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APPROXIMATION BY COMPLEX STANCU BETA OPERATORS OF SECOND KIND IN SEMIDISKS*

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Abstract. In this paper, the exact order of simultaneous approximation and Voronovskaja kind results with quantitative estimate for the complex Stancu Beta operator of second kind attached to analytic functions of exponential growth in semidisks of the right half-plane are obtained. In this way, we show the overconvergence phenomenon for this operator, namely the extensions of approximation properties with upper and exact quantitative estimates, from the real subinterval $(0, r]$, to semidisks of the right half-plane of the form $SD^r(0, r] = \{z \in \mathbb{C} : |z| \le r, 0 < \mathrm{Re}(z) \le r\}$ and to subsets of semidisks of the form $SD^r[a, r] = \{z \in \mathbb{C} : |z| \le r, a \le \mathrm{Re}(z) \le r\}$, with $r \ge 1$ and $0 < a < r$.

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1. INTRODUCTION

If $f : G \to \mathbb{C}$ is an analytic function in the open set $G \subset \mathbb{C}$, with $\overline{D}_1 \subset G$ (where $D_1 = \{z \in \mathbb{C} : |z| < 1\}$), then S. N. Bernstein proved that the complex Bernstein polynomials converges uniformly to f in $\overline{D}_1$ (see e.g., Lorentz [13], p. 88). Exact quantitative estimates and quantitative Voronovskaja-type results for these polynomials (see Gal [5]), together with similar results for complex Bernstein-Stancu polynomials (see also the papers Gal [6]-[7]), complex Kantorovich-Stancu polynomials (see also the paper Gal [8]), complex Favard-Szász-Mirakjan operators, Butzer's linear combinations of complex Bernstein polynomials, complex Baskakov operators and complex Balázs-Szabados operators were obtained by the first author in several recent papers collected by the recent book Gal [10].

The approximation properties of certain complex Durrmeyer-type operators were studied in Gal [4, 9], Agarwal and Gupta [3] and Mahmudov [14, 15].

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Furthermore, the approximation properties of the complex Beta operators of fist kind was studied in Gal-Gupta [11].

The aim of the present article is to obtain approximation results for the complex Stancu Beta operator of second kind, firstly introduced in the case of real variable in D. D. Stancu [17]. Then, Abel [1], Abel-Gupta [2] and Gupta-Abel-Ivan [12] obtained various estimates of the rate of convergence in the real variable case.

The complex Stancu Beta operators of second kind will be defined for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$ satisfying $0 < \mathrm{Re}(z)$, by

$$K_n(f,z) = \frac{1}{B(nz,n+1)} \int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} f(t)\mathrm{d}t, \tag{1.1}$$

where $B(\alpha,\beta)$ is the Euler's Beta function, defined by

$$B(\alpha,\beta) = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\mathrm{d}t, \qquad \alpha,\beta \in \mathbb{C},\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0$$

and f is supposed to be locally integrable and of polynomial growth on $(0,+\infty)$ as $t \to \infty$. This last hypothesis on f assures the existence of $K_n(f;z)$ for sufficiently large n, that is there exists n_0 depending on f such that $K_n(f;z)$ exists for all $n \geq n_0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$.

Note that because of the well-known formulas $B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, where Γ denotes the Euler's Gamma function, for all $z \in \mathbb{N}$ with $\mathrm{Re}(z) > 0$ and sufficiently large n we can easily deduce the form

$$K_n(f,z) = \frac{nz(nz+1)\ldots(nz+n)}{n!} \cdot \int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} f(t)\mathrm{d}t, \quad \mathrm{Re}(z) > 0. \tag{1.2}$$

The results in the present paper will show the overconvergence phenomenon for this complex Stancu Beta integral operator of second kind, that is the extensions of approximation properties with upper and exact quantitative estimates, from the real interval $(0,r]$ to semidisks of the right half-plane of the form

$$SD^r(0,r] = \{z \in \mathbb{C} : |z| \leq r,\ 0 < \mathrm{Re}(z) \leq r\}$$

and to subsets of semidisks of the form

$$SD^r[a,r] = \{z \in \mathbb{C} : |z| \leq r,\ a \leq \mathrm{Re}(z) \leq r\},$$

with $r \geq 1$ and $0 < a < r$.

It is worth noting that due to the special form of the complex Stancu Beta operators of second kind, the methods of proof are different from those used in the cases of the other complex operators studied by the papers mentioned in References.

2. AUXILIARY RESULT

In the sequel, we shall need the following auxiliary results.

LEMMA 2.1. *For all* $e_p = t^p, p \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ *with* $0 < \mathrm{Re}(z)$, *we have* $K_n(e_0, z) = 1$, $K_n(e_1)(z) = e_1(z)$ *and*

$$K_n(e_{p+1}, z) = \tfrac{nz+p}{n-p} K_n(e_p, z), \text{ for all } n > p.$$

Here $e_k(z) = z^k$.

Proof. By the relationship (1.1) of the Stancu Beta operators of second kind, it is obvious that $K_n(e_0, z) = 1$ and $K_n(e_1)(z) = e_1(z)$ (see [1], Proposition 2). Next

$$\begin{aligned} K_n(e_{p+1}, z) &= \tfrac{1}{B(nz,n+1)} B(nz + p + 1, n - p) \\ &= K_n(e_p, z) \cdot \tfrac{B(nz+p+1,n-p)}{B(nz+p,n-p+1)} = K_n(e_p, z) \cdot \tfrac{nz+p}{n-p}. \end{aligned}$$

Since $B(\alpha, \beta)$ is only defined for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$, it follows that the above recurrence is valid only for $n - p > 0$. This completes the proof of Lemma 2.1. $\square$

3. MAIN RESULTS

The first main result one refers to upper estimate.

