive. Let $v, z \in X_{t_0}$. Setting $\alpha = v - z$, we discover that

$$|(R(v) - R(z))(x,t)| \leq \left| \int_0^t \int_\Omega J(x-y)(\alpha(y,s) - \alpha(x,s))\mathrm{d}y\mathrm{d}s \right| + \left| \int_0^t (f(v(x,s)) - f(z(x,s)))\mathrm{d}s \right|.$$

Use Taylor's expansion to obtain

$$|(R(v) - R(z))(x,t)| \leq 2\|\alpha\|_{Y_{t_0}} t + t\|v - z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

where β is an intermediate function between v and z. We deduce that

$$\|R(v)-R(z)\|_{Y_{t_0}} \le 2\|\alpha\|_{Y_{t_0}} t_0 + t_0\|v-z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

which implies that $\|R(v)-R(z)\|_{Y_{t_0}} \le (2t_0 + t_0 f'(b_0))\|v-z\|_{Y_{t_0}}$. If

$$t_0 \le \tfrac{1}{4+2f'(b_0)}, \tag{6}$$

then $\|R(v)-R(z)\|_{Y_{t_0}} \le \frac{1}{2}\|v-z\|_{Y_{t_0}}$. Hence, we see that $R(v)$ is a strict contraction in Y_{t_0} and the proof is complete. □

It follows from the contraction mapping principle that for appropriately chosen $t_0 > 0$, R has a unique fixed point $u(x,t) \in Y_{t_0}$ which is a solution of (1)-(2). If $\|u\|_{Y_{t_0}} < b$, then taking as initial datum $u(\cdot, t_0) \in C^0(\overline{\Omega})$ and arguing as before, it is possible to extend the solution up to some interval $[0, t_1)$ for certain $t_1 > t_0$. Now, to end this section, we shall provide some results about the maximum principle tailored to our study. The following lemma is a version of the maximum principle for nonlocal problems.

Lemma 2. *Let $a \in C^0(\overline{\Omega} \times [0,T))$, and let $u \in C^{0,1}(\overline{\Omega} \times [0,T))$ satisfying the following inequalities*

$$u_t - \int_\Omega J(x-y)(u(y,t)-u(x,t))\mathrm{d}y + a(x,t)u(x,t) \ge 0 \quad \text{in} \quad \overline{\Omega} \times (0,T), \tag{7}$$

$$u(x,0) \ge 0 \quad \text{in} \quad \overline{\Omega}. \tag{8}$$

Then, we have $u(x,t) \ge 0$ in $\overline{\Omega} \times (0,T)$.

Proof. Let T_0 be any positive quantity satisfying $T_0 < T$. Since $a(x,t)$ is bounded in $\overline{\Omega} \times [0,T_0]$, then there exists λ such that $a(x,t) - \lambda > 0$ in $\overline{\Omega} \times [0,T]$. Define $z(x,t) = e^{\lambda t} u(x,t)$ and let $m = \min_{x\in\overline{\Omega}, t\in[0,T_0]} z(x,t)$. Due to the fact that z is continuous in $\overline{\Omega} \times [0,T_0]$, then it achieves its minimum in $\overline{\Omega} \times [0,T_0]$. Consequently, there exists $(x_0,t_0) \in \overline{\Omega} \times [0,T_0]$ such that $m = z(x_0,t_0)$. We get $z(x_0,t_0) \le z(x_0,t)$ for $t \le t_0$ and $z(x_0,t_0) \le z(y,t_0)$ for $y \in \Omega$. This implies that

$$z_t(x_0,t_0) \le 0, \qquad \int_\Omega J(x_0-y)(z(y,t_0) - z(x_0,t_0))\mathrm{d}y \ge 0. \tag{9}$$

With the aid of the first inequality of the lemma, it is not hard to see that

$$z_t(x_0,t_0) - \int_\Omega J(x_0-y)(z(y,t_0) - z(x_0,t_0))\mathrm{d}y + a(x_0,t_0) - \lambda)z(x_0,t_0) \ge 0.$$

We deduce from (9) that $(a(x_0,t_0) - \lambda)z(x_0,t_0 \ge 0$. Since $a(x_0,t_0) - \lambda > 0$, we get $z(x_0,t_0) \ge 0$. This implies that $u(x,t) \ge 0$ in $\overline{\Omega} \times [0,T_0]$, and the proof is complete. □

An immediate consequence of the above lemma is the following comparison lemma. Its proof is straightforward.

