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CONVERGENCE OF HALLEY'S METHOD UNDER CENTERED LIPSCHITZ CONDITION ON THE SECOND FRÉCHET DERIVATIVE

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Abstract. We present a semi-local as well as a local convergence analysis of Halley's method for approximating a locally unique solution of a nonlinear equation in a Banach space setting. We assume that the second Fréchet-derivative satisfies a centered Lipschitz condition. Numerical examples are used to show that the new convergence criteria are satisfied but earlier ones are not satisfied.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is twice Fréchet-differentiable operator defined on a nonempty open and convex subset of a Banach space X with values in a Banach space Y.

Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modelling [1, 2, 6]. The solutions of these equations can be rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

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In the present study we provide a convergence analysis for Halley's method defined by [3, 4, 5, 8]

$$x_{n+1} = x_n - \Gamma_F(x_n)F'(x_n)^{-1}F(x_n), \text{ for each } n = 0, 1, 2, \ldots, \tag{1.2}$$

where, $\Gamma_F(x) = (I - L_F(x))^{-1}$ and $L_F(x) = \frac{1}{2}F'(x)^{-1}F''(x)F'(x)^{-1}F(x)$. The convergence of Halley's method has a long history and has been studied by many authors (cf [1-5,7,8] and the references therein). The most popular conditions for the semi-local convergence of Halley's method are given by

(C_1) There exists $x_0 \in D$ such that $F'(x_0)^{-1} \in L(Y, X)$, the space of bounded linear operator from Y into X;

(C_2) $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$;

(C_3) $\|F'(x_0)^{-1}F''(x)\| \leq M$ for each x in D;

(C_4) $\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq K\|x - y\|$ for each x and y in D.

The corresponding sufficient convergence condition is given by

$$\eta \leq \frac{4K + M^2 - M\sqrt{M^2 + 2K}}{3K(M + \sqrt{M^2 + 2K})}. \tag{1.3}$$

There are simple examples show that (C_4) is not satisfied. As an example, let $X = Y = \mathbb{R}$, $D = [0, +\infty)$ and define $F(x)$ on D by

$$F(x) = \frac{4}{15}x^{\frac{5}{2}} + x^2 + x + 1.$$

Then, we have that

$$|F''(x) - F''(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}.$$

Therefore, there is no constant K satisfying (C_4). Other examples where (C_4) is not satisfied can be found in [2]. We shall use the weaker than (C_3) and (C_4) conditions given by

$(C_3)'$ $\|F'(x_0)^{-1}F''(x_0)\| \leq \beta$;

$(C_4)'$ $\|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| \leq L\|x - x_0\|$ for each x in D.

Note that in this case for $x_0 > 0$

$$|F''(x) - F''(x_0)| \leq \frac{|x-x_0|}{\sqrt{x_0}} \quad \textit{for each } x \textit{ in } D.$$

Hence, we can choose $L = |F'(x_0)^{-1}|\frac{1}{\sqrt{x_0}}$. A semi-local convergence under conditions $(C_1), (C_2), (C_3)'$ and $(C_4)'$ has been given by Xu in [8] using recurrent relations. However, the semi-local analysis is false under the stated hypotheses. In fact, the following semi-local convergence theorem was established in Ref. [8].

Theorem 1. *Let $F : D \subset X \to Y$ be continuously twice Fréchet differentiable, D open and convex. Assume that there exists a starting point $x_0 \in D$ such that $F'(x_0)^{-1}$ exists, and the following conditions hold:*

(C_2) $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$;

$(C_3)'$ $\|F'(x_0)^{-1}F''(x_0)\| \leq \beta$;

condition $(C_4)'$ *is true;*
$\frac{1}{2}\beta\eta < \tau$, *where*

$$\tau = \tfrac{3s^\star+1-\sqrt{7s^\star+1}}{9s^\star-1} = 0.134065\dots, \tag{1.4}$$

$s^\star = 0.800576\dots$ *such that* $q(s^\star) = 1$, *and*

$$q(s) = \tfrac{(6s+2)-2\sqrt{7s+1}}{(6s-2)+\sqrt{7s+1}}(1 + \tfrac{s}{1-s^2}); \tag{1.5}$$

$\overline{U}(x_0, R) \subset D$, *where* R *is the positive solution of*

$$Lt^2 + \beta t - 1 = 0. \tag{1.6}$$

Then, the Halley sequence $\{x_k\}$ *generated by* (1.2) *remains in the open ball* $U(x_0, R)$, *and converges to the unique solution* $x^\star \in \overline{U}(x_0, R)$ *of Eq.* (1.1). *Moreover, the following error estimate holds*

$$\|x^\star - x_k\| \le \tfrac{a}{c(1-\tau)\gamma} \sum_{i=k+1}^{\infty} \gamma^{2^i}, \tag{1.7}$$

where $a = \beta\eta$, $c = \frac{1}{R}$ *and* $\gamma = \frac{a(a+4)}{(2-3a)^2}$.

We provide an example to show the results of the above theorem does not hold under the stated hypotheses.

Example 2. Let us define a scalar function $F(x) = 20x^3 - 54x^2 + 60x - 23$ on $D = (0, 3)$ with initial point $x_0 = 1$. Then, we have that

$$F'(x) = 12(5x^2 - 9x + 5), \quad F''(x) = 12(10x - 9). \tag{1.8}$$

Hence, we obtain $F(x_0) = 3$, $F'(x_0) = 12$, $F''(x_0) = 12$. We can choose $\eta = \frac{1}{4}$ and $\beta = 1$ in Theorem 1.1. Moreover, we have for any $x \in D$ that

$$|F'(x_0)^{-1}[F''(x) - F''(x_0)]| = 10|x - x_0|. \tag{1.9}$$

That is, the center Lipschitz condition $(C_4)'$ is true for constant $L = 10$. We can also verify condition $\frac{1}{2}\beta\eta = \frac{1}{8} < \tau = 0.134065\dots$ is true. By (1.6), we get

$$R = \tfrac{\sqrt{\beta^2+4L}-\beta}{2L} = \tfrac{\sqrt{41}-1}{20} = 0.270156\dots. \tag{1.10}$$

Then, condition $\overline{U}(x_0, R) = [x_0 - R, x_0 + R] \approx [0.729844, 1.270156] \subset D$ is also true. Hence, all conditions in Theorem 1.1 are satisfied. However, we can verify that the point x_1 generated by the Halley's method (1.2) doesn't remain in the open ball $U(x_0, R)$. In fact, we have that

$$|x_1 - x_0| = \tfrac{|F'(x_0)^{-1}F(x_0)|}{|1-\frac{1}{2}F'(x_0)^{-1}F''(x_0)F'(x_0)^{-1}F(x_0)|} = \tfrac{2}{7} = 0.285714\dots > R. \tag{1.11}$$

Clearly, the rest of the conclusions of Theorem 1.1 cannot be reached. $\square$

We use a different approach than recurrent relations in our semi-local convergence analysis. The paper is organized as follows: Section 2 contains the semi-local convergence of Halley's method, in Section 3 the local convergence is given, whereas the numerical examples are presented in the concluding Section 4.