THEOREM 3.1. *Let* $D_R = \{z \in \mathbb{C} : |z| < R\}$ *be with* $1 < R < \infty$ *and suppose that* $f : [R, \infty) \bigcup \overline{D}_R \to C$ *is continuous in* $[R, \infty) \bigcup \overline{D}_R$, *analytic in* D_R *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$, *for all* $z \in D_R$, *and* $f(t)$ *is of polynomial growth on* $(0, +\infty)$ *as* $t \to \infty$. *In addition, suppose that there exist* $M > 0$ *and* $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ *such that* $|c_k| \le M \cdot \frac{A^k}{k!}$, *for al* $k = 0, 1, 2, \ldots$, *(which implies* $|f(z)| \le M e^{A|z|}$ *for all* $z \in D_R$*).*

Let $1 \le r < \frac{1}{2A}$. *There exists* $n_0 \in \mathbb{N}$ *(depending only on* f*) such that* $K_n(f, z)$ *is analytic in* $SD^r(0, r]$ *for all* $n \ge n_0$ *and*

$$|K_n(f, z) - f(z)| \le \tfrac{C}{n}, \quad \text{for all } n \ge n_1 \text{ and } z \in SD^r[a, r],$$

for any $a \in (0, r)$. *Here* $C > 0$ *is independent of* n *and* z *but depends on* f, r *and* a, *and* n_1 *depends on* f, r *and* a.

Proof. In the definition of $K_n(f, z)$ in (1.1), for $z = x + \mathrm{i}y$ with $x > 0$, note that it follows $t^{nz-1} = \mathrm{e}^{(nz-1)\ln(t)} = \mathrm{e}^{(nx-1)\ln(t)} \cdot \mathrm{e}^{\mathrm{i}ny\ln(t)}$ and $|t^{nz-1}| = t^{nx-1}$, which implies

$$\begin{aligned} |K_n(f, z)| &\le \tfrac{1}{|B(nz,n+1)|} \int_0^{+\infty} \left| \tfrac{t^{nz-1}}{(1+t)^{nz+n+1}} \right| \cdot |f(t)| \mathrm{d}t \\ &= \tfrac{1}{|B(nz,n+1)|} \int_0^{+\infty} \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}} \cdot |f(t)| \mathrm{d}t. \end{aligned}$$

But it is well-known that because $f(t)$ is of polynomial growth as $t \to +\infty$, the last integral exists finite for sufficiently large n.

Therefore, there exists n_0 depending only on f, such that $K_n(f, z)$ is well-defined for sufficiently large n and for z with $Re(z) > 0$.

It remains to prove that $K_n(f, z)$ is in fact analytic for $\mathrm{Re}(z) > 0$ and n sufficiently large. For this purpose, from a standard result in the theory of improper integrals depending on a parameter, it suffices to prove that for any $\delta > 0$, the improper integral

$$\int_0^\infty \left[\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right]_z' \cdot f(t)\mathrm{d}t$$

is uniformly convergent for $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large.

But by simple calculation we obtain

$$\left[\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right]_z' = \left[\frac{\mathrm{e}^{(nz-1)\ln(t)}}{\mathrm{e}^{(nz+n+1)\ln(1+t)}}\right]_z' = n[\ln(t) - \ln(1+t)] \cdot \frac{t^{nz-1}}{(1+t)^{nz+n+1}},$$

and since $\ln(1 + t) \leq 1 + t$ for all $t \geq 0$, it easily follows that it remains to prove that the integral $\int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot \ln(t)f(t)dt$ is uniformly convergent for $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large.

By $\ln(t) < t$ for all $t \geq 1$ and by

$$\int_0^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot \ln(t)f(t)\mathrm{d}t = \int_0^1 \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot \ln(t)f(t)\mathrm{d}t + \int_1^\infty \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot \ln(t)f(t)\mathrm{d}t,$$

clearly that it remains to prove the uniform convergence, for all $\mathrm{Re}(z) \geq \delta > 0$ and n sufficiently large, for the integral $\int_0^1 \frac{t^{nz-1}}{(1+t)^{nz+n+1}} \cdot \ln(t)f(t)\mathrm{d}t$. But this follows immediately from the estimate

$$\left|\frac{t^{nz-1}}{(1+t)^{nz+n+1}}\right| \cdot |\ln(t)| \cdot |f(t)| \leq Ct^{n\delta-1}|\ln(t)|,$$

(see e.g. [16], p. 19, Exercise 1.51), where $|f(t)| \leq C$ for all $t \in [0, 1]$.

In what follows we deal with the approximation property. For this purpose, firstly let us define $S_n(z) = \sum_{k=0}^n c_k z^k$ if $|z| \leq r$ and $S_n(t) = f(t)$ if $t \in (r, +\infty)$, where $1 \leq r < \frac{1}{2A}$. Evidently that for each $n \in \mathbb{N}$, S_n is piecewise continuous on $[0, +\infty)$ (more exactly, has a discontinuity point of first kind at $x = r$), but locally integrable on $[0, +\infty)$ and of polynomial growth as $t \to \infty$.

Clearly, $f(z) - S_n(z) = \sum_{k=n+1}^\infty c_k z^k$ if $|z| \leq r$ and $f(t) - S_n(t) = 0$ if $t \in (r, \infty)$. Also, it is immediate that $K_n(S_n)(z)$ is well-defined for all $n \in \mathbb{N}$.

Therefore, for sufficiently large n and for $z \in SD^r(0,r]$ we have

$$\begin{aligned}
&|K_n(f,z) - f(z)| \leq \\
&\leq |K_n(f,z) - K_n(S_n,z)| + |K_n(S_n,z) - S_n(z)| + |S_n(z) - f(z)| \\
&\leq |K_n(f - S_n,z)| + \sum_{k=0}^{n} |c_k| \cdot |K_n(e_k,z) - e_k(z)| + |S_n(z) - f(z)|,
\end{aligned}$$

where $e_k(z) = z^k$.

Firstly we will obtain an estimate for $|S_n(z) - f(z)|$. Let $1 \leq r < \frac{1}{2A} < r_1 < R$. By the hypothesis, we can make such of choice for r_1.