LEMMA 3. *Let $a \in C^0(\overline{\Omega} \times [0,T))$ and let u, $v \in C^{0,1}(\overline{\Omega} \times [0,T))$ satisfying the following inequalities*

$$u_t - \int_\Omega J(x-y)(u(y,t) - u(x,t))\mathrm{d}y + a(x,t)u(x,t)$$

$$\geq v_t - \int_\Omega J(x-y)(v(y,t) - v(x,t))\mathrm{d}y + a(x,t)v(x,t) \quad in \quad \overline{\Omega} \times (0,T),$$

$$u(x,0) \geq v(x,0) \quad in \quad \overline{\Omega}.$$

Then, we have $u(x,t) \geq v(x,t)$ in $\overline{\Omega} \times (0,T)$.

REMARK 4. Invoking the mean value theorem and Lemma 2.2, it is not hard to see that $v(x,t) \leq u(x,t)$ as long as all of them are defined. We infer that $T_h \geq T$. □

3. THE QUENCHING TIME

In this section, under some conditions, we show that the solution v of (3)–(4) quenches in a finite time and estimate its quenching time. We demonstrate in passing that, if the L^∞ norm of the initial datum is large enough, then the solution v of (3)–(4) quenches in a finite time and its quenching time goes to that of the solution of a differential equation as $\|u_0^h\|_\infty$ goes to b. Finally, we gather some results that we deem useful to prove the continuity of the quenching time time. Our first result says that the solution v of (3)–(4) always quenches in a finite time if the initial datum is nonnegative. It is stated in the following theorem.

THEOREM 5. *Let v be the solution of* (3)–(4). *Then v quenches in a finite time, and its quenching time T_h obeys the following estimate*

$$T_h \leq \int_A^b \frac{\mathrm{d}s}{f(s)},$$

where $A = \frac{1}{|\Omega|}\int_\Omega u_0^h(x)\mathrm{d}x$.

Proof. Since $(0,T_h)$ is the maximal time interval on which $\|v(.,t)\|_\infty < b$, our aim is to show that T_h is finite and satisfies the above inequality. Due to the fact that the initial datum u_0^h is nonnegative in $\overline{\Omega}$, we know from Lemma 2.1 that the solution v is also nonnegative in $\overline{\Omega} \times (0,T_h)$. Integrating both sides of (3) over $(0,t)$, we find that

$$v(x,t) - u_0^h(x) = \int_0^t \int_\Omega J(x-y)(v(y,s) - v(x,s))\mathrm{d}y\mathrm{d}s$$

$$+ \int_0^t f(v(x,s))\mathrm{d}s \quad \text{for} \quad t \in (0,T_h).$$

Integrate again in the x variable and apply Fubini's theorem to obtain

$$\int_\Omega v(x,t)\mathrm{d}x - \int_\Omega u_0^h(x)\mathrm{d}x = \int_0^t \Big(\int_\Omega f(v(x,s)\Big)\mathrm{d}s \quad \text{for} \quad t\in(0,T_h). \tag{10}$$

Set

$$w(t) = \tfrac{1}{|\Omega|}\int_\Omega v(x,t)\mathrm{d}x \quad \text{for} \quad t\in[0,T_h).$$

Taking the derivative of w in t and using (10), we arrive at

$$w'(t) = \int_\Omega \tfrac{1}{|\Omega|} f(v(x,t))\mathrm{d}x, \quad \text{for} \quad t\in(0,T_h)$$

It follows from Jensen's inequality that $w'(t) \geq f(w(t))$ for $t\in(0,T_h)$, or equivalently

$$\tfrac{\mathrm{d}w}{f(w)} \geq \mathrm{d}t \quad \text{for} \quad t\in(0,T_h).$$

Integrate the above inequality over $(0,T_h)$ to obtain

$$T_h \leq \int_{w(0)}^b \tfrac{\mathrm{d}s}{f(s)}.$$

Since the quantity on the right hand side of the above inequality is finite, we deduce that v quenches in a finite time at the time T_h which obeys the above inequality. Use the fact that $w(0) = A$ to complete the rest of the proof. □

The above theorem allows us to obtain an estimate which depends on the L^1 norm of the initial datum. This kinds of estimation is not interesting in order to obtain the continuity of the quenching time as a function of the initial datum. Therefore, we shall give another result which reveals an estimate of the of the quenching time that depends on the L^∞ norm of the initial datum. This result is stated in the theorem below.