2. SEMI-LOCAL CONVERGENCE ANALYSIS

We present the semi-local convergence analysis of Halley's method in a different way than in [1]. Let $\eta > 0$, $\beta \geq 0$ and $L > 0$. Set $R = \frac{2}{\beta+\sqrt{\beta^2+4L}}$. Then, we have that

$$LR^2 + \beta R = 1$$

and

$$Lt^2 + \beta t < 1, \text{ for any } t \in (0, R).$$

Suppose that

$$\eta < \frac{R}{1+\frac{\beta R}{2}} = \frac{2}{2\beta+\sqrt{\beta^2+4L}}, \tag{2.1}$$

which is equivalent to

$$\eta_0 < R,$$

where

$$\eta_0 = \frac{\eta}{1-a}, \quad a = \frac{1}{2}\beta\eta < 1.$$

Define function $\phi(t)$ on $[0, R]$ by

$$\begin{aligned} \phi(t) &= 2t^2[1-(Lt+\beta)t]^2 - 2t^2[1-(Lt+\beta)t](Lt+\beta)\eta_0 \\ &\quad -2t[1-(Lt+\beta)t]^2\eta_0 - (Lt+\beta)t\eta_0^2 + (Lt+\beta)\eta_0^3 \\ &= 2t^2[1-(Lt+\beta)t]^2 - 2t[1-(Lt+\beta)t]\eta_0 - (Lt+\beta)t\eta_0^2 + (Lt+\beta)\eta_0^3. \end{aligned}$$

Suppose function ϕ has zeros on (η_0, R), and let R_0 be the smallest such zero. Define

$$\alpha = (LR_0 + \beta)R_0.$$

Then, we have that

$$\alpha \in (0, 1).$$

Assume further that

$$(LR_0 + \beta)\eta_0^2 \leq 4R_0^2\beta(1-\alpha)^2 \tag{2.2}$$

and

$$(LR_0 + \beta)\eta_0^2 < 2R_0(1-\alpha)^2. \tag{2.3}$$

By the definition of R_0, we have that

$$b = \frac{2R_0(1-\alpha)(LR_0+\beta)\eta_0}{2R_0(1-\alpha)^2-(LR_0+\beta)\eta_0^2} = 1 - \frac{\eta_0}{R_0} \in (0, 1). \tag{2.4}$$

We shall refer to (C_1), (C_2), $(C_3)'$, $(C_4)'$, (2.1), (2.2), (2.3) and the existence of R_0 on (η_0, R) as the (C) conditions. Let $U(x, R)$, $\overline{U}(x, R)$ stand, respectively, for the open and closed balls in X with center x and radius $R > 0$. Then, we can show the following semi-local convergence result for Halley's method (1.2).

THEOREM 3. *Let $F : D \subset X \to Y$ be continuously twice Fréchet differentiable, where X, Y are Banach spaces and D is open and convex. Suppose the* (C) *conditions and $\overline{U}(x_0, R) \subset D$. Then, the Halley sequence $\{x_n\}$ generated by* (1.2) *is well defined, remains in $U(x_0, R_0)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, R_0)$ of equation $F(x) = 0$. Furthermore, $x^\star$ is the only solution limit point of equation $F(x) = 0$ in $\overline{U}(x_0, R)$. Moreover, the following error estimate holds for any $n \geq 1$*

$$\|x_{n+2} - x_{n+1}\| \leq \frac{(LR_0+\beta)\|x_{n+1}-x_n\|^2}{(1-\alpha)\left[1-\frac{(LR_0+\beta)\|x_{n+1}-x_n\|^2}{2R_0(1-\alpha)^2}\right]} \leq b\|x_{n+1} - x_n\|. \tag{2.5}$$

Proof. We shall show using induction that (2.5) and the following hold for $n \geq 0$:

$$\|(I - L_F(x_{n+1}))^{-1}\| \leq \frac{1}{1-\|L_F(x_{n+1})\|}, \tag{2.6}$$

$$x_{n+2} \in U(x_0, R_0), \tag{2.7}$$

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \leq \frac{1}{1-(L\|x_{n+1}-x_0\|+\beta)\|x_{n+1}-x_0\|} < \frac{1}{1-\alpha}, \tag{2.8}$$

$$\|F'(x_0)^{-1}F''(x_{n+1})\| \leq L\|x_{n+1} - x_0\| + \beta < LR_0 + \beta < \frac{1}{R_0}, \tag{2.9}$$

$$\begin{aligned}\|L_F(x_{n+1})\| &\leq \frac{(LR_0+\beta)\|x_{n+1}-x_n\|^2}{2R_0[1-(L\|x_{n+1}-x_0\|+\beta)\|x_{n+1}-x_0\|]^2}\\ &\leq \frac{LR_0+\beta}{2R_0(1-\alpha)^2}\|x_{n+1} - x_n\|^2 < 1,\end{aligned} \tag{2.10}$$

$$\frac{\|L_F(x_{n+1})\|}{2R_0} \leq \beta. \tag{2.11}$$

We have

$$\begin{aligned}\|I - (I - L_F(x_0))\| &= \|L_F(x_0)\| = \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_0)F'(x_0)^{-1}F(x_0)\|\\ &\leq \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_0)\|\|F'(x_0)^{-1}F(x_0)\| \leq \tfrac{1}{2}\beta\eta = a < 1.\end{aligned} \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible operators [2], [6] that $(I - L_F(x_0))^{-1}$ exists, so that

$$\|(I - L_F(x_0))^{-1}\| \leq \frac{1}{1-\|L_F(x_0)\|} \leq \frac{1}{1-a}$$

and

$$\begin{aligned}\|x_1 - x_0\| &= \|(I - L_F(x_0))^{-1}F'(x_0)^{-1}F(x_0)\|\\ &\leq \|(I - L_F(x_0))^{-1}\|\|F'(x_0)^{-1}F(x_0)\|\\ &\leq \frac{\eta}{1-a} = \eta_0 < R_0.\end{aligned}$$