Denoting $M_{r_1}(f) = \max\{|f(z)| : |z| \leq r_1\}$ and $\rho = \frac{r}{r_1}$, by $0 < \rho = \frac{r}{r_1} < 2Ar < 1$ and by the Cauchy's estimate (see e.g. [18], p. 184, Lemma 10.5) we get $|c_k| = \frac{|f^{(k)}(0)|}{k!} \leq \frac{1}{k!} \cdot \frac{M_{r_1}(f)k!}{r_1^k} = \frac{M_{r_1}(f)}{r_1^k}$, which implies

$$\begin{aligned}
|S_n(z) - f(z)| &\leq \sum_{k=n+1}^{\infty} |c_k| \cdot |z|^k \leq \sum_{k=n+1}^{\infty} \frac{M_{r_1}(f)}{r_1^k} \cdot |z|^k \leq \sum_{k=n+1}^{\infty} M_{r_1}(f) \frac{r^k}{r_1^k} \\
&= M_{r_1}(f) \rho^{n+1} \sum_{k=0}^{\infty} \rho^k = \frac{M_{r_1}(f)}{1-\rho} \cdot \rho^{n+1},
\end{aligned}$$

for all $|z| \leq r$ and $n \in \mathbb{N}$.

By using now Lemma 2.1 and taking into account the inequalities

$$\frac{1}{n-p} \leq \frac{2(p+1)}{n+p}, \ \frac{1}{n-p} \leq \frac{p+1}{n}, \ n \geq p+1,$$

for all $z \in SD^r(0,r]$ and $n \geq p+1$ we get

$$\begin{aligned}
&|K_n(e_{p+1},z) - e_{p+1}(z)| = \\
&= \left| \frac{nz+p}{n-p} K_n(e_p,z) - \frac{nz+p}{n-p} e_p(z) + \frac{nz+p}{n-p} e_p(z) - e_{p+1}(z) \right| \\
&\leq \frac{|nz+p|}{n-p} |K_n(e_p,z) - e_p(z)| + |e_p(z)| \cdot \left| \frac{nz+p}{n-p} - z \right| \\
&\leq |nz+p| \cdot \frac{2(p+1)}{n+p} \cdot |K_n(e_p,z) - e_p(z)| + r^p \cdot \frac{|p(1+z)|}{n-p} \\
&\leq \frac{nr+p}{n+p} \cdot 2(p+1)|K_n(e_p,z) - e_p(z)| + \frac{r^p \cdot 2pr \cdot (p+1)}{n} \\
&\leq 2r(p+1) \left[|K_n(e_p,z) - e_p(z)| + \frac{pr^p}{n} \right],
\end{aligned}$$

for all $p = 0, 1, ..., n-1$.

Therefore, denoting $E_{p,n}(z) = |K_n(e_p,z) - e_p(z)|$, we have obtained

$$E_{p+1,n}(z) \leq r(2p+2) \left[E_{p,n}(z) + p \cdot \frac{r^p}{n} \right],$$

for all $p = 0, 1, ..., n-1$.

Since $E_{0,n}(z) = E_{1,n}(z) = 0$, for $p = 1$ in the above inequality we get

$$E_{2,n}(z) \leq r(2 \cdot 1 + 2) \left[E_{1,n}(z) + \frac{r}{n} \right] \leq \frac{1 \cdot r^2}{n} \cdot (2 \cdot 1 + 2).$$

In what follows we will use the obvious inequality $p \le 2(p-1)+2$, valid for all $p \ge 1$.

For $p = 2$ in the above recurrence inequality it follows

$$\begin{aligned} E_{3,n}(z) &\le r(2\cdot 2+2)\left[E_{2,n}(z) + 2\cdot \tfrac{r^2}{n}\right] \\ &\le \tfrac{r^3}{n}\left[(2\cdot 2+2)(2\cdot 1+2) + 2\cdot(2\cdot 2+2)\right] \\ &\le \tfrac{r^3}{n}\left[(2\cdot 2+2)(2\cdot 1+2) + (2\cdot 1+2)(2\cdot 2+2)\right] \\ &\le \tfrac{2r^3}{n}(2\cdot 2+2)(2\cdot 1+2). \end{aligned}$$

For $p = 3$ in the above recurrence inequality we get

$$\begin{aligned} E_{4,n}(z) &\le r(2\cdot 3+2)\left[E_{3,n}(z) + 3\cdot \tfrac{r^3}{n}\right] \\ &\le r(2\cdot 3+2)\cdot\left[\tfrac{2r^3}{n}(2\cdot 1+2)(2\cdot 2+2) + (2\cdot 2+2)\cdot \tfrac{r^3}{n}\right] \\ &\le \tfrac{3r^4}{n}(2\cdot 3+2)(2\cdot 2+2)(2\cdot 1+2). \end{aligned}$$

By mathematical induction we easily obtain

$$E_{p,n}(z) \le \tfrac{(p-1)\cdot r^p}{n}\prod_{i=1}^{p-1} 2(i+1) = \tfrac{(p-1)\cdot 2^{p-1}r^p}{n}\cdot p! \le \tfrac{p\cdot p!(2r)^p}{2n},$$

for all $n \ge p+1$ and $z \in SD^r(0,r]$.

Therefore, we obtain

$$\sum_{k=0}^{n} |c_k|\cdot|K_n(e_k,z) - e_k(z)| \le \tfrac{M}{2n}\sum_{k=0}^{n} k(2Ar)^k \le \tfrac{M}{2n}\sum_{k=0}^{\infty} k(2Ar)^k,$$

where the hypothesis on f obviously implies that $\sum_{k=0}^{\infty} k\cdot(2Ar)^k < \infty$.