Theorem 6. *Let* $A = \int_0^b \frac{\mathrm{d}\sigma}{f(\sigma)}$. *If* $A < 1$, *then the solution* v *of* (3)–(4) *quenches in a finite time, and its quenching time* T_h *obeys the following estimate*

$$T_h \leq \tfrac{1}{1-A}\int_{\|u_0^h\|_\infty}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)}.$$

Proof. Since $(0,T_h)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h is finite and satisfies the above inequality. As in the proof of Theorem 3.1, an application of Lemma 2.1 reveals that the solution v is nonnegative in $\overline{\Omega}\times(0,T_h)$. Due to the fact that $J(z)$ is nonnegative for $z\in\mathbb{R}^N$, and

$$\int_\Omega J(x-y)\mathrm{d}y \leq \int_{\mathbb{R}^N} J(x-y)\mathrm{d}y = 1 \quad \text{for} \quad x\in\overline{\Omega},$$

we note that

$$v_t(x,t) \geq -v(x,t) + f(v(x,t)) \quad \text{in} \quad \overline{\Omega}\times(0,T_h),$$

which implies that

$$v_t(x,t) \geq f(v(x,t))\left(1 - \tfrac{v(x,t)}{f(v(x,t))}\right) \quad \text{in} \quad \overline{\Omega} \times (0, T_h).$$

It is not hard to see that

$$\int_0^b \tfrac{\mathrm{d}\sigma}{f(\sigma)} \geq \sup_{0 \leq t < b} \int_0^t \tfrac{\mathrm{d}\sigma}{f(\sigma)} \geq \sup_{0 \leq t < b} \tfrac{t}{f(t)},$$

because $f(s)$ is nondecreasing for $s \in [0, b)$. We infer that

$$v_t(x,t) \geq (1 - A) f(v(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T_h),$$

or equivalently

$$\tfrac{\mathrm{d}v}{f(v)} \geq (1 - A)\mathrm{d}t \quad \text{in} \quad \overline{\Omega} \times (0, T_h). \tag{11}$$

Integrate the above inequality over $(0, T_h)$ to obtain

$$(1 - A) T_h \leq \int_{u_0^h(x)}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)} \quad \text{in} \quad \overline{\Omega}.$$

It follows that

$$T_h \leq \tfrac{1}{1-A} \int_{\|u_0^h\|_\infty}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)}.$$

We conclude that the solution v of (3)–(4) quenches in a finite time, because the quantity on the right hand side of the above inequality is finite. This finishes the proof. □

Remark 7. Let $t_0 \in (0, T_h)$. Integrating the inequality (11) over (t_0, T_h), we find that

$$(1 - A)(T_h - t_0) \leq \int_{v(x,t_0)}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)} \quad \text{for} \quad x \in \overline{\Omega},$$

which implies that

$$T_h - t_0 \leq \tfrac{1}{1-A} \int_{\|v(\cdot,t_0)\|_\infty}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)}. \quad \square$$

It is worth noting that the above estimate is crucial to obtain the continuity of the quenching time as a function of the initial datum. Let us notice that the condition $A < 1$ of the above theorem is very restrictive in certain situations. By the following theorem, we avoid this condition in the case where the L^∞ norm of the initial datum is large enough.

Theorem 8. *Let v be the solution of* (3)–(4)*, and suppose that the initial datum at* (4) *obeys the following condition $f(\|u_0^h\|_\infty) > b$. Then, the solution v quenches in a finite time, and its quenching time T_h is estimated as follows*

$$T_h \leq \tfrac{f(\|u_0^h\|_\infty)}{f(\|u_0^h\|_\infty) - b} \int_{\|u_0^h\|_\infty}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)}.$$

Proof. Since $(0,T_h)$ is the maximal time interval of existence of the solution v, our aim is to show that T_h is finite and satisfies the above inequality. Owing to Lemma 2.1, we know that the solution v is nonnegative in $\overline{\Omega}\times(0,T_h)$ because the initial datum u_0^h is nonnegative in $\overline{\Omega}$. We note that

$$\int_\Omega J(x-y)\mathrm{d}y \le \int_{\mathbb{R}^N} J(x-y)\mathrm{d}y = 1 \quad \text{for} \quad x\in\overline{\Omega},$$

which implies that

$$v_t(x,t) \ge -v(x,t) + f(v(x,t)) \quad \text{in} \quad \overline{\Omega}\times(0,T_h). \tag{12}$$

Let $x_0(t)\in\overline{\Omega}$ be such that

$$U(t) = \max_{x\in\overline{\Omega}} v(x,t) = v(x_0(t),t) \quad \text{for} \quad t\in(0,T_h).$$

It is easy to see that

$$U'(t) = \max_{x\in\overline{\Omega}} v_t(x,t) \quad \text{for} \quad t\in(0,T_h).$$

Consequently, replacing x by $x_0(t)$ in (12), we have

$$U'(t) \ge v_t(x,t) \ge -U(t) + f(U(t)) \quad \text{for} \quad t\in(0,T_h).$$

This estimate may be rewritten as follows

$$U'(t) \ge f(U(t))\left(1 - \tfrac{U(t)}{f(U(t))}\right) \quad \text{for} \quad t\in(0,T_h).$$

