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ON FATOU TYPE CONVERGENCE OF HIGHER DERIVATIVES OF CERTAIN NONLINEAR SINGULAR INTEGRAL OPERATORS

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Abstract. The present paper concerns with the Fatou type convergence properties of the $r-th$ and $(r+1)-th$ derivatives of the nonlinear singular integral operators defined as

$$(I_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t))\,\mathrm{d}t, \quad x \in (a,b),$$

acting on functions defined on an arbitrary interval (a,b), where the kernel K_λ satisfies some suitable assumptions. The present study is a continuation and extension of the results established in the paper [7].

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1. INTRODUCTION

Let Λ be a nonempty set of indices with a topology and λ_0 be an accumulation point of Λ in this topology. By $\mathcal{U}(\theta)$ we denote the family of all neighborhoods of the neutral element θ of $\mathbb{R}$, and x_0 is a fixed accumulation point of $\mathbb{R}$. We take a family $\mathcal{K}$ of functions $K_\lambda : \mathbb{R}\,\mathrm{x}\,\mathbb{R}\to\mathbb{R}$, where $K_\lambda(t,0) = 0$ for all $t \in \mathbb{R}$ and $\lambda \in \Lambda$, such that $K_\lambda(t,u)$ is integrable over $\mathbb{R}$ with respect to t, in the sense of Lebesgue measure, for all values of the index λ and second variable u. The family $\mathcal{K}$ will be called a kernel. In addition, if the kernel function $K_\lambda(t,u)$ is continuous in $\mathbb{R}$ for every $t \in \mathbb{R}$, then the kernel function is called Carathéodory kernel function.

In this paper we are concerned with the Fatou type pointwise convergence of $r-th$ and $(r+1)-th$ derivatives of certain family of nonlinear singular

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integral operators $(I_\lambda f)$ of the form

(1) $$\left(I_\lambda f\right)(x) = \int_a^b K_\lambda(t-x, f(t))\,\mathrm{d}t, \quad x \in (a,b)\,,$$

acting on functions defined on an arbitrary interval (a,b), where the kernel function K_λ satisfies some suitable assumptions. In these theorems the convergence is restricted to some subsets of the plane, i.e. the Fatou type convergence is discussed, whenever the first parameter tends to an accumulation point x_0, at which the function f has finite $r-th$ and $(r+1)-th$ derivatives, whereas the second one tends to an accumulation point λ_0 of a given index set Λ.

Further results on convergence of the operators (1) and its linear cases can be found in [1]-[12].

In particular, we obtain the rate of the Fatou type pointwise convergence for the nonlinear family of singular integral operators (1) to the point x_0, at which the function f has finite $r-th$ and $(r+1)-th$ derivatives, as $(x,\lambda) \to (x_0, \lambda_0)$. The results presented in this paper are the continuation and extension of those established in [7] in which the kernel function K_λ satisfies

$$\left[\tfrac{\partial}{\partial x} K_\lambda(t-x,u) - \tfrac{\partial}{\partial x} K_\lambda(t-x,v)\right] = \tfrac{\partial}{\partial x} L_\lambda(t-x)\,[u-v]$$

for every $t,u,v \in \mathbb{R}$ and for any $\lambda \in \Lambda$.

Throughout this paper we assume that the function $K_\lambda : \mathbb{R} \text{ x } \mathbb{R} \to \mathbb{R}$ satisfies the following conditions;

a) Let $L_\lambda(t)$ be an integrable function such that for any fixed $r \in \mathbb{N}$

(2) $$\left[\tfrac{\partial^r}{\partial x^r} K_\lambda(t-x,u) - \tfrac{\partial^r}{\partial x^r} K_\lambda(t-x,v)\right] = \tfrac{\partial^r}{\partial x^r} L_\lambda(t-x)\,[u-v]\,,$$

holds for every $t\ u,v \in \mathbb{R}$ and for any $\lambda \in \Lambda$.

b) $\lim\limits_{\lambda \to \lambda_0} \int_{\mathbb{R}\setminus U} L_\lambda(t)\mathrm{d}t = 0$, for every $U \in \mathcal{U}(0)$.

c) $\lim\limits_{\lambda \to \lambda_0} \left[\sup\limits_{|t| \geq \delta} L_\lambda(t)\right] = 0$, for every $\delta > 0$.

d) $\lim\limits_{\lambda \to \lambda_0} \int_{\mathbb{R}} L_\lambda(t)\,\mathrm{d}t = 1$.