We need an estimate on

$$
\begin{aligned}
&\|I - F'(x_0)^{-1}F'(x_1)\| = \\
&= \left\|F'(x_0)^{-1}\int_0^1 F''(x_0+\theta(x_1-x_0))(x_1-x_0)\mathrm{d}\theta\right\| \\
&= \Big\|F'(x_0)^{-1}\int_0^1 [F''(x_0+\theta(x_1-x_0)) - F''(x_0)](x_1-x_0)\mathrm{d}\theta \\
&\quad + F'(x_0)^{-1}F''(x_0)(x_1-x_0)\Big\| \\
&\le \int_0^1 \|F'(x_0)^{-1}[F''(x_0+\theta(x_1-x_0)) - F''(x_0)](x_1-x_0)\mathrm{d}\theta\| \\
&\quad + \|F'(x_0)^{-1}F''(x_0)(x_1-x_0)\| \\
&\le \int_0^1 L\theta\|x_1-x_0\|^2 d\theta + \beta\|x_1-x_0\| = (\tfrac{L}{2}\|x_1-x_0\|+\beta)\|x_1-x_0\| \\
&< (LR_0+\beta)R_0 = \alpha < 1.
\end{aligned}
$$

Hence, $F'(x_1)^{-1}$ exists and

$$\|F'(x_1)^{-1}F'(x_0)\| \le \frac{1}{1-\left(\frac{L}{2}\|x_1-x_0\|+\beta\right)\|x_1-x_0\|} \le \frac{1}{1-(L\|x_1-x_0\|+\beta)\|x_1-x_0\|} < \frac{1}{1-\alpha}.$$

In view of Halley's iteration we can write

$$[I - L_F(x_0)](x_1-x_0) + F'(x_0)^{-1}F(x_0) = 0$$

or

$$F(x_0) + F'(x_0)(x_1-x_0) - \tfrac{1}{2}F''(x_0)F'(x_0)^{-1}F(x_0)(x_1-x_0) = 0.$$

It then follows from the integral form of the mean theorem that

$$\|F'(x_0)^{-1}[F(x_1)-F(x_0)-F'(x_0)(x_1-x_0)-\tfrac{1}{2}F''(x_0)(x_1-x_0)^2]\| \le \tfrac{L}{6}\|x_1-x_0\|^3$$

and

$$\|\tfrac{1}{2}F'(x_0)^{-1}F''(x_0)[F'(x_0)^{-1}F(x_0)+(x_1-x_0)](x_1-x_0)\| \le \tfrac{\beta}{2}\|L_F(x_0)\|\|x_1-x_0\|^2.$$

Hence, we get that

$$
\begin{aligned}
&\|F'(x_0)^{-1}F(x_1)\| = \\
&= \|F'(x_0)^{-1}[F(x_1)-F(x_0)-F'(x_0)(x_1-x_0)-\tfrac{1}{2}F''(x_0)(x_1-x_0)^2] \\
&\quad + \tfrac{1}{2}F'(x_0)^{-1}F''(x_0)[F'(x_0)^{-1}F(x_0)+(x_1-x_0)](x_1-x_0)\| \\
&\le (\tfrac{L}{6}\|x_1-x_0\| + \tfrac{\beta}{2}\|L_F(x_0)\|)\|x_1-x_0\|^2 \le (LR_0+\beta)\|x_1-x_0\|^2 \\
&\le (LR_0+\beta)R_0\|x_1-x_0\| \le \alpha\eta_0
\end{aligned}
$$

and

$$
\begin{aligned}
\|F'(x_0)^{-1}F''(x_1)\| &\le \|F'(x_0)^{-1}(F''(x_1)-F''(x_0))\| + \|F'(x_0)^{-1}F''(x_0)\| \\
&\le L\|x_1-x_0\| + \beta < LR_0+\beta < \tfrac{1}{R_0}.
\end{aligned}
$$

Hence, we get that

$$\begin{aligned}
&\|L_F(x_1)\| = \\
&= \tfrac{1}{2}\|F'(x_1)^{-1}F'(x_0)F'(x_0)^{-1}F''(x_1)F'(x_1)^{-1}F'(x_0)F'(x_0)^{-1}F(x_1)\| \\
&\le \tfrac{1}{2}\|F'(x_1)^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}F''(x_1)\|\|F'(x_0)^{-1}F(x_1)\| \\
&\le \tfrac{(LR_0+\beta)\|x_1-x_0\|^2}{2R_0[1-\|x_1-x_0\|(L\|x_1-x_0\|+\beta)]^2} \le \tfrac{(LR_0+\beta)\|x_1-x_0\|^2}{2R_0(1-\alpha)^2} \le \tfrac{(LR_0+\beta)\eta_0^2}{2R_0(1-\alpha)^2} < 1
\end{aligned}$$

and

$$\tfrac{1}{2R_0}\|L_F(x_1)\| \le \beta$$

by (2.2) and (2.3). Then, $(I - L_F(x_1))^{-1}$ exists,

$$\|(I - L_F(x_1))^{-1}\| \le \tfrac{1}{1-\|L_F(x_1)\|}.$$

So, x_2 is well defined, and using (1.2) and (2.4) we get

$$\begin{aligned}
\|x_2 - x_1\| &\le \frac{\|F'(x_1)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_1)\|}{1 - \|L_F(x_1)\|} \\
&\le \frac{(LR_0+\beta)\|x_1 - x_0\|^2}{(1-\alpha)(1 - \frac{(LR_0+\beta)\|x_1-x_0\|^2}{2R_0(1-\alpha)^2})} \le b\|x_1 - x_0\|.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\|x_2 - x_0\| &\le \|x_2 - x_1\| + \|x_1 - x_0\| \\
&\le b\|x_1 - x_0\| + \|x_1 - x_0\| = (1+b)\|x_1 - x_0\| \\
&= \tfrac{1-b^2}{1-b}\|x_1 - x_0\| \le \tfrac{\|x_1-x_0\|}{1-b} \le \tfrac{\eta_0}{1-b} = R_0 < R.
\end{aligned}$$

Hence, we have $x_2 \in U(x_0, R_0)$. The rest will be shown by induction. Assume (2.5)-(2.11) are true for all natural integers $n \le k$, where $k \ge 0$ is a fixed integer. Then we have that

$$\begin{aligned}
&\|I - F'(x_0)^{-1}F'(x_{k+2})\| = \\
&= \left\|F'(x_0)^{-1}\int_0^1 F''(x_0 + \theta(x_{k+2} - x_0))(x_{k+2} - x_0)\mathrm{d}\theta\right\| \\
&= \left\|F'(x_0)^{-1}\int_0^1 [F''(x_0 + \theta(x_{k+2} - x_0)) - F''(x_0)](x_{k+2} - x_0)\mathrm{d}\theta \right.\\
&\quad \left. + F'(x_0)^{-1}F''(x_0)(x_{k+2} - x_0)\right\| \\
&\le \int_0^1 \|F'(x_0)^{-1}[F''(x_0 + \theta(x_{k+2} - x_0)) - F''(x_0)](x_{k+2} - x_0)\|\mathrm{d}\theta \\
&\quad + \|F'(x_0)^{-1}F''(x_0)\|\|x_{k+2} - x_0\| \\
&\le \int_0^1 L\theta\|x_{k+2} - x_0\|^2\mathrm{d}\theta + \beta\|x_{k+2} - x_0\| \\
&\le (L\|x_{k+2} - x_0\| + \beta)\|x_{k+2} - x_0\| < (LR_0 + \beta)R_0 = \alpha < 1.
\end{aligned}$$