According to the fact that $U(t)\le b$, the above estimate becomes

$$U'(t) \ge f(U(t))\left(1 - \tfrac{b}{f(U(t))}\right) \quad \text{for} \quad t\in(0,T_h). \tag{13}$$

We note that $U'(0)>0$, and we claim that $U'(t)>0$ for $t\in(0,T_h)$. To prove the claim, we argue by contradiction. Let t_0 be the first $t\in(0,T_h)$ such that $U'(t)>0$ for $t\in(0,t_0)$, but $U'(t_0)=0$. This implies that $U(t_0)\ge U(0) = \|u_0^h\|_\infty$. Therefore, we get

$$0 = U'(t_0) \ge f(\|u_0^h\|_\infty)\left(1 - \tfrac{b}{f(\|u_0^h\|_\infty)}\right) > 0,$$

which is a contradiction, and the claim is proved. In view of the claim, we find that $U(t)\ge\|u_0^h\|_\infty$ for $t\in(0,T_h)$, and making use of (13), we arrive at

$$U'(t) \ge \left(1 - \tfrac{b}{f(\|u_0^h\|_\infty)}\right) f(U(t)) \quad \text{for} \quad t\in(0,T_h),$$

or equivalently

$$\tfrac{dU}{f(U)} \ge \left(1 - \tfrac{b}{f(\|u_0^h\|_\infty)}\right) dt \quad \text{for} \quad t\in(0,T_h). \tag{14}$$

Integrating the above estimate over $(0,T_h)$ we obtain

$$\left(1 - \tfrac{b}{f(\|u_0^h\|_\infty)}\right) T_h \le \int_{\|u_0^h\|_\infty}^b \tfrac{\mathrm{d}\sigma}{f(\sigma)},$$

which implies that

$$T_h \leq \frac{f(\|u_0^h\|_\infty)}{f(\|u_0^h\|_\infty)-b} \int_{\|u_0^h\|_\infty}^{b} \frac{d\sigma}{f(\sigma)}.$$

We use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

REMARK 9. Let $t_0 \in (0, T_h)$. Integrating the estimate (14) over (t_0, T_h), we discover that

$$T_h - t_0 \leq \frac{f(\|u_0^h\|_\infty)}{f(\|u_0^h\|_\infty)-b} \int_{\|v(\cdot,t_0)\|_\infty}^{b} \frac{d\sigma}{f(\sigma)}. \quad \square$$

Up to now, the results obtained allow us to see some upper bounds of the quenching time. In the theorem below, we derive a lower bound of the quenching time when quenching occurs.

THEOREM 10. *Suppose that the solution v of* (3)–(4) *quenches in a finite time T_h. Then, we have*

$$T_h \geq \int_{\|u_0^h\|_\infty}^{b} \frac{d\sigma}{f(\sigma)}.$$

Proof. Let $\alpha(t)$ be the solution of the following ordinary differential equation

$$\alpha'(t) = f(\alpha(t)), \quad t \in (0, T_e), \quad \alpha(0) = \|u_0^h\|_\infty,$$

where $(0, T_e)$ is the maximal time interval of existence of the solution $\alpha(t)$. By a routine computation, one easily sees that $T_e = \int_{\|u_0^h\|_\infty}^{b} \frac{d\sigma}{f(\sigma)}$. Now, let us introduce the function z defined as follows

$$z(x,t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times [0, T_e).$$

A straightforward calculation yields

$$z_t(x,t) = \int_\Omega J(x-y)(z(y,t) - z(x,t))dy + f(z(x,t)) \quad \text{in} \quad \overline{\Omega} \times (0, T_e),$$

$$z(x,0) \geq v(x,0) \quad \text{in} \quad \overline{\Omega}.$$

Set

$$w(x,t) = z(x,t) - v(x,t) \quad \text{in} \quad \overline{\Omega} \times [0, T_*),$$

where $T_* = \min\{T_h, T_e\}$. Making use of the mean value theorem, we find that

$$w_t(x,t) \geq \int_\Omega J(x-y)(w(y,t) - w(x,t))dy + f'(\xi(x,t))w(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, T_*),$$

$$w(x,0) \geq 0 \quad \text{in} \quad \overline{\Omega},$$

where $\xi(x,t)$ is an intermediate value between $v(x,t)$ and $z(x,t)$. It follows from Lemma 2.1 that

$$w(x,t) \geq 0 \quad \text{in} \quad \overline{\Omega} \times (0, T_*),$$

or equivalently

$$v(x,t) \leq \alpha(t) \quad \text{in} \quad \overline{\Omega} \times (0, T_*). \tag{15}$$