Now, let us estimate $|K_n(f - S_n, z)|$. By the definition of S_n and by (1.2), we easily get

$$K_n(f - S_n, z) = \tfrac{nz(nz+1)\dots(nz+n)}{n!}\cdot\int_0^r \tfrac{t^{nz-1}}{(1+t)^{nz+n+1}}(f(t) - S_n(t))\mathrm{d}t,$$

for all $z \in SD^r[a,r]$, $z = x + \mathrm{i}y$, and $n \in \mathbb{N}$. Passing to the absolute value, it follows

$$\begin{aligned} |K_n(f - S_n, z)| &\le \tfrac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r \left|\tfrac{t^{nz-1}}{(1+t)^{nz+n+1}}\right|\cdot|f(t) - S_n(t)|\mathrm{d}t \\ &\le \|f - S_n\|_{C[0,r]}\cdot\tfrac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t \\ &\le C_{r,r_1,f}\cdot\rho^{n+1}\cdot\tfrac{nr(nr+1)\dots(nr+n)}{n!}\cdot\int_0^r \tfrac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t. \end{aligned}$$

Now, let us estimate the integral $\int_0^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t$. For sufficiently large n (such that $na-1 \geq 1$) we have

$$\begin{aligned}\int_0^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t &= \int_0^1 \frac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t + \int_1^r \frac{t^{nx-1}}{(1+t)^{nx+n+1}}\mathrm{d}t\\ &\leq \int_0^1 \frac{t^{nx-1}}{(1+t)^{nx-1}}\mathrm{d}t + \int_1^r \frac{t^{nx+n+1}}{(1+t)^{nx+n+1}}\mathrm{d}t\\ &\leq \left(\tfrac{1}{2}\right)^{nx-1} + (r-1)\left(\tfrac{r}{r+1}\right)^{nx+n+1}\\ &\leq r\left(\tfrac{r}{r+1}\right)^{nx-1} \leq r\left(\tfrac{r}{r+1}\right)^{na-1},\end{aligned}$$

which immediately implies the estimate for $n \geq n_0$ (with n_0 depending only on f and a) and $z \in SD^r[a,r]$

$$|K_n(f-S_n,z)| \leq C_{r,r_1,f}\cdot\rho^{n+1}\cdot\tfrac{nr(nr+1)...(nr+n)}{n!}\cdot r\left(\tfrac{r}{r+1}\right)^{na-1}.$$

Collecting all the above estimates, for sufficiently large n and $z \in S^r[a,r]$, we get

$$\begin{aligned}|K_n(f,z)-f(z)| &\leq C_{r,r_1,f}\rho^{n+1} + \tfrac{M}{n}\sum_{k=0}^{\infty} k(2Ar)^k\\ &+ C_{r,r_1,f}\cdot\rho^{n+1}\cdot\tfrac{nr(nr+1)...(nr+n)}{n!}\cdot r\left(\tfrac{r}{r+1}\right)^{na-1}.\end{aligned}\tag{3.1}$$

In (3.1) we need to choose $n \geq 2/a$.

Now, denote

$$a_n = \tfrac{1}{n^2}\cdot\tfrac{nr(nr+1)...(nr+n)}{n!} = \tfrac{r}{n}\cdot\tfrac{(nr+1)...(nr+n)}{n!}.$$

We can write

$$\rho^{n+1}\cdot\tfrac{nr(nr+1)...(nr+n)}{n!}\cdot\left(\tfrac{r}{r+1}\right)^{na-1} = (n\cdot\rho^{n+1})\cdot a_n\cdot\left[n\left(\tfrac{r}{r+1}\right)^{na-1}\right].$$

Note that because $0<\rho<1$ and $0<r/(r+1)<1$, clearly that for sufficiently large n we have $n\cdot\rho^{n+1} \leq \frac{c_1}{n}$ and $n\left(\frac{r}{r+1}\right)^{na-1} \leq \frac{c_2}{n}$, where $c_1>0$ and $c_2>0$ are independent of n and z. On the other hand, by simple calculation we get

$$\begin{aligned}\tfrac{a_{n+1}}{a_n} &= \tfrac{n}{(n+1)^2}\cdot\left(1+\tfrac{r}{nr+1}\right)\left(1+\tfrac{r}{nr+2}\right)...\left(1+\tfrac{r}{nr+n}\right)\\ &< \tfrac{n}{(n+1)^2}\left(1+\tfrac{r}{nr+1}\right)^{n} < \tfrac{n}{(n+1)^2}\left(1+\tfrac{r}{nr+1}\right)^{(nr+1)/r} \leq \tfrac{3n}{(n+1)^2} \leq 1,\end{aligned}$$

for all $n \in \mathbb{N}$. We used here the inequality $e<3$. Therefore, the sequence $(a_n)_n$ is nonincreasing, which implies that it is bounded.

In conclusion, for sufficiently large n we have

$$\rho^{n+1}\cdot\tfrac{nr(nr+1)...(nr+n)}{n!}\cdot\left(\tfrac{r}{r+1}\right)^{na-1} \leq \tfrac{c_3}{n^2},$$

which coupled with (3.1) immediately implies the order of approximation $\mathcal{O}(1/n)$ in the statement of Theorem 3.1. □

The following Voronovskaja-type result with a quantitative estimate holds.

THEOREM 3.2. *Let $D_R = \{z \in \mathbb{C} : |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R, \infty) \bigcup \overline{D}_R \to C$ is continuous in $[R, \infty) \bigcup \overline{D}_R$, analytic in D_R i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and $f(t)$ is of polynomial growth on $(0, +\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ such that $|c_k| \leq M \cdot \frac{A^k}{k!}$, for al $k = 0, 1, 2, ...,$ (which implies $|f(z)| \leq M e^{A|z|}$ for all $z \in D_R$).*

Let $1 \leq r < \frac{1}{2A}$. There exists $n_1 \in \mathbb{N}$ (depending on f, r and a) such that for all $n \geq n_1$, $z \in SD^r[a, r]$ and $a \in (0, r)$ we have

$$\left| K_n(f, z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)} \right| \leq \frac{C}{n^2},$$

where $C > 0$ is independent of n and z but depends on f, r and a.