Hence, $F'(x_{k+2})^{-1}$ exists and

$$\|F'(x_{k+2})^{-1}F'(x_0)\| \leq \tfrac{1}{1-(L\|x_{k+2}-x_0\|+\beta)\|x_{k+2}-x_0\|} < \tfrac{1}{1-\alpha}.$$

Next, we shall estimate $\|F'(x_0)^{-1}F(x_{k+2})\|$. We have that

$$\begin{aligned}
&F(x_{k+2}) = \\
&= F(x_{k+2}) - F(x_{k+1}) - F'(x_{k+1})(x_{k+2}-x_{k+1}) \\
&\quad + \tfrac{1}{2}F''(x_{k+1})F'(x_{k+1})^{-1}F(x_{k+1})(x_{k+2}-x_{k+1}) \\
&= F(x_{k+2}) - F(x_{k+1}) - F'(x_{k+1})(x_{k+2}-x_{k+1}) - \tfrac{1}{2}F''(x_{k+1})(x_{k+2}-x_{k+1})^2 \\
&\quad + \tfrac{1}{2}F''(x_{k+1})[F'(x_{k+1})^{-1}F(x_{k+1}) + (x_{k+2}-x_{k+1})](x_{k+2}-x_{k+1}).
\end{aligned}$$

Hence, we get that

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_{k+2})\| \leq A_1 + A_2 = \\
&= |F'(x_0)^{-1}[F(x_{k+2}) - F(x_{k+1}) - F'(x_{k+1})(x_{k+2}-x_{k+1}) \\
&\quad - \tfrac{1}{2}F''(x_{k+1})(x_{k+2}-x_{k+1})^2\| \\
&\quad + \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_{k+1})[F'(x_{k+1})^{-1}F(x_{k+1}) \\
&\quad + (x_{k+2}-x_{k+1})](x_{k+2}-x_{k+1})]\|.
\end{aligned}$$

We have in turn that

$$\begin{aligned}
&A_1 = \\
&= \Big\|F'(x_0)^{-1}\int_0^1\int_0^1 [F''(x_{k+1}+s\theta(x_{k+2}-x_{k+1})) - F''(x_{k+1})](x_{k+2}-x_{k+1})^2\theta \mathrm{d}s\mathrm{d}\theta\Big\| \\
&= \Big\|F'(x_0)^{-1}\int_0^1\int_0^1 [F''(x_{k+1}+s\theta(x_{k+2}-x_{k+1})) - F''(x_0)](x_{k+2}-x_{k+1})^2\theta \mathrm{d}sd\theta \\
&\quad + F'(x_0)^{-1}\int_0^1\int_0^1 [F''(x_0) - F''(x_{k+1})](x_{k+2}-x_{k+1})^2\theta \mathrm{d}sd\theta\Big\| \\
&\leq \int_0^1\int_0^1 \|F'(x_0)^{-1}[F''(x_{k+1}+s\theta(x_{k+2}-x_{k+1})) - F''(x_0)]\|\|x_{k+2}-x_{k+1}\|^2\theta \mathrm{d}sd\theta \\
&\quad + \int_0^1\int_0^1 \|F'(x_0)^{-1}[F''(x_0) - F''(x_{k+1})]\|\|x_{k+2}-x_{k+1}\|^2\theta \mathrm{d}s\mathrm{d}\theta \\
&\leq \int_0^1\int_0^1 L\|x_{k+1}+s\theta(x_{k+2}-x_{k+1}) - x_0\|\|x_{k+2}-x_{k+1}\|^2\theta \mathrm{d}s\mathrm{d}\theta \\
&\quad + \int_0^1\int_0^1 L\|x_{k+1}-x_0\|\|x_{k+2}-x_{k+1}\|^2\theta \mathrm{d}s\mathrm{d}\theta \\
&\leq \Big[\int_0^1\int_0^1 L(s\theta\|x_{k+2}-x_0\| + (1-s\theta)\|x_{k+1}-x_0\| + \|x_{k+1}-x_0\|)\theta \mathrm{d}s\mathrm{d}\theta\Big]\|x_{k+2}-x_{k+1}\|^2 =
\end{aligned}$$

$$= (\tfrac{L}{6}\|x_{k+2} - x_0\| + \tfrac{L}{3}\|x_{k+1} - x_0\| + \tfrac{L}{2}\|x_{k+1} - x_0\|)\|x_{k+2} - x_{k+1}\|^2$$
$$\leq LR_0\|x_{k+2} - x_{k+1}\|^2$$

and

$$\begin{aligned} A_2 &= \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_{k+1})\big(- [I - L_F(x_{k+1})](x_{k+2} - x_{k+1}) \\ &\quad +(x_{k+2} - x_{k+1})\big)(x_{k+2} - x_{k+1})\| \\ &= \tfrac{1}{2}\|F'(x_0)^{-1}F''(x_{k+1})L_F(x_{k+1})(x_{k+2} - x_{k+1})^2\| \\ &\leq \tfrac{1}{2R_0}\|L_F(x_{k+1})\|\|x_{k+2} - x_{k+1}\|^2. \end{aligned}$$

Hence, summing up we get that

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+2})\| &\leq (LR_0 + \tfrac{1}{2R_0}\|L_F(x_{k+1})\|)\|x_{k+2} - x_{k+1}\|^2 \\ &\leq (LR_0 + \beta)\|x_{k+2} - x_{k+1}\|^2 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \|L_F(x_{k+2})\| &\leq \tfrac{1}{2R_0}\|F'(x_{k+2})^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}F(x_{k+2})\| \\ &\leq \tfrac{(LR_0+\beta)\|x_{k+2}-x_{k+1}\|^2}{2R_0(1-\alpha)^2} \leq \tfrac{(LR_0+\beta)\eta_0^2}{2R_0(1-\alpha)^2} < 1. \end{aligned}$$