We claim that $T_h \geq T_e$. To prove the claim, we argue by contradiction. Suppose that $T_h < T_e$. In view of (15), we see that $\|v(\cdot, T_h)\|_\infty \leq \alpha(T_h) < b$, which contradicts the fact that $(0, T_h)$ is the maximum time interval of existence of the solution v. This demonstrates the claim, and the proof is complete. □

REMARK 11. Combining Theorems 3.1 and 3.4, we note that, if the initial datum at (4) satisfies $u_0^h = \beta = \text{const} \geq 0$, then the solution v of (3)–(4) quenches in a finite time $T_h = \int_\beta^b \frac{d\sigma}{f(\sigma)}$. □

With the aid of Theorems 3.3 and 3.4, we can derive the following interesting result.

THEOREM 12. *Let v be the solution of (3)–(4), and suppose that the initial datum (4) obeys the following condition $f(\|u_0^h\|_\infty) > b$. Then, the solution v quenches in a finite time, and its quenching time T_h obeys the following estimates*

$$0 \leq T_h - T_e \leq \frac{bT_e}{f(\|u_0^h\|_\infty)} + o\left(\frac{T_e}{f(\|u_0^h\|_\infty)}\right) \quad \text{as} \quad \|u_0^h\|_\infty \to b,$$

where $T_e = \int_{\|u_0^h\|_\infty}^b \frac{d\sigma}{f(\sigma)}$.

Proof. Since $(0, T_h)$ is the maximal time interval of existence of the solution u, our aim is to show that T_h is finite and satisfies the above estimates. Making use of Theorems 3.3 and 3.4, we find that T_e is finite and obeys the following estimates

$$T_e \leq T_h \leq \frac{T_e}{1 - \frac{b}{f(\|u_0^h\|_\infty)}}. \tag{16}$$

Apply Taylor's expansion to obtain

$$\frac{1}{1 - \frac{b}{f(\|u_0^h\|_\infty)}} = 1 + \frac{b}{f(\|u_0^h\|_\infty)} + o\left(\frac{1}{f(\|u_0^h\|_\infty)}\right) \quad \text{as} \quad \|u_0^h\|_\infty \to b.$$

Use (16) and the above relation to complete the rest of the proof. □

REMARK 13. The estimates of Theorem 3.5 can be rewritten as follows

$$0 \leq \frac{T_h}{T_e} - 1 \leq \frac{b}{f(\|u_0^h\|_\infty)} + o\left(\frac{b}{f(\|u_0^h\|_\infty)}\right) \quad \text{as} \quad \|u_0^h\|_\infty \to b.$$

We infer that

$$\lim_{\|u_0^h\|_\infty \to b} \frac{T_h}{T_e} = 1. \quad \square$$

4. CONTINUITY OF THE QUENCHING TIME

In this section, under some assumptions, we show that the solution v of (3)–(4) quenches in a finite time, and its quenching time goes to that of the solution u of (1)–(2) when the parameter h goes to zero. In order to obtain the above result, we firstly reveal that the solution v approaches the solution u in any interval $\overline{\Omega} \times [0, T-\tau]$ where τ *in*$(0, T)$. This result is stated in the following theorem.

THEOREM 14. *Assume that the problem* (1)–(2) *has a solution* $u \in C^{0,1}(\overline{\Omega} \times [0, T))$ *such that* $\sup_{t\in[0,T-\tau]} \|u(\cdot,t)\|_\infty \le b-\alpha$, *where* $\alpha \in (0, b)$ *and* $\tau \in (0, T)$. *Suppose that the initial datum at* u_0^h *satisfies the following condition*

$$\|u_0^h - u_0\|_\infty = o(1) \quad \text{as} \quad h \to 0. \tag{17}$$

Then, the problem (3)–(4) *admits a unique solution* $v \in C^{0,1}(\overline{\Omega} \times [0, T_h))$, *and the following relation holds*

$$\sup_{t\in[0,T-\tau]} \|v(\cdot,t) - u(\cdot,t)\|_\infty = O(\|u_0^h - u_0\|_\infty) \quad \text{as} \quad h \to 0,$$

where $\tau \in (0, T)$.

