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THE APPROXIMATION OF BIVARIATE FUNCTIONS BY MODIFIED BIVARIATE OPERATORS AND GBS OPERATORS ASSOCIATED

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Abstract. In this paper we demonstrate a Voronovskaja-type theorem and approximation theorem for a class of modified operators and *Generalized Boolean Sum* (GBS) associated operators obtained (see (3)) from given operators.

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1. INTRODUCTION

In this paper we start from a class of linear and positive operators. We construct the bivariate operators and GBS operators associated for bivariate functions.

The aim of this paper is to modify these operators applying the G.H. Kirov idea (see [3]) and for the new class of operators, we demonstrate a Voronovskaja-type theorem and an approximation theorem.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this section we recall some notions and results which we will use in this paper.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following function sets: $B(I) = \big\{f | f : I \to \mathbb{R},\ f$ bounded on $I\big\}$, $(C(I) = \big\{f | f : I \to \mathbb{R},\ f$ continuous on $I\big\}$ and $C_B(I) = B(I) \cap C(I)$. For $x \in I$, let the function $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ for any $t \in I$ and let $e_0 : I \to \mathbb{R}$, $e_0(x) = 1$ for any $x \in I$.

If $I \subset \mathbb{R}$ is a given interval and $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f; \cdot) : [0, \infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\big\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\big\} \tag{1}$$

and then $\omega(-f, \delta) = \omega(f, \delta)$.

If $n \in \mathbb{N}$, $\delta \geq 0$ and $f_1, f_2, \ldots, f_n \in B(I)$, then

$$\omega(f_1 + f_2 + \cdots + f_n; \delta) \leq \omega(f_1;\delta) + \omega(f_2;\delta) + \cdots + \omega(f_n;\delta). \tag{2}$$

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals, $E(I_1 \times I_2)$, $F(J_1 \times J_2)$ which are subsets of the set of real functions defined on $I_1 \times I_2$, respectively $J_1 \times J_2$. In these sets,

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we have the classical operations of addition of functions and multiplication with scalars of functions.

Let $L : E(I_1 \times I_2) \to F(J_1 \times J_2)$ be a linear positive operator. The operator $UL : E(I_1 \times I_2) \to F((I_1 \cap J_1) \times (I_2 \cap J_2))$ defined for any function $f \in E(I_1 \times I_2)$, any $(x, y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ by

$$(ULf)(x,y) = (L((x,*) + f(\cdot,y) - f(\cdot,*)))(x,y) \tag{3}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [1]).

If $f \in E(I_1 \times I_2)$ and $(x, y) \in I_1 \times I_2$, let the functions $f_x = f(x, *)$, $f^y = f(\cdot, y) : I_1 \times I_2 \to \mathbb{R}$, $f_x(s,t) = f(x,t)$, $f^y(s,t) = f(s,y)$ for any $(s,t) \in I_1 \times I_2$. Then, we can consider that f_x, f^y are functions of real variable, $f_x : I_2 \to \mathbb{R}$, $f_x(t) = f(x,t)$ for any $t \in I_2$ and $f^y : I_1 \to \mathbb{R}$, $f^y(s) = f^y(s,y)$ for any $s \in I_1$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function. The function $\omega_{total}(f;\cdot,*) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0,\infty) \times [0,\infty)$ by

$$\begin{aligned}\omega_{total}(f;\delta_1,\delta_2) = \sup\big\{&|f(x,y) - f(x',y')| : (x,y),(x',y') \in I_1 \times I_2,\\ &|x - x'| \le \delta_1,\ |y - y'| \le \delta_2\big\}\end{aligned} \tag{4}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [10]).

If $(L_m)_{m\ge1}$ is a sequence of operators, $L_m : E(I) \to F(J)$, $m \in \mathbb{N}$, for $i \in \mathbb{N}_0$ define T_i by

$$(T_i L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) \tag{5}$$

for any $x \in I \cap J$ and $m \in \mathbb{N}$, where $E(I)$, $F(J)$ are subsets of the set of real functions defined on I, respectively J.

We consider that if $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}\backslash\{0,1,\dots,n\}$, then $\binom{n}{k} = 0$ and $\binom{0}{0} = 1$. The identities

$$\sum_{j=0}^{k} (-1)^j \binom{n}{j} = \begin{cases} (-1)^k \binom{n-1}{k}, & 0 \le k \le n-1,\ n \in \mathbb{N}\\ 1, & n = 0\\ 0, & \text{in other cases}\end{cases} \tag{6}$$

and

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \tag{7}$$

are known, where $k, m, n \in \mathbb{N}_0$.