Hence, $(I - L_F(x_{k+2}))^{-1}$ exists and

$$\|(I - L_F(x_{k+2}))^{-1}\| \leq \tfrac{1}{1-\|L_F(x_{k+2})\|}.$$

Therefore, x_{k+3} is well defined. Moreover, we obtain that

$$\begin{aligned} &\|x_{k+3} - x_{k+2}\| \leq \\ &\leq \|(I - L_F(x_{k+2}))^{-1}\|\|F'(x_{k+2})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+2})\| \\ &\leq \frac{(LR_0+\beta)\|x_{k+2}-x_{k+1}\|^2}{\left[1-\frac{(LR_0+\beta)\|x_{k+2}-x_{k+1}\|^2}{2R_0(1-\alpha)^2}\right][1-\|x_{k+2}-x_0\|(L\|x_{k+2}-x_0\|+\beta)]} \\ &\leq \frac{(LR_0+\beta)\|x_{k+2}-x_{k+1}\|^2}{\left[1-\frac{(LR_0+\beta)\eta_0^2}{2R_0(1-\alpha)^2}\right][1-R_0(LR_0+\beta)]} \\ &\leq \frac{(LR_0+\beta)\eta_0}{\left[1-\frac{(LR_0+\beta)\eta_0^2}{2R_0(1-\alpha)^2}\right](1-\alpha)}\|x_{k+2} - x_{k+1}\| \leq b\|x_{k+2} - x_{k+1}\|. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} &\|x_{k+3} - x_0\| \leq \\ &\leq \|x_{k+3} - x_{k+2}\| + \|x_{k+2} - x_{k+1}\| + \cdots + \|x_1 - x_0\| \\ &\leq (b^{k+2} + b^{k+1} + \cdots + 1)\|x_1 - x_0\| \\ &= \tfrac{1-b^{k+3}}{1-b} < \tfrac{\eta_0}{1-b} = R_0. \end{aligned}$$

Hence, we deduce that $x_{k+3} \in U(x_0, R_0)$

Let m be a natural integer. Then, we have that

$$\begin{aligned}
&\|x_{k+m}-x_k\| \le \\
&\le \|x_{k+m}-x_{k+m-1}\| + \|x_{k+m-1}-x_{k+m-2}\| + \cdots + \|x_{k+1}-x_k\| \\
&\le (b^{m-1}+\cdots+b+1)\|x_{k+1}-x_k\| \\
&\le \tfrac{1-b^m}{1-b} b^k \|x_1-x_0\|.
\end{aligned}$$

It follows that $\{x_k\}$ is Cauchy in a Banach space X and as such it converges to some $x^\star \in \overline{U}(x_0, R_0)$ (since $\overline{U}(x_0, R_0)$ is a closed set). By letting $k \to \infty$ in (2.13) we obtain $F(x^\star) = 0$. We also have

$$\|x^\star - x_k\| \le \tfrac{b^k}{1-b}\|x_1 - x_0\|.$$

To show the uniqueness part, let $y^\star$ be a solution equation $F(x) = 0$ in $\overline{U}(x_0, R_0)$. Let $T = \int_0^1 F'(x_0)^{-1}F'(x^\star + \theta(y^\star - x^\star))\mathrm{d}\theta$. We have in turn that

$$\begin{aligned}
&\|I-T\| = \\
&= \Big\| \int_0^1 F'(x_0)^{-1}[F'(x^\star+\theta(y^\star-x^\star)) - F'(x_0)]\mathrm{d}\theta \Big\| \\
&= \Big\| \int_0^1 \int_0^1 F'(x_0)^{-1}F''(x_0+s(x^\star+\theta(y^\star-x^\star)-x_0))(x^\star+\theta(y^\star-x^\star)-x_0)\mathrm{d}s\mathrm{d}\theta \Big\| \\
&\le \int_0^1 \int_0^1 \|F'(x_0)^{-1}F''(x_0+s(x^\star+\theta(y^\star-x^\star)-x_0))\|\mathrm{d}s \\
&\quad \cdot ((1-\theta)\|x^\star-x_0\| + \theta\|y^\star-x_0\|)\mathrm{d}\theta \\
&< R_0 \int_0^1 \int_0^1 \|F'(x_0)^{-1}F''(x_0+s(x^\star+\theta(y^\star-x^\star)-x_0))\|\mathrm{d}s\mathrm{d}\theta \\
&\le R_0 \int_0^1 \int_0^1 (L\|s((x^\star+\theta(y^\star-x^\star))-x_0)\| + \beta)\mathrm{d}s\mathrm{d}\theta \\
&= R_0 \int_0^1 (\tfrac{1}{2}L\|(1-\theta)(x^\star-x_0)+\theta(y^\star-x_0)\| + \beta)\mathrm{d}\theta \\
&< R_0(LR_0+\beta) = \alpha < 1.
\end{aligned}$$

It follows that T^{-1} exists. Using the identity

$$0 = F'(x_0)^{-1}(F(y^\star) - F(x^\star)) = F'(x_0)^{-1}T(y^\star - x^\star)$$

we deduce $y^\star = x^\star$. The proof of the theorem is complete. □

Remark 4. The conclusion of Theorem 2.1 holds in an another setting, where the conditions can be weaker. Indeed, let us introduce center-Lipschitz condition

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0)\| \le L_0\|x - x_0\| \text{ for all } x \in D.$$

Then, it follows from the proof of Theorem 2.1 that α, R, b can be replaced by α_1, R_1, b_1, where

$$\alpha_1 = L_0 R_0, \quad R_1 = \tfrac{1}{L_0}, \quad 0 < b_1 < 1 - \tfrac{\eta}{R_1}.$$

It is possible that

$$L_0 < LR_0 + \beta \text{ and } R_1 > R. \tag{2.14}$$

The proof of Theorem 2.1 goes through with α_1 replacing α and the results are finer in this case, since

$$\tfrac{1}{1-\alpha_1} < \tfrac{1}{1-\alpha}.$$

As an example, let us define polynomial f on $D = \overline{U}(1, 1-p)$ by

$$f(x) = x^3 - p,$$

where $p \in [2-\sqrt{3}, 1)$. Then, we have $\beta = L = 2$, $\eta = \frac{1-p}{3}$ and $L_0 = 3 - p$. Estimate (2.14) holds provided that b is chosen so that

$$\tfrac{p}{2+p} < b < 1 - \tfrac{1-p}{2+p}(1+\sqrt{3}),$$

where

$$\tfrac{p}{2+p} < 1 - \tfrac{1-p}{2+p}(1+\sqrt{3})$$

by the choice of p. Note also that $R_1 > R$ and $1 - \frac{\eta_0}{R_1} > 1 - \frac{\eta_0}{R}$. The uniqueness of the solution can be shown in larger ball $U(x_0, R_1)$, since

$$\begin{aligned} \|F'(x_0)^{-1}(T - F'(x_0))\| &\leq L_0 \int_0^1 \|x^\star + \theta(y^\star - x^\star) - x_0\| \mathrm{d}\theta \\ &\leq \tfrac{L_0}{2}(\|x^\star - x_0\| + \|y^\star - x_0\|) \\ &< \tfrac{L_0}{2}(R_0 + R_0) < L_0 R < L_0 R_1 = 1. \end{aligned}$$