Proof. The problem (3)–(4) admits a unique solution $v \in C^{0,1}(\overline{\Omega} \times [0, T_h))$. In Remark 2.1, we have mentioned that $T_h \ge T$. Let $t(h) \le T - \tau$ be the first t such that

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty < \tfrac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{18}$$

We know from (17) that $t(h) > 0$ for h small enough. An application of the triangle inequality yields

$$\|v(\cdot,t)\|_\infty \le \|u(\cdot,t)\|_\infty + \|v(\cdot,t) - u(\cdot,t)\|_\infty \quad \text{for} \quad t \in (0, t(h)),$$

which implies that

$$\|v(\cdot,t)\|_\infty \le b - \alpha + \tfrac{\alpha}{2} \le b - \tfrac{\alpha}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{19}$$

Introduce the error e defined as follows

$$e(x,t) = v(x,t) - u(x,t) \quad \text{in} \quad \overline{\Omega} \times [0, t(h)).$$

Making use of the mean value theorem, we find that

$$e_t(x,t) = \int_\Omega J(x-y)(e(y,t) - e(x,t))\mathrm{d}y + f'(\xi(x,t))e(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)),$$

$$e(x,0) = u_0^h(x) - u_0(x) \quad \text{in} \quad \overline{\Omega},$$

where $\xi(x,t)$ is an intermediate value between $v(x,t)$ and $u(x,t)$. Set

$$z(x,t) = e^{(L+1)t}\|u_0^h - u_0\|_\infty \quad \text{in} \quad \overline{\Omega} \times [0, T],$$

where $L = f'(b - \frac{\alpha}{2})$. Due the fact that $\|u(\cdot,t)\|_\infty \le b - \alpha$ for $t \in (0, t(h))$, having in mind (19), it is not hard to see that $L \ge f'(\xi(x,t)) \in \overline{\Omega} \times (0, t(h))$. A straightforward computation reveals that

$$z_t(x,t) \ge \int_\Omega J(x-y)(z(y,t)-z(x,t))\mathrm{d}y + f'(\xi(x,t))z(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)),$$

$$z(x,0) \ge e(x,0) \quad \text{in} \quad \overline{\Omega}.$$

Invoking Lemma 2.2, we obtain

$$z(x,t) \ge e(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)).$$

In the same way, we also prove that

$$z(x,t) \ge -e(x,t) \quad \text{in} \quad \overline{\Omega} \times (0, t(h)),$$

which implies that

$$\|v(\cdot,t) - u(\cdot,t)\|_\infty \le \mathrm{e}^{(L+1)t}\|u_0^h - u_0\|_\infty \quad \text{for} \quad t \in (0, t(h)). \tag{20}$$

Now, we claim that $t(h) = T - \tau$. To prove the claim, we argue by contradiction. Suppose that $t(h) < T - \tau$. In view of (18) and (20), it is easy to check that

$$\tfrac{\alpha}{2} \le \|v(\cdot, t(h)) - u(\cdot, t(h))\|_\infty \le \mathrm{e}^{(L+1)T}\|u_0^h - u_0\|_\infty.$$

Since the term on the right hand side of the above inequality goes to zero as h goes to zero, infer that $\frac{\alpha}{2} \le 0$, which is a contradiction. This demonstrates the claim, and the proof is complete. □

At the moment, we are in a position to prove the main result of this section.

Theorem 15. *Assume that the problem* (1)–(2) *has a solution u which quenches in a finite time T such that $u \in C^{0,1}(\overline{\Omega} \times [0,T))$. Suppose that the initial datum at u_0^h satisfies the condition* (17). *Then, under the assumption of* Theorem 3.2, *the problem* (3)–(4) *admits a unique solution v which quenches in a finite time, and the following relation holds*

$$\lim_{h\to 0} T_h = T.$$

Proof. Let $0 < \varepsilon < T/2$. There exists a positive constant $\alpha \in (0,b)$ such that

$$\frac{1}{1-A}\int_{b-\alpha}^{b} \frac{\mathrm{d}\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{21}$$

Since u quenches at the time T, then there exists a time $T_0 \in (T - \varepsilon/2, T)$ such that $\|u(\cdot,t)\|_\infty \ge b - \frac{\alpha}{2}$ for $t \in [T_0, T)$. Invoking Theorem 4.1, we note that the problem (3)–(4) admits a unique solution v, and the following estimate holds $\|v(\cdot,T_0) - u(\cdot,T_0)\|_\infty \le \frac{\alpha}{2}$. Making use of the triangle inequality, we find that

$$\|v(\cdot,T_0)\|_\infty \ge \|u(\cdot,T_0)\|_\infty - \|v(\cdot,T_0) - u(\cdot,T_0)\|_\infty,$$

which implies that

$$\|v(\cdot, T_0)\|_\infty \geq b - \tfrac{\alpha}{2} - \tfrac{\alpha}{2} = b - \alpha.$$