According to a) it is easy to see that $K_\lambda : \mathbb{R} \text{ x } \mathbb{R} \to \mathbb{R}$ is a kernel function.

We introduce a function $\widetilde{f} \in L_1(\mathbb{R})$ as

(3) $$\widetilde{f}(t) := \begin{cases} f(t), & t \in (a,b)\,, \\ 0, & t \notin (a,b)\,, \end{cases}$$

(See [6]-[9]).

Theorem 1. [3] *Let* $1 \leq p < \infty$ *and assume that* $K_\lambda(t,u)$ *is a Carathéodory kernel function. If* $f \in L_p(a,b)$*, then* $(I_\lambda f) \in L_p(a,b)$ *and*

$$\|I_\lambda f\|_{L_p(a,b)} \leq H(\lambda) \|f\|_{L_p(a,b)}$$

for every $\lambda \in \Lambda$.

This kind of existence theorem is also valid in general functional spaces (see e.g. [3]*).*

2. CONVERGENCE OF THE DERIVATIVES

Let us define, for any constants $C_\nu > 0$ $(\nu = 1,2,...,r)$, the set

$$D_r := \left\{ (x,\lambda) \in I \ x \ \Lambda : |x - x_0|^\nu \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-\nu} \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t) \right| \mathrm{d}t < C_\nu \right\}, \tag{4}$$

where $I = (a,b)$ is an arbitrary interval in $\mathbb{R}$.

Now, we are ready to investigate the approximation for finite r-th derivatives of the operator $(I_\lambda f)$ in $L_1(I)$.

Theorem 2. *Let the function* $L_\lambda(t)$ *and its derivatives* $\frac{\partial^\nu}{\partial t^\nu} L_\lambda(t)$, $(\nu = 1,2,...,r)$ *be continuous with respect to* t *on* $(-\infty,\infty)$ *and* $L_\lambda(t)$ *be integrable with respect to* t *for each fixed* $\lambda \in \Lambda$. *Suppose that the conditions* c) *and* d) *together with*

$$\sup_{\lambda \in \Lambda} \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^r \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t) \right| \mathrm{d}t < \infty \quad \text{and} \quad \lim_{\lambda \to \lambda_0} \sup_{0<\delta \leq |t|} \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t) \right| = 0 \tag{5}$$

hold for every $\delta > 0$, *are satisfied. Suppose that the function* $f \in L_1(I)$ *has at* x_0 *a finite* $r-th$ *derivative.*

Then

$$\lim \tfrac{\partial^r}{\partial x^r} (I_\lambda f)(x) = f^{(r)}(x_0) \tag{6}$$

as $(x,\lambda) \to (x_0,\lambda_0)$ *and* $(x,\lambda) \in D_r$.

Proof. Suppose that $a < x_0 < b$ and $0 < |x_0 - x| < \delta$ $(\delta > 0)$.

We construct a function

$$g(t) = f(x_0) + (t-x_0) f'(x_0) + ... + \tfrac{(t-x_0)^r}{r!} f^{(r)}(x_0) \tag{7}$$

so that $g^{(r)}(t) = f^{(r)}(x_0)$.