□

3. LOCAL CONVERGENCE OF HALLEY'S METHOD

In this section we present the local convergence of Halley's method (1.2). Let $c \geq 0$, $d \geq 0$ and $l > 0$. It is convenient for us to define polynomial p_0 on interval $[0, +\infty)$ by

$$p_0(t) = (c + dt)(1 + \tfrac{l}{2}t)t - 2(1 - lt)^2. \tag{3.1}$$

We have $p_0(0) = -2 < 0$ and $p_0(\frac{1}{l}) = (c + \frac{d}{l})(1 + \frac{1}{2})\frac{1}{l} > 0$. It follows from the intermediate value theorem that there exists a root of polynomial p_0 in $(0, \frac{1}{l})$. Denote by r_0 the smallest such root. Moreover, define functions g and h on $[0, r_0)$ by

$$g(t) = \tfrac{(c+dt)(1+\frac{l}{2}t)t}{2(1-lt)^2} \tag{3.2}$$

and

$$h(t) = (1 - g(t))^{-1}. \tag{3.3}$$

Note that functions g and h are well defined on $[0, r_0)$ and

$$g(t) \in [0, 1) \text{ for each } t \in [0, r_0). \tag{3.4}$$

Define polynomial p_1 on $[0, +\infty)$ by

$$p_1(t) = [10d(1-lt)+(2dt+3c)(c+dt)]t^2-6[2(1-lt)^2-(c+dt)(1+\tfrac{l}{2}t)t]. \tag{3.5}$$

We get $p_1(0) = -12$ and $p_1(\frac{1}{l}) = (\frac{2d}{l} + 3c)(c + \frac{d}{l})\frac{1}{l^2} + 6(c + \frac{d}{l})(1 + \frac{1}{2})\frac{1}{l} > 0$. Hence, there exists $r_1 \in (0, \frac{1}{l})$ such that $p_1(r_1) = 0$. Set

$$r = \min\{r_0, r_1\}. \tag{3.6}$$

Then, function q given by

$$q(t) = \tfrac{1}{12}\tfrac{h(t)}{1-lt}(10d + \tfrac{(2dt+3c)(c+dt)}{1-lt})t^2 \tag{3.7}$$

is well defined on $[0, r)$ and

$$q(t) \in [0, 1) \text{ for each } t \in [0, r). \tag{3.8}$$

We shall show the local convergence of Halley's method using the conditions (H) given by
(H_1) there exists $x^\star \in D$ such that $F'(x^\star) \in L(Y, X)$ and $F(x^\star) = 0$;
(H_2) $\|F'(x^\star)^{-1}(F'(x) - F'(x^\star))\| \leq l\|x - x^\star\|$ for each $x \in D$;
(H_3) $\|F'(x^\star)^{-1}F''(x^\star)\| \leq c$;
(H_4) $F'(x^\star)^{-1}(F''(x) - F''(x^\star))\| \leq d\|x - x^\star\|$ for each $x \in D$
and
(H_5) $U(x^\star, r) \subseteq D$.
Then, we can show:

Theorem 5. *Suppose that the (H) conditions hold. Then, sequence $\{x_n\}$ generated by Halley's method starting from $x_0 \in U(x^\star, r)$ is well defined, remains in $U(x^\star, r)$ for all $n \geq 0$ and converges to $x^\star$. Moreover, the following estimates hold*

$$\|x_{n+1} - x^\star\| \leq e_n\|x_n - x^\star\|^3 \text{ for each } n = 0, 1, 2, \ldots, \tag{3.9}$$

where

$$e_n = \tfrac{1}{12}\tfrac{h(\|x_n-x^\star\|)}{1-l\|x_n-x^\star\|}(10d + \tfrac{(2d\|x_n-x^\star\|+3c)(c+d\|x_n-x^\star\|)}{1-l\|x_n-x^\star\|}). \tag{3.10}$$

Proof. We have for $x \in U(x^\star, r)$, the choice of r and (H_2) that

$$\|F'(x^\star)^{-1}(F'(x) - F'(x^\star))\| \leq l\|x - x^\star\| < lr < 1. \tag{3.11}$$

It follows from (3.11) and the Banach lemma on invertible operators that $F'(x)^{-1} \in L(Y, X)$ and

$$\|F'(x)^{-1}F'(x^\star)\| \leq \tfrac{1}{1-l\|x-x^\star\|}. \tag{3.12}$$

Using the definition of operator L_F, function g, radius r, (3.4), (3.12), hypotheses (H_3) and (H_4) we have in turn that

$$\|L_F(x)\| \le \tfrac{1}{2}\|F'(x)^{-1}F'(x^\star)\|^2[\|F'(x^\star)^{-1}(F''(x)-F''(x^\star))+F'(x^\star)^{-1}F''(x^\star)\|]$$

$$(3.13)\qquad \cdot\Big\|\Big\{\int_0^1 F'(x^\star)^{-1}[F'(x^\star+\theta(x-x^\star))-F'(x^\star)]\mathrm{d}\theta + I\Big\}(x-x^\star)\Big\|$$

$$\le \tfrac{1}{2}\big(\tfrac{1}{1-l\|x-x^\star\|}\big)^2(c+d\|x-x^\star\|)(1+\tfrac{l}{2}\|x-x^\star\|)\|x-x^\star\|$$

$$= g(\|x-x^\star\|) \le g(r) < 1.$$

Hence, we get $\Gamma_F(x)$ exists and

$$(3.14)\qquad \|\Gamma_F(x)\| \le h(\|x-x^\star\|).$$

In view of (1.2) and $F(x^\star)=0$ we obtain the identity (cf [4])

$$x_{n+1}-x^\star = \Gamma_F(x_n)F'(x_n)^{-1}F'(x^\star)F'(x^\star)^{-1}$$

$$(3.15)\qquad \cdot\int_0^1 (1-\theta)[(F''(x_n+\theta(x^\star-x_n))-F''(x^\star))$$

$$+ (F''(x^\star)-F''(x_n))](x^\star-x_n)^2\mathrm{d}\theta$$

$$-\tfrac{1}{2}\Gamma_F(x_n)F'(x_n)^{-1}F'(x^\star)F'(x^\star)^{-1}(F''(x_n)-F''(x^\star)+F''(x^\star))$$

$$\cdot\Big[F'(x_n)^{-1}F'(x^\star)F'(x^\star)^{-1}\int_0^1(1-\theta)((F''(x_n+\theta(x^\star-x_n))-F''(x^\star))$$

$$+F''(x^\star))(x^\star-x_n)^2\mathrm{d}\theta\Big](x^\star-x_n).$$

Using (3.12), (3.13), (3.14) for $x=x_n$, (3.15), (H_3), (H_4) and the definition of r and q we get that

$$(3.16)\qquad \|x_{n+1}-x^\star\| \le$$

$$\le \tfrac{5}{6}\tfrac{dh(\|x_n-x^\star\|)}{1-l\|x_n-x^\star\|}\|x_n-x^\star\|^3$$

$$+\tfrac{2d\|x_n-x^\star\|+3c}{12}\tfrac{h(\|x_n-x^\star\|)}{(1-l\|x_n-x^\star\|)^2}(c+d\|x_n-x^\star\|)\|x_n-x^\star\|^3$$

$$= e_n\|x_n-x^\star\|^3 = q(\|x_n-x^\star\|)\|x_n-x^\star\| < \|x_n-x^\star\|.$$