2. PRELIMINARIES

In this section we recall the construction and the results from the paper [8].

Let $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$ and similarly is defined q_n, $n \in \mathbb{N}$.

Let $I_1, I_2, J_1, J_2 \subset \mathbb{R}$ be intervals with $I_1 \cap J_1 \neq \emptyset$ and $I_2 \cap J_2 \neq \emptyset$. For $m, n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, q_n\} \cap \mathbb{N}_0$, we consider $\varphi_{m,k} : J_1 \to \mathbb{R}$, $\varphi_{m,k}(x) \geq 0$ for any $x \in J_1$, $\psi_{n,j} : J_2 \to \mathbb{R}$, $\psi_{n,j}(y) \geq 0$ for any $y \in J_2$ and the linear positive functionals $A_{m,k} : E(I_1) \to \mathbb{R}$, $B_{n,j} : E_2(I_2) \to \mathbb{R}$.

For $m, n \in \mathbb{N}$ define the sequences of operators $(L_m)_{m\geq 1}$ and $(K_n)_{n\geq 1}$ by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{8}$$

$$(K_n g)(y) = \sum_{j=0}^{q_n} \psi_{n,j}(y) B_{n,j}(g) \tag{9}$$

for any $f \in E_1(I_1)$, $g \in E_2(I_2)$, $x \in J_1$ and $y \in J_2$, where $E_1(I_1)$, $E_2(I_2)$ are subsets of the set of real functions defined on I_1, respectively I_2.

The operators $(L_m)_{m\geq 1}$ and $(K_n)_{n\geq 1}$ are linear and positive on $E_1(I_1 \cap J_1)$, respectively $E_2(I_2 \cap J_2)$.

In the following let $s \in \mathbb{N}_0$, s even.

We suppose that the operators $(L_m)_{m\geq 1}$, $(K_n)_{n\geq 1}$ verify the conditions: there exist the smallest α_j, $\beta_j \in [0, \infty)$, $j \in \{0, 2, 4, \ldots, s+2\}$, such that

$$\lim_{m\to\infty} \tfrac{(T_j L_m)(x)}{m^{\alpha_j}} = \alpha_j(x) \tag{10}$$

for any $x \in I_1 \cap J_1$,

$$\lim_{n\to\infty} \tfrac{(T_j K_n)(y)}{n^{\beta_j}} = b_j(y) \tag{11}$$

for any $y \in I_2 \cap J_2$ and if we note

$$\gamma_s = \max\left\{\alpha_{s-2l} + \beta_{2l} : l \in \{0, 1, \ldots, \tfrac{s}{2}\}\right\}, \tag{12}$$

then

$$\begin{cases} \alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2 < 0 \\ \alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4 < 0 \end{cases} \tag{13}$$

where $l \in \{0, 1, 2, \ldots \frac{s}{2}\}$.

In the following we consider the set $E(I_1 \times I_2) = \big\{f | f : I_1 \times I_2 \to \mathbb{R}$, $f_x \in E_2(I_2)$ for any $x \in I_1$ and $f^y \in E_1(I_1)$ for any $y \in I_2\big\}$.

For $m, n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I_1 \times I_2) \to \mathbb{R}$ with the property

$$A_{m,n,k,j}\left((\cdot - x)^i (* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right) B_{n,j}\left((* - y)^l\right) \tag{14}$$

for any $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, q_n\} \cap \mathbb{N}_0$, $i, l \in \{0, 1, \ldots, s\}$ and $x \in I_1$, $y \in I_2$.

Let $m, n \in \mathbb{N}$. The operator $L^*_{m,n}$ defined for any function $f \in E(I_1 \times I_2)$ and any $(x, y) \in J_1 \times J_2$ by

$$(15)\qquad \left(L^*_{m,n} f\right)(x, y) = \sum_{k=0}^{p_m} \sum_{j=0}^{q_n} \varphi_{m,k}(x)\psi_{n,j}(y)A_{m,n,k,j}(f)$$

is named the bivariate operator of LK-type.

The operators $\left(L^*_{m,n}\right)_{m,n\geq 1}$ are linear and positive on $E\left((I_1 \cap J_1) \times (I_2 \cap J_2)\right)$.

In the following we consider that

$$(16)\qquad (T_0 L_m)(x) = A_{m,0}(e_0) = 1$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$ and

$$(17)\qquad (T_0 K_n)(y) = B_{n,0}(e_0) = 1$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$.