In Remark 2.1 of the paper, we have revealed that $T_h \geq T$. We infer from (21) and Remark 3.1 that

$$0 \leq T_h - T \leq T_h - T_0 + T_0 - T \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. □

Remark 16. If in Theorem 4.2 we replace the assumption of Theorem 3.2 by that of Theorem 3.3, then the result of Theorem 4.2 remains valid. □

5. NUMERICAL RESULTS

In this section, we give some computational experiments to confirm the theory given in the previous section. We consider the problem (1)-(2) in the case where $\Omega = (-1, 1)$, $f(u) = (1-u)^p$ with $p > 1$,

$$J(x) = \begin{cases} \frac{3}{2}x^2, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

$u_0(x) = \gamma(\frac{2-\varepsilon(\sin(\pi x))^2}{4})$ with $\gamma > 0$, $\varepsilon \in (0, 1]$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer, and let $h = 2/I$. Define the grid $x_i = -1 + ih$, $0 \leq i \leq I$, and approximate the solution u of (1)-(2) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\begin{aligned} \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} =& \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)}) + (1 - U_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \\ U_i^{(0)} =& \varphi_i, \quad 0 \leq i \leq I, \end{aligned}$$

where $\varphi_i = \gamma(\frac{2-\varepsilon(\sin(\pi x_i))^2}{4})$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}$$

with $\|U_h^{(n)}\|_\infty = \max_{0 \leq i \leq I} |U_i^{(n)}|$. Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution. We also approximate the solution u of (1)–(2) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\begin{aligned} \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} =& \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n+1)} - U_i^{(n+1)}) + (1 - U_i^{(n)})^{-p}, \quad 0 \leq i \leq I, \\ U_i^{(0)} =& \varphi_i, \quad 0 \leq i \leq I. \end{aligned}$$

As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{p+1}.$$

Let us again remark that for the above implicit scheme, existence and nonnegativity of the discrete solution are also guaranteed using standard methods (see, for instance [9]). We need the following definition.

Definition 17. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if* $\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = 1$, *and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the numerical quenching time of the discrete solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Tables 1–8 show the numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit and implicit Euler method.

I	t_n	n	CPU time	s
16	0.1267303	74	1.1	-
32	0.1254584	295	1.8	-
64	0.1251387	1179	6.26	1.992
128	0.1250586	4715	78	1.996

Table 1. Explicit Euler method, $p = 1$, $\gamma = 1$, $\varepsilon = 1/100$

I	t_n	n	CPU time	s
16	0.1267353	75	1.8	-
32	0.1254588	295	2.1	-
64	0.1251389	1179	6.26	1.995
128	0.1250587	4715	78	1.996

Table 2. The implicit Euler method, $p = 1$, $\gamma = 1$, $\varepsilon = 1/100$

I	t_n	n	CPU time	s
16	0.0326408	35	2.2	-
32	0.0319626	139	15.6	-
64	0.0317912	554	74	1.98
128	0.0317480	2215	79	2.01

Table 3. The explicit Euler method, $p = 1$, $\gamma = 1.5$, $\varepsilon = 1$

I	t_n	n	CPU time	s
16	0.0326418	35	2.4	-
32	0.0319629	139	16.1	-
64	0.0317915	554	75.4	1.981
128	0.0317481	2215	79	1.985

Table 4. The implicit Euler method, $p = 1$, $\gamma = 1.5$, $\varepsilon = 1$

I	t_n	n	CPU time	s
16	0.0321300	35	1.4	-
32	0.0314844	137	6.1	-
64	0.0313305	548	62	2.89
128	0.0312703	2189	96	1.35

Table 5. The explicit Euler method, $p = 1$, $\gamma = 1.5$, $\varepsilon = 1/100$

I	t_n	n	CPU time	s
16	0.0321305	35	1.6	-
32	0.0314846	137	6.1	-
64	0.0313325	548	63	2.08
128	0.0312713	2189	98	1.31

Table 6. The implicit Euler method, $p = 1$, $\gamma = 1.5$, $\varepsilon = 1/100$

I	t_n	n	CPU time	s
16	3.684595 e-4	4	1.5	-
32	3.371334 e-4	13	9.6	-
64	3.190056 e-4	52	78	2.02
128	3.145623 e-4	207	499	0.78

Table 7. The explicit Euler method, $p = 1$, $\gamma = 1.95$, $\varepsilon = 1$

I	t_n	n	CPU time	s
16	3.684596 e-4	4	2.5	-
32	3.371394 e-4	13	10.6	-
64	3.190086 e-4	52	79	2.02
128	3.145624 e-4	207	499	0.78