That is $x_{n+1}\in U(x^\star,r)$ and $\lim_{n\to\infty}x_n = x^\star$. The proof of the theorem is complete. □

Remark 6. It follows from the estimate

$$(3.17)\qquad \|F'(x^\star)^{-1}(F'(x)-F'(x^\star))\| =$$

$$= \Big\|\int_0^1 F'(x^\star)^{-1}[(F''(x^\star+\theta(x-x^\star))-F''(x^\star))$$

$$+F''(x^\star)](x-x^\star)\mathrm{d}\theta\Big\|$$

$$\le (\tfrac{d}{2}\|x-x^\star\|+c)\|x-x^\star\|$$

that condition (H_2) can be dropped from the computation leading to (3.12), which can be replaced by

$$\|F'(x)^{-1}F'(x^\star)\| \leq \frac{1}{1-(\frac{d}{2}\|x-x^\star\|+c)\|x-x^\star\|}.$$

The rest stays the same. In this case to obtain the corresponding to Theorem 3.1 result simply replace l by $m(t) = \frac{d}{2}t + c$ and $\frac{1}{l}$ by the only positive root of polynomial

$$p_2(t) = m(t)t - 1. \tag{3.18}$$

This can improve the choice of r if

$$\frac{d}{2}t + c < \frac{1}{l} \quad for\ t \in [0, \tfrac{1}{l}). \tag{3.19}$$

□

4. NUMERICAL EXAMPLES

In this section, we will give some examples to show the application of our theorem.

Example 7. Let us define a scalar function $F(x) = x^3 - 2.25x^2 + 3x - 1.585$ on $D = (0, 3)$ with initial point $x_0 = 1$. Then, we have that

$$F'(x) = 3x^2 - 4.5x + 3, \quad F''(x) = 6x - 4.5. \tag{4.1}$$

So, $F(x_0) = 0.165$, $F'(x_0) = 1.5$, $F''(x_0) = 1.5$. We can choose $\eta = 0.11$ and $\beta = 1$ in Theorem 2.1. Moreover, we have for any $x \in D$ that

$$|F'(x_0)^{-1}[F''(x) - F''(x_0)]| = 4|x - x_0|. \tag{4.2}$$

Hence, the weak Lipschitz condition (1.3) is true for constant $L = 4$. By (1.6), we get

$$R = \frac{\sqrt{\beta^2+4L}-\beta}{2L} = \frac{\sqrt{17}-1}{8} = 0.390388\ldots. \tag{4.3}$$

Then, condition $\overline{U}(x_0, R) = [x_0 - R, x_0 + R] \approx [0.609612, 1.390388] \subset D$ is true. We can also verify that function ϕ has the minimized zero $R_0 = 0.169896107$ on (η_0, R), and conditions

$$\eta = 0.11 < \frac{R}{1+\frac{\beta}{2}} = 0.326631635,$$

$$(LR_0 + \beta)\eta_0^2 = 0.02275745 \leq 4R_0^2\beta(1-\alpha)^2 = 0.058966824,$$

$$(LR_0 + \beta)\eta_0^2 = 0.02275745 < 2R_0(1-\alpha)^2 = 0.029483412$$

are satisfied. Hence, all conditions in Theorem 2.1 are satisfied, and our theorem applies. □

Example 8. In this example we provide an application of our results to a special nonlinear Hammerstein integral equation of the second kind. Consider the integral equation

$$u(s) = f(s) + \lambda \int_{a'}^{b'} k(s,t) u(t)^{2+\frac{1}{n}} \mathrm{d}t, \quad \lambda \in \mathbb{R}, n \in \mathbb{N}, \tag{4.4}$$

where f is a given continuous function satisfying $f(s) > 0$ for $s \in [a', b']$ and the kernel is continuous and positive in $[a', b'] \times [a', b']$.

Let $X = Y = C[a', b']$ and $D = \{u \in C[a', b'] : u(s) \geq 0, s \in [a', b']\}$. Define $F : D \to Y$ by

$$F(u)(s) = u(s) - f(s) - \lambda \int_{a'}^{b'} k(s,t) u(t)^{2+\frac{1}{n}} \mathrm{d}t, \quad s \in [a', b']. \tag{4.5}$$

We use the max-norm, The first and second derivatives of F are given by

$$F'(u)v(s) = v(s) - \lambda(2 + \tfrac{1}{n}) \int_{a'}^{b'} k(s,t) u(t)^{1+\frac{1}{n}} v(t) \mathrm{d}t, \quad v \in D, s \in [a', b'], \tag{4.6}$$

and

$$F''(u)(vw)(s) = -\lambda(1 + \tfrac{1}{n})(2 + \tfrac{1}{n}) \int_{a'}^{b'} k(s,t) u(t)^{\frac{1}{n}} (vw)(t) \mathrm{d}t, \tag{4.7}$$

where $v, w \in D, s \in [a', b']$, respectively.

Let $x_0(t) = f(t)$, $\gamma = \min_{s\in[a',b']} f(s)$, $\delta = \max_{s\in[a',b']} f(s)$ and $M = \max_{s\in[a',b']} \int_{a'}^{b'} |k(s,t)| \mathrm{d}t$. Then, for any $v, w \in D$,