From (16), (17) it results immediately that

$$(18)\qquad \sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$$

for any $x \in I_1 \cap J_1$, $m \in \mathbb{N}$,

$$(19)\qquad \sum_{j=0}^{q_n} \psi_{n,j}(y) = 1$$

for any $y \in I_2 \cap J_2$, $n \in \mathbb{N}$ and then

$$(20)\qquad \alpha_0 = \beta_0 = 0.$$

In the following, we note by t, τ the first, respectively second variable of function.

Theorem 1. *Let $f : I_1 \times I_2 \to \mathbb{R}$ be a bivariate function. If $(x, y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x, y), then*

$$(21)\qquad \lim_{m\to\infty} m^{s-\gamma_s} \Bigg[\left(L^*_{m,m} f\right)(x, y) - \\ - \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x, y)\left(T_{i-l} L_m\right)(x)\left(T_l K_m\right)(y) \Bigg] = 0.$$

If f admits partial derivatives of order s continuous on $(I_1 \cap J_1) \times (I_2 \cap J_2)$ and there exist the intervals $K_1 \subset I_1 \cap J_1$, $K_2 \subset I_2 \cap J_2$ such that there exist $m(s) \in \mathbb{N}$ and $a_{2l}, b_{2l} \in \mathbb{R}$ depending on K_1, respectively K_2, so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and for any $(x, y) \in K_1 \times K_2$ we have

$$(22)\qquad \frac{(T_{2l} L_m)(x)}{m^{\alpha_{2l}}} \leq a_{2l},$$

$$\frac{(T_{2l}K_m)(y)}{m^{\beta_{2l}}} \leq b_{2l}, \tag{23}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2} + 1\right\}$, *then the convergence given in* (21) *is uniform on* $K_1 \times K_2$ *and*

$$\begin{aligned} m^{s-\gamma_s} \Bigg| & \left(L^*_{m,m} f\right)(x,y) - \\ & - \sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l}(x,y)\left(T_{i-l}L_m\right)(x)\left(T_l K_m\right)(y)\Bigg| \leq \\ \leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} & \binom{\frac{s}{2}}{l} (a_{s-2l} + a_{s-2l+2})(b_{2l} + b_{2l+2}) \cdot \\ & \cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\delta_s}}}, \frac{1}{\sqrt{m^{\delta_s}}}\right) \end{aligned} \tag{24}$$

for any $(x, y) \in K_1 \times K_2$, *any* $m \in \mathbb{N}$ *with* $m \geq m(s)$, *where*

$$\begin{aligned} \delta_s = -\max \Big\{ & \alpha_{s-2l} + \beta_{2l+2} - \gamma_s - 2, \alpha_{s-2l+2} + \beta_{2l} - \gamma_s - 2, \\ & \tfrac{1}{2}\left(\alpha_{s-2l+2} + \beta_{2l+2} - \gamma_s - 4\right) : l \in \{0, 1, \ldots, \tfrac{s}{2}\} \Big\}. \end{aligned} \tag{25}$$

In the following, in addition we suppose that

$$\alpha_{s+2} < \alpha_s + 2, \tag{26}$$

$$\beta_{s+2} < \beta_s + 2 \tag{27}$$

and for any $f \in E(I_1 \times I_2)$ we have

$$A_{m,n,k,j}(f_x) = B_{n,j}(f_x), \tag{28}$$

$$A_{m,n,k,j}(f^y) = A_{m,k}(f^y), \tag{29}$$

$$A_{m,n,k,j}(f) = A_{m,k}(B_{n,j}(f_x)) = B_{n,j}(A_{m,k}(f^y)) \tag{30}$$

for any $x \in I_1$, $y \in I_2$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, $j \in \{0, 1, \ldots, q_n\} \cap \mathbb{N}_0$, $m, n \in \mathbb{N}$.

Now, let $\left(UL^*_{m,n}\right)_{m,n\geq 1}$ be the GBS operators associated to the $\left(L^*_{m,n}\right)_{m,n\geq 1}$ operators.

Lemma 2. *If* $m, n \in \mathbb{N}$, *then* $UL^*_{m,n}$ *have the form*

$$\left(UL^*_{m,n} f\right)(x,y) = (K_n f_x)(y) + (L_m f^y)(x) - \left(L^*_{m,n} f\right)(x,y) \tag{31}$$

for any $(x, y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$, *any* $f \in E(I_1 \times I_2)$.