Table 8. The implicit Euler method, $p = 1$, $\gamma = 1.95$, $\varepsilon = 1$

REMARK 18. If we consider the problem (1)–(2) in the case where $f(u) = (1-u)^{-1}$ and $u_0(x) = 1/2$, then we know from Remark 3.3 that the quenching time of the solution u equals 0.125. We observe from Tables 1 to 2 that when ε is small enough, then the numerical quenching time is approximately equal to 0.125. This fact confirms the result established within the framework of the continuity. On the other hand, the quenching time T_e of the solution of the following differential equation $\alpha'(t) = (1-\alpha(t))^{-1}$, $t > 0$, $\alpha(0) = \frac{\gamma}{2}$ is given explicitly by $T_e = (2-\gamma)^2/8$, and

$$T_e = \left\{ \begin{array}{ccc} 3.145e-4 & \text{when} & \gamma = 1.95 \end{array} \right\}.$$

We note from Tables 3 to 4 that, when $\gamma = 1.95$, then numerical quenching time of the discrete solution is approximately equal that to T_e. These results illustrate the idea of Theorem 3.5. □

For other illustrations, in what follows, we shall give some plots. In the following figures, we can appreciate that the discrete solution quenches in a finite time.

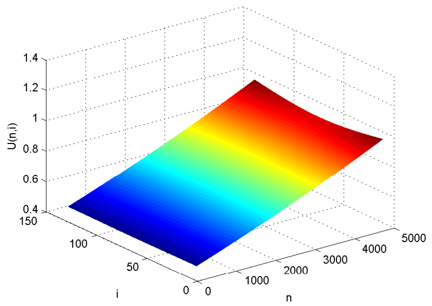

Fig. 1. Evolution of the discrete solution, $\gamma = 1$, $\varepsilon = \frac{1}{100}$ (explicit scheme).

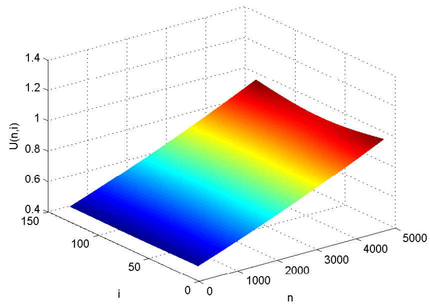

Fig. 2. Evolution of the discrete solution, $\gamma = 1$, $\varepsilon = \frac{1}{100}$ (implicit scheme).

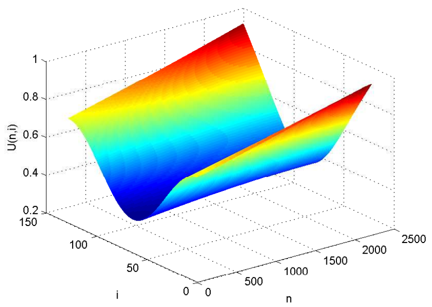

Fig. 3. Evolution of the discrete solution, $\gamma = 1.5$, $\varepsilon = 1$ (explicit scheme).

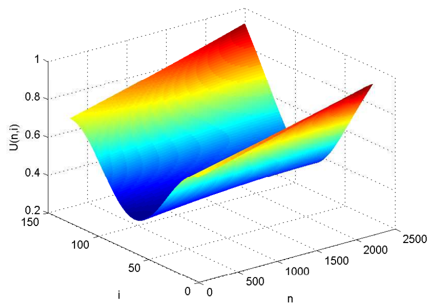

Fig. 4. Evolution of the discrete solution, $\gamma = 1.5$, $\varepsilon = 1$ (implicit scheme).

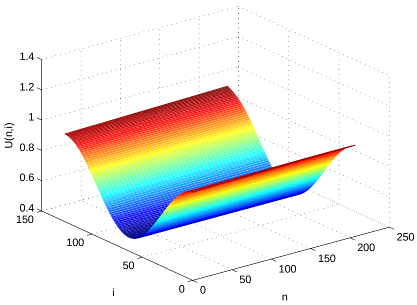

Fig. 5. Evolution of the discrete solution, $\gamma = 1.95$, $\varepsilon = 1$ (explicit scheme).

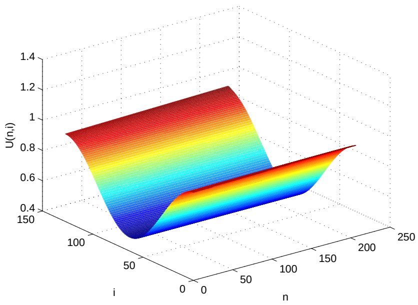

Fig. 6. Evolution of the discrete solution, $\gamma = 1.95$, $\varepsilon = 1$ (implicit scheme).

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