Proof. Keeping the notations in the proof of Theorem 3.1, we can write

$$\begin{aligned}
&\left| K_n(f, z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)} \right| = \\
&= \left| \left(K_n(f - S_n, z) - (f(z) - S_n(z)) - \frac{z(1+z)[f(z) - S_n(z)]''}{2(n-1)} \right) \right. \\
&\quad \left. + \left(K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)} \right) \right| \\
&\leq \left| K_n(f - S_n, z) - (f(z) - S_n(z)) - \frac{z(1+z)[f(z) - S_n(z)]''}{2(n-1)} \right|
\end{aligned}$$

We get

$$\begin{aligned}
A &\leq |K_n(f - S_n, z)| + |f(z) - S_n(z)| + \left| \frac{z(1+z)[f(z) - S_n(z)]''}{2(n-1)} \right| \\
&\leq |K_n(f - S_n, z)| + |f(z) - S_n(z)| + \frac{r(1+r)|f''(z) - S_n''(z)|}{2(n-1)} \\
&:= A_1 + A_2 + A_3.
\end{aligned}$$

From the proof of Theorem 3.1, for all $z \in SD^r[a, r]$ with $a \in (0, r)$ and for sufficiently large n, we have

$$A_1 \leq \frac{C_1}{n^2} \text{ and } A_2 \leq C_2 \rho^{n+1},$$

where $C_1 > 0$, $C_2 > 0$ are independent of n and z but may depend on f, r and a and $0 < \rho < 1$.

In order to estimate A_3, let $0 < a_1 < a < r$, $1 \leq r < r_1 < \frac{1}{2A}$ and denote by $\Gamma = \Gamma_{a_1, r_1} = S_1 \bigcup L_1$ the closed curve composed by the segment in $\mathbb{C}$

$$S_1 = \left\{ z = x + iy \in \mathbb{C} : x = a_1 \text{ and } -\sqrt{r_1^2 - a_1^2} \leq y \leq \sqrt{r_1^2 - a_1^2} \right\},$$

and by the arc

$$L_1 = \{z \in \mathbb{C} : |z| = r_1, \mathrm{Re}(z) \geq a_1\}.$$

Clearly that Γ together with its interior is exactly $SD^{r_1}[a_1, r_1]$ and that from $r < r_1$ we have $SD^r[a, r] \subset SD^{r_1}[a_1, r_1]$, the inclusion being strictly.

By the Cauchy's integral formula for derivatives, we have for all $z \in SD^r[a, b]$ and $n \in \mathbb{N}$ sufficiently large

$$f''(z) - S_n''(z) = \frac{2!}{2\pi i} \int_\Gamma \frac{f(u)-S_n(u)}{(u-z)^3} \mathrm{d}u,$$

which by the estimate of $||f - S_n(\cdot)||_{SD^{r_1}[a_1,r_1]}$ in the proof of Theorem 3.1 and by the inequality $|u - z| \geq d = \min\{r_1 - r, a - a_1\}$ valid for all $z \in SD^r[a, r]$ and $u \in \Gamma$, implies

$$\begin{aligned} ||f''(z) - S_n''(\cdot)||_{SD^r[a,r]} &\leq \frac{2!}{2\pi} \cdot \frac{l(\Gamma)}{d^3} ||f - S_n(\cdot)||_{SD^{r_1}[a_1,r_1]} \\ &\leq \frac{M_{r_1}(f)}{1-\rho} \cdot \rho^{n+1} \cdot \frac{C_{r_1}(f) 2! l(\Gamma)}{2\pi d^3}, \end{aligned}$$

with $\rho = \frac{r}{r_1}$.

Note that here, by simple geometrical reasonings, for the length $l(\Gamma)$ of Γ, we get

$$l(\Gamma) = l(S_1) + l(L_1) = 2\sqrt{r_1^2 - a_1^2} + 2r_1 \cdot \arccos(a_1/r_1),$$

where $\arccos(\alpha)$ is considered expressed in radians.

Therefore, collecting all the above estimates we easily get $A \leq \frac{C}{n^2}$ for sufficiently large n, with $C > 0$ independent of n and z (but depending on f, r and a).

In the last part of the prof, we will obtain an estimate of the order $\mathcal{O}(1/n^2)$ for $B = \left| K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)} \right|$ too, which will implies the estimate in the statement.

Denoting $\pi_{k,n}(z) = K_n(e_k)(z)$ and

$$E_{k,n}(z) = \pi_{k,n}(z) - e_k(z) - \frac{z^{k-1}(1+z)k(k-1)}{2(n-1)},$$

firstly it is clear that $E_{0,n}(z) = E_{1,n}(z) = 0$. Then, we can write

$$\left| K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)} \right| \leq \sum_{k=2}^{n} |c_k| \cdot |E_{k,n}(z)|,$$

so it remains to estimate $E_{k,n}(z)$ for $2 \leq k \leq n$, by using the recurrence in Lemma 2.1.

In this sense, simple calculation based on Lemma 2.1 too, leads us to the formula

$$\begin{aligned} E_{k,n}(z) &= \tfrac{nz+k-1}{n-k+1}\cdot \pi_{k-1,n}(z) - z^k - \tfrac{z^{k-1}(1+z)k(k-1)}{2(n-1)} \\ &= \tfrac{nz+k-1}{n-k+1}\left[E_{k-1,n}(z) + z^{k-1} + \tfrac{z^{k-2}(1+z)(k-1)(k-2)}{2(n-1)}\right] \\ &\quad - z^k - \tfrac{z^{k-1}(1+z)k(k-1)}{2(n-1)} \\ &= \tfrac{nz+k-1}{n-k+1}E_{k-1,n}(z) + \tfrac{(k-1)(k-2)z^{k-2}(1+z)[(1+z)k+(z-1)]}{2(n-1)(n-k+1)}. \end{aligned}$$