THEOREM 3. *Let $f : I_1 \times I_2 \to \mathbb{R}$ be a bivariate function. If $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then*

$$\lim_{m\to\infty} m^{s-\gamma_s}\Bigg\{ \left(UL^*_{m,m}f\right)(x,y) - \sum_{i=0}^{s} \frac{1}{m^i i!}\Bigg[\Bigg(\frac{\partial^i f}{\partial \tau^i}(x,y)\left(T_iK_m\right)(y) + \frac{\partial^i f}{\partial t^i}(x,y)\left(T_iL_m\right)(x)\Bigg) - \sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial \tau^l}(x,y)\left(T_{i-l}L_m\right)(x)\left(T_lK_m\right)(y)\Bigg]\Bigg\} = 0. \tag{32}$$

If f admits partial derivatives of order s continuous on $(I_1 \cap J_1) \times (I_2 \cap J_2)$ and there exist the intervals $K_1 \subset I_1 \cap J_1$, $K_2 \subset I_2 \cap J_2$ such that there exist $m(s) \in \mathbb{N}$ and $a_{2l}, b_{2l} \in \mathbb{R}$ depending on K_1, respectively K_2 so that for any $m \in \mathbb{N}$, $m \geq m(s)$ and any $(x,y) \in K_1 \times K_2$ we have

$$\frac{(T_{2l}L_m)(x)}{m^{\alpha_{2l}}} \leq a_{2l}, \tag{33}$$

$$\frac{(T_{2l}K_m)(y)}{m^{\beta_{2l}}} \leq b_{2l} \tag{34}$$

where $l \in \{0, 1, \ldots, \frac{s}{2}+1\}$, then the convergence given in (32) *is uniform on $K_1 \times K_2$ and*

(35)

$$m^{s-\gamma_s}\Bigg|(UL^*_{m,m}f)(x,y) - \sum_{i=0}^{s}\frac{1}{m^i i!}\Bigg[\frac{\partial^i f}{\partial \tau^i}(x,y)(T_iK_m)(y) + \frac{\partial^i f}{\partial t^i}(x,y)(T_iL_m)(x) - \sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(T_{i-l}L_m)(x)(T_lK_m)(y)\Bigg]\Bigg| \leq$$

$$\leq \frac{1}{s!}\Bigg[(b_s+b_{s+2})\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2+\beta_s-\beta_{s+2}}}}\right) + (a_s+a_{s+2})\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right) + \sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(a_{s-2l}+a_{s-2l+2})(b_{2l}+b_{2l+2})\sum_{i=0}^{s}\binom{s}{i}\cdot\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m^{\delta_s}}},\frac{1}{\sqrt{m^{\delta_s}}}\right)\Bigg]$$

for any $(x,y) \in K_1 \times K_2$, any $m \in \mathbb{N}$, $m \geq m(s)$, where δ_s is given in (25).

3. MAIN RESULTS

Let the fixed number $r \in \mathbb{N}_0$ and $\left(L^*_{m,n}\right)_{m,n\geq 1}$ the operators which we construct in Preliminaries.

We denote by $E^r(I_1 \times I_2)$ the set of all function $f \in E(I_1 \times I_2)$ with the partial derivatives $\frac{\partial^k f}{\partial t^{k-i}\partial \tau^i}$, $i \in \{0,1,\ldots,k\}$ and $k \in \{0,1,\ldots,r\}$ belonging to $E(I_1 \times I_2)$.

For $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$, $\alpha \in \{0,1,\ldots,r\}$ we consider the functions $g_{x,y,\alpha}$, $g_{x,y} : I_1 \times I_2 \to \mathbb{R}$, defined by

$$g_{x,y,\alpha}(t,\tau) = \left(\tfrac{\partial}{\partial t}\,(x-t) + \tfrac{\partial}{\partial \tau}\,(y-\tau)\right)^{\alpha} f(t,\tau) = \\ = \sum_{\beta=0}^{\alpha} \tbinom{\alpha}{\beta} \tfrac{\partial^\alpha f}{\partial t^{\alpha-\beta}\partial\tau^\beta}\,(t,\tau)(x-t)^{\alpha-\beta}(y-\tau)^{\beta}, \tag{36}$$

$$g_{x,y}(t,\tau) = \sum_{\alpha=0}^{r} \tfrac{1}{\alpha!}\, g_{x,y,\alpha}(t,\tau) \tag{37}$$

for any $(t,\tau) \in I_2 \times I_2$ and $f \in E^r(I_1 \times I_2)$.