$$\begin{aligned}
&\|[F''(x) - F''(x_0)](vw)\| \leq \\
&\leq |\lambda|(1 + \tfrac{1}{n})(2 + \tfrac{1}{n}) \max_{s\in[a',b']} \int_{a'}^{b'} |k(s,t)| \cdot \left|x(t)^{\frac{1}{n}} - f(t)^{\frac{1}{n}}\right| \mathrm{d}t \|vw\| \\
&= |\lambda|(1+\tfrac{1}{n})(2+\tfrac{1}{n}) \max_{s\in[a',b']} \int_{a'}^{b'} |k(s,t)| \frac{|x(t) - f(t)|}{x(t)^{\frac{n-1}{n}} + x(t)^{\frac{n-2}{n}} f(t)^{\frac{1}{n}} + \cdots + f(t)^{\frac{n-1}{n}}} \mathrm{d}t \|vw\| \\
&\leq |\lambda|(1 + \tfrac{1}{n})(2 + \tfrac{1}{n}) \max_{s\in[a',b']} \int_{a'}^{b'} |k(s,t)| \frac{|x(t) - f(t)|}{f(t)^{\frac{n-1}{n}}} \mathrm{d}t \|vw\| \\
&\leq \frac{|\lambda|(1+\frac{1}{n})(2+\frac{1}{n})}{\gamma^{\frac{n-1}{n}}} \max_{s\in[a',b']} \int_{a'}^{b'} |k(s,t)| \cdot |x(t) - f(t)| \mathrm{d}t \|vw\| \\
&\leq \frac{|\lambda|(1+\frac{1}{n})(2+\frac{1}{n})M}{\gamma^{\frac{n-1}{n}}} \|x - x_0\| \|vw\|,
\end{aligned} \tag{4.8}$$

which means

$$\|F''(x) - F''(x_0)\| \le \frac{|\lambda|(1+\frac{1}{n})(2+\frac{1}{n})M}{\gamma^{\frac{n-1}{n}}}\|x - x_0\|. \tag{4.9}$$

Next, we give a bound for $\|F'(x_0)^{-1}\|$. Using (4.6), we have that

$$\|I - F'(x_0)\| \le |\lambda|(2 + \tfrac{1}{n})\delta^{1+\frac{1}{n}}M. \tag{4.10}$$

It follows from the Banach theorem that $F'(x_0)^{-1}$ exists if $|\lambda|(2+\frac{1}{n})\delta^{1+\frac{1}{n}}M < 1$, and

$$\|F'(x_0)^{-1}\| \le \frac{1}{1-|\lambda|(2+\frac{1}{n})\delta^{1+\frac{1}{n}}M}. \tag{4.11}$$

On the other hand, we have from (4.5) and (4.7) that $\|F(x_0)\| \le |\lambda|\delta^{2+\frac{1}{n}}M$ and $\|F''(x_0)\| \le |\lambda|(1 + \frac{1}{n})(2 + \frac{1}{n})\delta^{\frac{1}{n}}M$. Hence, if $|\lambda|(2 + \frac{1}{n})\delta^{1+\frac{1}{n}}M < 1$, the weak Lipschitz condition (1.3) is true for

$$L = \frac{|\lambda|(1+\frac{1}{n})(2+\frac{1}{n})M}{\gamma^{\frac{n-1}{n}}[1-|\lambda|(2+\frac{1}{n})\delta^{1+\frac{1}{n}}M]} \tag{4.12}$$

and constants η and β in Theorem 2.1 can be given by

$$\eta = \frac{|\lambda|\delta^{2+\frac{1}{n}}M}{1-|\lambda|(2+\frac{1}{n})\delta^{1+\frac{1}{n}}M}, \quad \beta = \frac{|\lambda|(1+\frac{1}{n})(2+\frac{1}{n})\delta^{\frac{1}{n}}M}{1-|\lambda|(2+\frac{1}{n})\delta^{1+\frac{1}{n}}M}. \tag{4.13}$$

Next we let $[a', b'] = [0, 1]$, $n = 2$, $f(s) = 1$, $\lambda = 0.8$ and $k(s,t)$ is the Green kernel on $[0,1] \times [0,1]$ defined by

$$G(s,t) = \begin{cases} t(1-s), & t \le s; \\ s(1-t), & s \le t. \end{cases} \tag{4.14}$$

Consider the following particular case of (4.4):

$$u(s) = f(s) + 0.8\int_0^1 G(s,t)u(t)^{\frac{5}{2}}\mathrm{d}t, \quad s \in [0,1]. \tag{4.15}$$

Then, $\gamma = \delta = 1$ and $M = \frac{1}{8}$. Moreover, we have that

$$\eta = \tfrac{2}{15}, \quad \beta = \tfrac{1}{2}, \quad L = \tfrac{1}{2}. \tag{4.16}$$

By (1.6), we get

$$R = \frac{\sqrt{\beta^2+4L}-\beta}{2L} = 1. \tag{4.17}$$

Hence, $\overline{U}(x_0, R) \subset D$. We can also verify that function ϕ has the minimized zero $R_0 = 0.15173576$ on (η_0, R), and conditions

$$\eta = 0.137931034 < \frac{R}{1+\frac{\beta}{2}} = 0.8,$$

$$(LR_0 + \beta)\eta_0^2 = 0.010955869 \le 4R_0^2\beta(1-\alpha)^2 = 0.076703659,$$

$$(LR_0+\beta)\eta_0^2 = 0.010955869 < 2R_0(1-\alpha)^2 = 0.25275406$$

are satisfied. Hence, all conditions in Theorem 2.1 are satisfied. Consequently, sequence $\{x_n\}$ generated by Halley's method (1.2) with initial point x_0 converges to the unique solution $x^\star$ of Eq. (4.5) on $\overline{U}(x_0,1)$. □

EXAMPLE 9. Let $X = Y = \mathbb{R}$, $D = (-1, 1)$ and define F on D by

$$F(x) = e^x - 1. \tag{4.18}$$

Then, $x^\star = 0$ is a solution of Eq. (1.1), and $F'(x^\star) = 1$. Note that for any $x \in D$, we have

$$\begin{aligned} |F'(x^\star)^{-1}(F'(x)-F'(x^\star))| &= |F'(x^\star)^{-1}(F''(x)-F''(x^\star))| \\ &= |\mathrm{e}^x - 1| = |x(1+\tfrac{x}{2!}+\tfrac{x^2}{3!}+\cdots)| \\ &\le |x(1+\tfrac{1}{2!}+\tfrac{1}{3!}+\cdots)| = (\mathrm{e}-1)|x-x^\star|. \end{aligned} \tag{4.19}$$

Then, we can choose $d = l = \mathrm{e} - 1$ in Theorem 3.1. It is easy to get $c = 1$, $r_0 = 0.2837798914$, $r_1 = 0.2575402082$ and $r = r_1$. Then, all conditions of Theorem 3.1 are satisfied. Let us choose $x_0 = 0.25$. Suppose sequence $\{x_n\}$ is generated by Halley's method (1.2). Table 1 gives a comparison results of error estimates for Example 4.3, which shows that error estimates (3.9) are true.

Table 1. The comparison results of error estimates for Example 4.3

n	the left-side of (3.9)	the right-side of (3.9)
0	1.29e-03	1.84e-01
1	1.81e-10	3.66e-09
2	4.91e-31	9.90e-30
3	9.84e-93	1.99e-91
4	7.93e-278	1.60e-276
5	4.16e-833	8.39e-832

□

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