Taking into account the inequalities valid for all $2 \le k \le n$ and $r \ge 1$

$$\tfrac{1}{n-k+1} \le \tfrac{2k}{n+k-1} \le \tfrac{2k}{n+k},\ \tfrac{nr+k-1}{n+k-1} \le r,$$

$$2(n-1)(n+k) \ge n^2,\ k(1+r)+(r-1) \le (k+1)(1+r),$$

this immediately implies, for all $2 \le k \le n$ and $|z| \le r$ with $a \le \mathrm{Re}(z) \le r$

$$\begin{aligned} |E_{k,n}(z)| &\le \left|\tfrac{nz+k-1}{n-k+1}\right| \cdot |E_{k-1,n}(z)| + \left|\tfrac{(k-1)(k-2)z^{k-2}(1+z)[(1+z)k+(z-1)]}{2(n-1)(n-k+1)}\right| \\ &\le \tfrac{nr+k-1}{n-k+1}\cdot |E_{k-1,n}(z)| + \tfrac{(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n-k+1)} \\ &\le \tfrac{2k(nr+k-1)}{n+k-1}\cdot |E_{k-1,n}(z)| \\ &\quad + \tfrac{2k(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n+k)} \\ &\le 2kr\cdot |E_{k-1,n}(z)| + \tfrac{2k(k-1)(k-2)r^{k-2}(1+r)[(1+r)k+(r-1)]}{2(n-1)(n+k)} \\ &\le 2rk|E_{k-1,n}(z)| + \tfrac{r^{k-2}(1+r)2k(k-1)(k-2)}{n^2}\cdot [k(1+r)+(r-1)] \\ &\le 2rk|E_{k-1,n}(z)| + \tfrac{r^{k-1}(1+r)^2k(k-1)(k-2)(k+1)}{n^2}. \end{aligned}$$

Denoting $A(k,r) = 2(1+r)^2(k+1)k(k-1)(k-2)$, we have obtained

$$|E_{k,n}(z)| \le 2rk|E_{k-1,n}(z)| + \tfrac{r^{k-1}}{n^2}\cdot A(k,r).$$

Obviously $E_{0,n}(z) = E_{1,n}(z) = E_{2,n} = 0$. Take in the last inequality, $k = 3, 4, ..., n$.

For $k = 3$ we obtain $|E_{3,n}(z)| \le \frac{r^2}{n^2}\cdot A(3,r)$.

For $k = 4$ it follows

$$|E_{4,n}(z)| \le r(2\cdot 4)\cdot |E_{3,n}(z)| + \tfrac{r^3}{n^2}A(4,r) \le \tfrac{r^3}{n^2}\cdot (2\cdot 4)\left[A(3,r)+A(4,r)\right].$$

For $k = 5$ we analogously get

$$\begin{aligned} |E_{5,n}(z)| &\le r(2\cdot 5)\cdot |E_{4,n}(z)| + \tfrac{r^4}{n^2}A(5,r) \\ &\le r(2\cdot 5)\left[(2\cdot 4)\tfrac{r^3}{n^2}[A(3,r)+A(4,r)]\right] + \tfrac{r^4}{n^2}A(5,r) \\ &\le \tfrac{r^4}{n^2}\cdot (2\cdot 4)(2\cdot 5)[A(3,r)+A(4,r)+A(5,r)]. \end{aligned}$$

Reasoning by mathematical induction, finally we easily obtain

$$|E_{k,n}(z)| \leq \frac{r^{k-1}}{n^2} \cdot (2 \cdot 4)(2 \cdot 5)...(2 \cdot k) \cdot \sum_{j=3}^{k} A(j,r) = \frac{r^{k-1}}{n^2} \cdot \frac{2^{k-1}}{2^2} \cdot \frac{k!}{3!} \cdot \sum_{j=3}^{k} A(j,r)$$

$$= \frac{(2r)^{k-1}}{24n^2} \cdot k! \cdot 2(1+r)^2 \sum_{j=3}^{k} (j-2)(j-1)j(j+1)$$

$$\leq \frac{k!(2r)^{k-1}}{12n^2} \cdot (1+r)^2 (k-2)^2 (k-1)k(k+1).$$

We conclude that

$$B := \left| K_n(S_n, z) - S_n(z) - \frac{z(1+z)S_n''(z)}{2(n-1)} \right| \leq \sum_{k=2}^{n} |c_k| \cdot |E_{k,n}|$$

$$\leq \frac{MA(1+r)^2}{12n^2} \cdot \sum_{k=2}^{n} (k-2)^2 (k-1)k(k+1)(2rA)^{k-1}$$

$$\leq \frac{MA(1+r)^2}{12n^2} \cdot \sum_{k=2}^{\infty} (k-2)^2 (k-1)k(k+1)(2rA)^{k-1},$$

where since $2Ar < 1$ by hypothesis, we get that

$$\sum_{k=2}^{\infty} (k-2)^2 (k-1)k(k+1)(2rA)^{k-1} < +\infty.$$

Indeed, the fact that the last series is convergent follows form the uniform convergence of the series $\sum_{k=0}^{\infty} z^k$ and its derivative of order 5, for $|z| < 1$. This finishes the proof of the theorem. □

In what follows, we obtain the exact order in approximation by the complex Stancu Beta operators of second kind and by their derivatives. In this sense, we present the following three results.

THEOREM 3.3. *Let $D_R = \{z \in \mathbb{C} : |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R, \infty) \bigcup \overline{D}_R \to C$ is continuous in $[R, \infty) \bigcup \overline{D}_R$, analytic in D_R i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, and $f(t)$ is of polynomial growth on $(0, +\infty)$ as $t \to \infty$. In addition, suppose that there exist $M > 0$ and $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ such that $|c_k| \leq M \cdot \frac{A^k}{k!}$, for al $k = 0, 1, 2, ...$, (which implies $|f(z)| \leq M e^{A|z|}$ for all $z \in D_R$).*

Let $1 \leq r < \frac{1}{2A}$. If f is not a polynomial of degree ≤ 1, then there exists $n_1 \in \mathbb{N}$ (depending on f, r and a) such that for all $n \geq n_1$, $z \in SD^r[a, r]$ and $a \in (0, r)$ we have

$$||K_n(f, \cdot) - f||_{SD^r[a,r]} \geq \frac{C_{r,a}(f)}{n},$$

where $C_{r,a}(f)$ depends only on f, a and r. Here $\|\cdot\|_{SD^r[a,r]}$ denotes the uniform norm on $SD^r[a, r]$.