Definition 4. For $m,n \in \mathbb{N}$ define the operator $L_{m,n,r}$ for any $f \in E^r(I_1 \times I_2)$, any $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ by

$$(L_{m,n,r}f)(x,y) = \sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(x)\psi_{n,j}(y) \sum_{\alpha=0}^{r} \tfrac{1}{\alpha!}\, A_{m,n,k,j}(g_{x,y,\alpha}) = \\ = \sum_{k=0}^{p_m}\sum_{j=0}^{q_n} \varphi_{m,k}(x)\psi_{n,j}(y) \sum_{\alpha=0}^{r}\sum_{\beta=0}^{\alpha} \tfrac{1}{\alpha!}\tbinom{\alpha}{\beta}\cdot \\ \cdot A_{m,n,k,j}\left(\tfrac{\partial^\alpha f}{\partial t^{\alpha-\beta}\partial\tau^\beta}(x-\cdot)^{\alpha-\beta}(y-*)^{\beta}\right). \tag{38}$$

Remark 1. If in (15) we replace the function f by the function $g_{x,y}$, $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$, we obtain the operators from (38). □

Remark 2. The operators $(L_{m,n,r})_{m,n\geq 1}$ are construct about G. H. Kirov idea, but for the bivariate functions. □

Remark 3. For $r = 0$ we obtain $L_{m,n,0} = L^*_{m,n}$ for any $m,n \in \mathbb{N}$. □

Theorem 5. *a) Let $f : I_1 \times I_2 \to \mathbb{R}$ be a function, $f \in E^r(I_1 \times I_2)$. If $(x,y) \in (I_1 \cap J_1) \times (I_2 \cap J_2)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then*

$$\lim_{m\to\infty} (L_{m,m,r}f)(x,y) = f(x,y), \tag{39}$$

$$\lim_{m\to\infty} (UL_{m,m,r}f)(x,y) = f(x,y) \tag{40}$$

if $s=0$ *and*

(41) $$\lim_{m\to\infty} m^{s-\gamma_s}\left[(L_{m,m,r}f)(x,y)-f(x,y)-\right.$$
$$\left.-(-1)^r\sum_{i=1}^{s}\frac{1}{m^i i!}\binom{i-1}{r}\sum_{l=0}^{i}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(T_{i-l}L_m)(x)(T_lK_m)(y)\right]=0$$

and

(42) $$\lim_{m\to\infty} m^{s-\gamma_s}\left\{(UL_{m,m,r}f)(x,y)-f(x,y)-\right.$$
$$-(-1)^r\sum_{i=1}^{s}\frac{1}{m^i i!}\binom{i-1}{r}\left[\left(\frac{\partial^i f}{\partial\tau^i}(x,y)(T_iK_m)(y)+\frac{\partial^i f}{\partial t^i}(x,y)(T_iL_m)(x)\right)-\right.$$
$$\left.\left.-\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(T_{i-l}L_m)(x)(T_lK_m)(y)\right]\right\}=0.$$

b) Assume that f *admits partial derivatives of order* s *continuous on* $(I_1\cap J_1)\times(I_2\cap J_2)$ *and there exist the intervals* $K_1\subset I_1\cap J_1$, $K_2\subset I_2\cap J_2$ *such that there exist* $m(s)\in\mathbb{N}$ *and* $a_{2l},b_{2l}\in\mathbb{R}$ *depending on* K_1*, respectively* K_2*, so that for any* $m\in\mathbb{N}$, $m\geq m(s)$ *and for any* $(x,y)\in K_1\times K_2$ *we have*

(43) $$\frac{(T_{2l}L_m)(x)}{m^{\alpha_{2l}}}\leq a_{2l},$$

(44) $$\frac{(T_{2l}K_m)(y)}{m^{\beta_{2l}}}\leq b_{2l},$$

where $l\in\left\{0,1,\ldots,\frac{s}{2}+1\right\}$.
c) In hypothesis b), then the convergence given in $(39)--(42)$ *are uniform on* $K_1\times K_2$ *and*

(45) $$|(L_{m,m,r}f)(x,y)-f(x,y)|\leq(1+a_2)(1+b_2)\omega_{total}\left(f;\frac{1}{\sqrt{m^{\delta_0}}},\frac{1}{\sqrt{m^{\delta_0}}}\right),$$

(46) $$|(UL_{m,m,r}f)(x,y)-f(x,y)|\leq(1+b_2)\omega\left(f_x;\frac{1}{\sqrt{m^{2-\beta_2}}}\right)+$$
$$+(1+a_2)\left(f^y;\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+(1+a_2)(1+b_2)\omega_{total}\left(f;\frac{1}{\sqrt{m^{\delta_0}}},\frac{1}{\sqrt{m^{\delta_0}}}\right)\leq$$
$$\leq(1+b_2)\omega\left(f_x;\frac{1}{\sqrt{m^{\delta_0}}}\right)+(1+a_2)\omega\left(f^y;\frac{1}{\sqrt{m^{\delta_0}}}\right)+$$
$$+(1+a_2)(1+b_2)\omega\left(f;\frac{1}{\sqrt{m^{\delta_0}}},\frac{1}{\sqrt{m^{\delta_0}}}\right)$$

for any $(x,y)\in K_1\times K_2$*, any* $m\in\mathbb{N}$, $m\geq m(0)$ *if* $s=0$*, where*