Proof. For all $|z| \leq r$ and $n > n_0$ (with n_0 depending only on f), we have

$$K_n(f,z) - f(z) = \frac{1}{(n-1)}\left[\frac{z(1+z)f''(z)}{2}\right.$$
$$\left. + \frac{1}{(n-1)}\left\{(n-1)^2\left(K_n(f,z) - f(z) - \frac{z(1+z)f''(z)}{2(n-1)}\right)\right\}\right].$$

Also, we have

$$||F+G||_{SD^r[a,r]} \geq \left|||F||_{SD^r[a,r]} - ||G||_{SD^r[a,r]}\right| \geq ||F||_{SD^r[a,r]} - ||G||_{SD^r[a,r]}.$$

It follows

$$||K_n(f,\cdot) - f||_{SD^r[a,r]} \geq \frac{1}{(n-1)}\left[\left\|\frac{e_1(1+e_1)}{2}f''\right\|_{SD^r[a,r]}\right.$$
$$\left. - \frac{1}{(n-1)}\left\{(n-1)^2\left\|K_n(f,\cdot) - f - \frac{e_1(1+e_1)}{2(n-1)}f''\right\|_{SD^r[a,r]}\right\}\right].$$

Taking into account that by hypothesis f is not a polynomial of degree ≤ 1 in D_R, we get $\left\|\frac{e_1(1+e_1)}{2}f''\right\|_{SD^r[a,r]} > 0$.

Indeed, supposing the contrary it follows that $\frac{z(1+z)}{2}f''(z) = 0$ for all $z \in \overline{D}_R$, which implies that $f''(z) = 0$, for all $z \in \overline{D}_R \setminus \{0, -1\}$. Because f is analytic, by the uniqueness of analytic functions we get $f''(z) = 0$, for all $z \in D_R$, that is f is a polynomial of degree ≤ 1, which contradicts the hypothesis.

Now by Theorem 3.2, for sufficiently large n we have

$$(n-1)^2\left\|K_n(f,\cdot) - f - \frac{e_1(1+e_1)}{2(n-1)}f''\right\|_{SD^r[a,r]} \leq \frac{C(n-1)^2}{n^2} \leq M_r(f).$$

Therefore there exists an index $n_1 > n_0$ depending only on f, a and r, such that for any $n \geq n_1$, we have

$$\left\|\frac{e_1(1+e_1)}{2}f''\right\|_{SD^r[a,r]} - \frac{1}{(n-1)}\left\{(n-1)^2\left\|K_n(f,\cdot) - f - \frac{e_1(1+e_1)}{2(n-1)}f''\right\|_{SD^r[a,r]}\right\}$$
$$\geq \frac{1}{4}\|e_1(1+e_1)f''\|_{SD^r[a,r]},$$

which immediately implies

$$||K_n(f,\cdot) - f||_{SD^r[a,r]} \geq \frac{1}{4n}||e_1(1+e_1)f''||_{SD^r[a,r]}, \quad \forall n \geq n_1.$$

This completes the proof. □

As a consequence of Theorem 3.1 and Theorem 3.3, we immediately get the following:

COROLLARY 3.4. *Under the hypothesis in the statement of Theorem* 3.3, *if f is not a polynomial of degree ≤ 1, then there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$, $z \in SD^r[a,r]$ and $a \in (0,r)$, we have*

$$||K_n(f,\cdot) - f||_{SD^r[a,r]} \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f, a and r.

Our last result is in simultaneous approximation and can be stated as follows.

THEOREM 3.5. *Let* $D_R = \{z \in \mathbb{C} : |z| < R\}$ *be with* $1 < R < \infty$ *and suppose that* $f : [R, \infty) \bigcup \overline{D}_R \to C$ *is continuous in* $[R, \infty) \bigcup \overline{D}_R$*, analytic in* D_R *i.e.* $f(z) = \sum_{k=0}^{\infty} c_k z^k$*, for all* $z \in D_R$*, and* $f(t)$ *is of polynomial growth on* $(0, +\infty)$ *as* $t \to \infty$*. In addition, suppose that there exist* $M > 0$ *and* $A \in \left(\frac{1}{2R}, \frac{1}{2}\right)$ *such that* $|c_k| \le M \cdot \frac{A^k}{k!}$*, for al* $k = 0, 1, 2, ...$*, (which implies* $|f(z)| \le M e^{A|z|}$ *for all* $z \in D_R$*).*

Let $1 \le r < r_1 < \frac{1}{2A}$*,* $0 < a_1 < a < r$ *and* $p \in \mathbb{N}$*. If* f *is not a polynomial of degree* $\le \max\{1, p-1\}$*, then there exists* $n_1 \in \mathbb{N}$ *(depending on* f*,* r *and* a*) such that for all* $n \ge n_1$ *and* $z \in SD^r[a, r]$ *we have*

$$||K_n^{(p)}(f, \cdot) - f^{(p)}||_{SD^r[a,r]} \sim \tfrac{1}{n},$$

where the constants in the equivalence depend only on f, r, r_1, a, a_1 *and* p*.*

Proof. Denote by $\Gamma = \Gamma_{a_1, r_1} = S_1 \bigcup L_1$ the closed curve composed by the segment in $\mathbb{C}$

$$S_1 = \left\{ z = x + \mathrm{i}y \in \mathbb{C} : x = a_1 \text{ and } -\sqrt{r_1^2 - a_1^2} \le y \le \sqrt{r_1^2 - a_1^2} \right\},$$

and by the arc

$$L_1 = \{z \in \mathbb{C} : |z| = r_1, \mathrm{Re}(z) \ge a_1\}.$$

Clearly that Γ together with its interior is exactly $SD^{r_1}[a_1, r_1]$ and that from $r < r_1$ we have $SD^r[a, r] \subset SD^{r_1}[a_1, r_1]$, the inclusion being strictly.