(47) $$\delta_0=-\max\left\{\beta_2-2,\alpha_2-2,\frac{1}{2}(\alpha_2+\beta_2-4)\right\}$$

and

$$(48)\qquad m^{s-\gamma_s}\Bigg|(L_{m,m,r}f)(x,y)-f(x,y)-$$

$$-(-1)^r\sum_{i=1}^{s}\tfrac{1}{m^i i!}\tbinom{i-1}{r}\sum_{l=0}^{i}\tfrac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(T_{i-l}L_m)(x)(T_lK_m)(y)\Bigg|\le$$

$$\le\tfrac{1}{s!}\sum_{\alpha=0}^{r}\tbinom{s}{\alpha}\sum_{l=0}^{\frac{s}{2}}\tbinom{\frac{s}{2}}{l}(a_{s-2l}+a_{s-2l+2})(b_{2l}+b_{2l+2})\sum_{i=0}^{s}\tbinom{s}{i}\cdot$$

$$\cdot\,\omega_{total}\left(\tfrac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\tfrac{1}{\sqrt{m^{\delta_s}}},\tfrac{1}{\sqrt{m^{\delta_s}}}\right),$$

$$(49)\ \ m^{s-\gamma_s}\Bigg|(UL_{m,m,r}f)(x,y)-f(x,y)-$$

$$-(-1)^r\sum_{i=1}^{s}\tfrac{1}{m^i i!}\tbinom{i-1}{r}\Bigg[\Bigg(\tfrac{\partial^i f}{\partial\tau^i}(x,y)(T_iK_m)(y)+\tfrac{\partial^i f}{\partial t^i}(x,y)(T_iL_m)(x)\Bigg)-$$

$$-\sum_{l=0}^{i}\tbinom{i}{l}\tfrac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)(T_{i-l}L_m)(x)(T_lK_m)(y)\Bigg]\Bigg|\le$$

$$\le\tfrac{1}{s!}\Bigg\{\tbinom{s-1}{r}\Bigg[(b_s+b_{s+2})\omega\left(\tfrac{\partial^s f_x}{\partial\tau^s};\tfrac{1}{\sqrt{m^{2+\beta_s-\beta_{s+2}}}}\right)+$$

$$+(a_s+a_{s+2})\omega\left(\tfrac{\partial^s f^y}{\partial t^s};\tfrac{1}{\sqrt{m^{2+a_s-a_{s+2}}}}\right)\Bigg]+\sum_{\alpha=0}^{s}\tbinom{s}{\alpha}\sum_{l=0}^{\frac{s}{2}}\tbinom{\frac{s}{2}}{l}\cdot$$

$$\cdot(a_{s-2l}+a_{s-2l+2})(b_{2l}+b_{2l+2})\sum_{i=0}^{s}\tbinom{s}{i}\omega_{total}\left(\tfrac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\tfrac{1}{\sqrt{m^{\delta_s}}},\tfrac{1}{\sqrt{m^{\delta_s}}}\right)\Bigg\}$$

for any $(x,y)\in K_1\times K_2$, any $m\in\mathbb{N}$, $m\ge m(s)$, if $s\ge 2$, where δ_s is given in (25).

Proof. For $(t,\tau)\in I_1\times I_2$ and $\alpha\in\{0,1,\dots,r\}$, we have from (36)

$$\tfrac{\partial^i g_{x,y,\alpha}}{\partial t^{i-l}\partial\tau^l}(t,\tau)=\sum_{\beta=0}^{\alpha}\tbinom{\alpha}{\beta}\tfrac{\partial^i}{\partial t^{i-l}\partial\tau^l}\left(\tfrac{\partial^\alpha f}{\partial t^{\alpha-\beta}\partial\tau^\beta}(t,\tau)(x-t)^{\alpha-\beta}(y-\tau)^\beta\right)=$$

$$=\sum_{\beta=0}^{\alpha}\tbinom{\alpha}{\beta}\tfrac{\partial^l}{\partial\tau^l}\left[\sum_{i_1=0}^{i-l}\tbinom{i-l}{i_1}\tfrac{\partial^{\alpha+i-l-i_1}f}{\partial t^{\alpha-\beta+i-l-i_1}\partial\tau^\beta}(t,\tau)((x-t)^{\alpha-\beta})^{(i_1)}(y-\tau)^\beta\right],$$

from where

$$\frac{\partial^i g_{x,y,\alpha}}{\partial t^{i-l}\partial\tau^l}(t,\tau)=\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\sum_{i_1=0}^{i-l}\sum_{i_2=0}^{l}\binom{i-l}{i_1}\binom{l}{i_2}\cdot$$
$$\cdot\frac{\partial^{\alpha+i-i_1-i_2}f}{\partial t^{\alpha-\beta+i-l-i_1}\partial\tau^{\beta+l-i_2}}(t,\tau)((x-t)^{\alpha-\beta})^{(i_1)}((y-\tau)^{\beta})^{(i_2)}.$$

Taking into account that the function $((x-t)^{\alpha-\beta})^{(i_1)}((y-\tau)^{\beta})^{(i_2)}$ of variable (t,τ) takes nonzero value in (x,y) if and only if $\alpha-\beta=i_1$ and $\beta=i_2$, we have that

$$\frac{\partial^i g_{x,y,\alpha}}{\partial t^{i-l}\partial\tau^l}(x,y)=(-1)^{\alpha}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\binom{i-l}{\alpha-\beta}\binom{l}{\beta}(\alpha-\beta)!\beta!\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)$$

so

$$\frac{\partial^i g_{x,y}}{\partial t^{i-l}\partial\tau^l}(x,y)=\left(\sum_{\alpha=0}^{r}\sum_{\beta=0}^{\alpha}(-1)^{\alpha}\binom{i-l}{\alpha-\beta}\binom{l}{\beta}\right)\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y) \tag{50}$$

for $i\in\{0,1,\dots,s\}$.

If $i\in\{0,1,\dots,s\}$ and $l=i$, from (50) we have that

$$\frac{\partial^i g_{x,y}}{\partial\tau^i}(x,y)=\sum_{\alpha=0}^{r}\sum_{\beta=0}^{\alpha}(-1)^{\alpha}\binom{0}{\alpha-\beta}\binom{i}{\beta}\frac{\partial^i f}{\partial\tau^i}(x,y)=$$
$$=\sum_{\alpha=0}^{r}(-1)^{\alpha}\binom{i}{\alpha}\frac{\partial^i f}{\partial\tau^i}(x,y).$$

But $\sum_{\alpha=0}^{r}(-1)^{\alpha}\binom{i}{\alpha}=(-1)^r\binom{i-1}{r}$, then

$$\frac{\partial^i g_{x,y}}{\partial\tau^i}(x,y)=(-1)^r\binom{i-1}{r}\frac{\partial^i f}{\partial\tau^i}(x,y) \tag{51}$$

and similarly

$$\frac{\partial^i g_{x,y}}{\partial t^i}(x,y)=(-1)^r\binom{i-1}{r}\frac{\partial^i f}{\partial t^i}(x,y), \tag{52}$$

$i\in\{1,\dots,s\}$.

If $i=0$ then

$$g(x,y)=f(x,y). \tag{53}$$

Applying Theorem 1 and Theorem 3 for the function $g_{x,y}$ and taking (6), (7) and (50)–(53) into account, we obtain the assertions from Theorem 5. $\square$

THEOREM 6. *If $m,n\in\mathbb{N}$, then $UL_{m,n,r}$ have the form*

$$(UL_{m,n,r}f)(x,y)=\sum_{\alpha=0}^{r}\frac{1}{\alpha!}\left[\left(K_n\frac{\partial^\alpha f_x}{\partial\tau^\alpha}(y-*)^\alpha\right)(y)+\right.$$
$$\left.+\left(L_m\frac{\partial^\alpha f^y}{\partial t^\alpha}(x-\cdot)^\alpha\right)(x)\right]-(L_{m,n,r}f)(x,y) \tag{54}$$

for any $(x,y)\in(I_1\cap J_1)\times(I_2\cap J_2)$ and $f\in E^r(I_1\times I_2)$.

Proof. From (36) we have that $g_{x,y,\alpha}(x,\tau)=\frac{\partial^\alpha f}{\partial\tau^\alpha}(x,\tau)$, so $g_{x,y,\alpha}(x,\tau)=\frac{\partial^\alpha f_x}{\partial\tau^\alpha}(\tau)$ and then

$$g_{x,y}(x,\tau)=\sum_{\alpha=0}^{r}\frac{1}{\alpha!}g_{x,y,\alpha}(x,\tau)=\sum_{\alpha=0}^{r}\frac{1}{\alpha!}\frac{\partial^\alpha f_x}{\partial\tau^\alpha}(\tau).$$