By the Cauchy's integral formula for derivatives, we have for all $z \in SD^r[a, b]$ and $n \in \mathbb{N}$ sufficiently large

$$f^{(p)}(z) - K_n^{(p)}(f, z) = \tfrac{p!}{2\pi i} \int_{\Gamma} \tfrac{f(u) - K_n(f,u)}{(u-z)^{p+1}} \mathrm{d}u,$$

which by the estimate in Theorem 3.1 and by the inequality $|u - z| \ge d = \min\{r_1 - r, a - a_1\}$ valid for all $z \in SD^r[a, r]$ and $u \in \Gamma$, implies

$$\begin{aligned} ||f^{(p)}(z) - K_n^{(p)}(f, \cdot)||_{SD^r[a,r]} &\le \frac{p!}{2\pi} \cdot \frac{l(\Gamma)}{d^{p+1}} ||f - K_n(f, \cdot)||_{SD^{r_1}[a_1, r_1]} \\ &\le \frac{C}{n} \cdot \frac{C_{r_1}(f) p! l(\Gamma)}{2\pi d^{p+1}}. \end{aligned}$$

Note that here, by simple geometrical reasonings, for the length $l(\Gamma)$ of Γ, we get

$$l(\Gamma) = l(S_1) + l(L_1) = 2\sqrt{r_1^2 - a_1^2} + 2r_1 \cdot \arccos(a_1/r_1),$$

where $\arccos(\alpha)$ is considered expressed in radians.

It remains to prove the lower estimation for $||K_n^{(p)}(f, \cdot) - f^{(p)}||_{SD^r[a,r]}$.

By the proof of Theorem 3.3, for all $u \in \Gamma$ and $n \geq n_1$, we have

$$K_n(f,z) - f(z) = \frac{1}{(n-1)}\left[\frac{z(1+z)(z)}{2}f'' + \frac{1}{(n-1)}\left\{(n-1)^2\left(K_n(f,z) - f(z) - \frac{z(1+z)(z)}{2(n-1)}f''\right)\right\}\right].$$

Substituting it in the above Cauchy's integral formula, we get

$$K_n^{(p)}(f,z) - f^{(p)}(z) = \frac{1}{n-1}\left[\left(\frac{z(1+z)}{2}f''(z)\right)^{(p)} + \frac{1}{n-1}\cdot\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n-1)^2\left(K_n(f,u)-f(u)-\frac{u(1+u)}{2(n-1)}f''(u)\right)}{(u-z)^{p+1}}\mathrm{d}u\right].$$

Thus

$$\left\|K_n^{(p)}(f,\cdot) - f^{(p)}\right\|_{SD^r[a,r]} \geq \frac{1}{(n-1)}\left[\left\|\left(\frac{e_1(1+e_1)}{2}f''\right)^{(p)}\right\|_{SD^r[a,r]} - \frac{1}{(n-1)}\left\|\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n-1)^2\left(K_n(f,u)-f(u)-\frac{u(1+u)}{2(n-1)}f''(u)\right)}{(u-\cdot)^{p+1}}\mathrm{d}u\right\|_{SD^r[a,r]}\right].$$

Applying Theorem 3.2 too, it follows

$$\left\|\frac{p!}{2\pi \mathrm{i}}\int_\Gamma \frac{(n-1)^2\left(K_n(f,u)-f(u)-\frac{u(1+u)}{2(n-1)}f''(u)\right)}{(u-\cdot)^{p+1}}du\right\|_{SD^r[a,r]} \leq$$

$$\leq \frac{p!}{2\pi}\frac{l(\Gamma)n^2}{d^{p+1}}\left\|K_n(f,\cdot) - f - \frac{e_1(1+e_1)}{2(n-1)}f''\right\|_{SD^{r_1}[a_1,r_1]} \leq \frac{M_{r_1}(f)l(\Gamma)p!}{2\pi d^{p+1}}.$$

But by the hypothesis on f, we necessarily have

$$||[e_1(1+e_1)f''/2]^{(p)}||_{SD^r[a,r]} > 0.$$

Indeed, supposing the contrary we get that $e_1(1+e_1)f''$ is a polynomial of degree $\leq p-1$ in $SD^r[a,r]$, which by the uniqueness of analytic functions implies that

$$z(1+z)f''(z) = Q_{p-1}(z), \quad \text{for all } z \in D_R,$$

where $Q_{p-1}(z)$ is a polynomial of degree $\leq p-1$.

Now, if $p = 1$ and $p = 2$, then the analyticity of f in D_R easily implies that f necessarily is a polynomial of degree $\leq 1 = \max\{1, p-1\}$. If $p > 2$, then the analyticity of f in D_R easily implies that f necessarily is a polynomial of degree $\leq p-1 = \max\{1, p-1\}$. Therefore, in all the cases we get a contradiction with the hypothesis.

In conclusion, $||[e_1(1+e_1)f''/2]^{(p)}||_{SD^r[a,r]} > 0$ and furthermore, reasoning exactly as in the proof of Theorem 3.3, but for $\|K_n^{(p)}(f,\cdot)-f^{(p)}\|_{SD^r[a,r]}$ instead of $\|K_n(f,\cdot) - f\|_{SD^r[a,r]}$, we immediately get the desired conclusion. □

Remark 3.6. Comparing the error estimate in Theorem 3.1 with that in the real case, one sees that the overconvergence phenomenon holds (that is, the approximation from the real line is maintained in the complex plane for subclasses of analytic functions of exponential growth), with the same order of approximation $\mathcal{O}\left(\frac{1}{n}\right)$. Also, note that moreover, with respect to real approximation where these kind of results are missing, in the case of complex approximation the exact order $\mathcal{O}\left(\frac{1}{n}\right)$ is obtained including the case of simultaneous approximation and a quantitative estimate in the Voronovskaja's theorem of order $\mathcal{O}\left(\frac{1}{n^2}\right)$ is proved. □

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