Similarly $g_{x,y}(t,y)=\sum\limits_{\alpha=0}^{r}\frac{1}{\alpha!}\frac{\partial^\alpha f^y}{\partial t^\alpha}(t)$ and if we replace the function f by the function $g_{x,y}$ in (31), we obtain relation (54). □

Now, we give an application.
If $I_1=J_1=[0,1]$, $I_2=J_2=[0,\infty)$, $E_1(I_1)=C([0,1])$, $E_2(I_2)=C_2([0,\infty))$, $p_m=m$, $q_n=\infty$, $\varphi_{m,k}(x)=p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}$, $\psi_{n,j}(y)=e^{-ny}\frac{(ny)^j}{j!}$, $A_{m,k}(f^y)=f\left(\frac{k}{m},y\right)$, $B_{n,j}(f_x)=f\left(x,\frac{j}{n}\right)$, $A_{m,n,k,j}(f)=f\left(\frac{k}{m},\frac{j}{n}\right)$ for any $(x,y)\in[0,1]\times[0,\infty)$, $m,n\in\mathbb{N}$, $k\in\{0,1,\ldots,m\}$, $j\in\mathbb{N}_0$ and $f\in E([0,1]\times[0,\infty))$. Then starting with $r\in\mathbb{N}_0$, $(B_m)_{m\geq1}$ the Bernstein operators and $(S_n)_{n\geq1}$ the Mirakjan-Favard-Szász operators, we obtain the operators $(L_{m,n,r})_{m,n\geq1}$ and $(UL_{m,n,r})_{m,n\geq1}$ defined for any function $f\in E([0,1]\times[0,\infty))$, $(x,y)\in[0,1)\times[0,\infty)$ and $m,n\in\mathbb{N}$ by

$$(L_{m,n,r}f)(x)=\sum_{k=0}^{m}\sum_{j=0}^{\infty}\binom{m}{k}x^k(1-x)^{m-k}e^{-ny}\frac{(ny)^j}{j!}\cdot$$
$$\cdot\sum_{\alpha=0}^{r}\frac{1}{\alpha!}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\frac{\partial^\alpha f}{\partial t^{\alpha-\beta}\partial\tau^\beta}\left(\frac{k}{m},\frac{j}{n}\right)\left(x-\frac{k}{m}\right)^{\alpha-\beta}\left(y-\frac{j}{n}\right)^\beta, \tag{55}$$

$$(UL_{m,n,r}f)(x,y)=\sum_{\alpha=0}^{r}\frac{1}{\alpha!}\left[e^{-ny}\sum_{j=0}^{\infty}\frac{(ny)^j}{j!}\frac{\partial^\alpha f_x}{\partial\tau^\alpha}\left(y-\frac{j}{n}\right)^\alpha+\right.$$
$$\left.+\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}\frac{\partial^\alpha f^y}{\partial t^\alpha}\left(x-\frac{k}{m}\right)^\alpha\right]-(L_{m,n,r}f)(x,y). \tag{56}$$

We have $\alpha_2=\beta_2=1$, $\delta_0=1$, $K_1=[0,1]$, $K_2=[0,b]$, $b>0$, $a_2=\frac{5}{4}$, $b_2=b$ and $m(0)=1$ (see [8]).

THEOREM 7. *Let* $f : [0,1] \times [0,\infty)$ *be a function,* $f \in E^r([0,1] \times [0,\infty))$*. If* $(x,y) \in [0,1] \times [0,\infty))$ *and* f *is continuous in* (x,y)*, then*

$$\lim_{m\to\infty} (L_{m,n,r}f)(x,y) = f(x,y) \tag{57}$$

and

$$\lim_{m\to\infty} (UL_{m,n,r}f)(x,y) = f(x,y). \tag{58}$$

If f *is continuous in* $[0,1] \times [0,\infty)$*, then the convergence given in* (57)*,* (58) *are uniform on* $[0,1] \times [0,b]$*,* $b > 0$ *and*

$$|(L_{m,n,r}f)(x,y) - f(x,y)| \le \tfrac{9}{4}(1+b)\omega_{total}\left(f; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{m}}\right), \tag{59}$$

$$\begin{aligned} \left|(UL_{m,n,r}f)(x,y) - f(x,y)\right| \le{}& (1+b)\omega\left(f_x; \tfrac{1}{\sqrt{m}}\right) + \tfrac{9}{4}\omega\left(f^y; \tfrac{1}{\sqrt{m}}\right) \\ &+ \tfrac{9}{4}(1+b)\omega_{total}\left(f; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{m}}\right) \end{aligned} \tag{60}$$

for any $(x,y) \in [0,1] \times [0,b]$*, any* $m \in \mathbb{N}$*.*

Proof. It results from Theorem 5. □

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