Firstly we shall prove this theorem for the function $g(t)$. For this purpose we introduce a function $\widetilde{g} \in L_1(\mathbb{R})$ as follows;

$$\widetilde{g}(t) := \begin{cases} g(t), & t \in (a,b), \\ 0, & t \notin (a,b). \end{cases} \tag{8}$$

Applying the operator I_λ to the function $g(t)$, we have

$$\left(I_\lambda g\right)(x) = \int\limits_a^b K_\lambda(t-x, g(t))\mathrm{d}t$$

and according to (8) we can rewrite the last equality as follows;

$$\left(I_\lambda g\right)(x) = \int\limits_{\mathbb{R}} K_\lambda(t-x, \widetilde{g}(t))\mathrm{d}t = \left(I_\lambda \widetilde{g}\right)(x) \tag{9}$$

and hence

$$\begin{aligned}\tfrac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x) &= \tfrac{\partial^r}{\partial x^r}\int\limits_{\mathbb{R}} K_\lambda(t-x, \widetilde{g}(t))\mathrm{d}t = \int\limits_{R} \widetilde{g}(t)\tfrac{\partial^r}{\partial x^r}L_\lambda(t-x)\mathrm{d}t \\ &= (-1)^r \int\limits_{\mathbb{R}} \widetilde{g}(t)\tfrac{\partial^r}{\partial t^r}L_\lambda(t-x)\mathrm{d}t = \int\limits_{\mathbb{R}} \widetilde{g}^{(r)}(t)L_\lambda(t-x)\mathrm{d}t.\end{aligned}$$

In the above, substituting (7), we get

$$\begin{aligned}\left|\tfrac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x) - f^{(r)}(x_0)\right| &= \left|\int\limits_a^b f^{(r)}(x_0)L_\lambda(t-x)\mathrm{d}t - f^{(r)}(x_0)\right| \\ &= \left|f^{(r)}(x_0)\right|\left|\int\limits_a^b L_\lambda(t-x)\mathrm{d}t - 1\right|.\end{aligned}$$

Hence, from condition $d)$ one has

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \tfrac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x) = f^{(r)}(x_0). \tag{10}$$

We denote $I_\lambda(x) := \frac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x) - \frac{\partial^r}{\partial x^r}\left(I_\lambda f\right)(x)$.

Thanks to (10), for completing the proof of the theorem, it is sufficient to show that

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |I_\lambda(x)| = 0,$$

which gives (6).

It is easy to see that

$$
\begin{aligned}
|I_\lambda(x)| &= \left| \int_{\mathbb{R}} \widetilde{f}(t) \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \mathrm{d}t - \int_{\mathbb{R}} \widetilde{g}(t) \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \mathrm{d}t \right| \\
&= \left| \int_a^b [f(t)-g(t)] \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \mathrm{d}t \right| \le \int_a^{x_0-\delta} |f(t)-g(t)| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t \\
&\quad + \int_{x_0-\delta}^{x_0+\delta} |f(t)-g(t)| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t + \int_{x_0+\delta}^{b} |f(t)-g(t)| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t \\
&=: I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda).
\end{aligned}
$$

To estimate $I_1(x,\lambda)$ and $I_3(x,\lambda)$, we use the following method:

$$
\begin{aligned}
I_1(x,\lambda) &= \int_a^{x_0-\delta} |f(t)-g(t)| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t \\
&\le \sup_{0<\delta\le|t|} \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \int_a^b |f(t)-g(t)| \, \mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
I_3(x,\lambda) &= \int_{x_0+\delta}^{b} |f(t)-g(t)| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t \\
&\le \sup_{0<\delta\le|t|} \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \int_a^b |f(t)-g(t)| \, \mathrm{d}t.
\end{aligned}
$$

Now we shall rewrite $I_2(x,\lambda)$ as follows,

$$
I_2(x,\lambda) = \int_{x_0-\delta}^{x_0+\delta} \left| \tfrac{f(t)-g(t)}{(t-x_0)^r} \right| |(t-x_0)^r| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t.
$$

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that;

$$
I_2(x,\lambda) \le \varepsilon \int_{x_0-\delta}^{x_0+\delta} |(t-x_0)^r| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t.
$$

Let

$$
I_{2,1}(x,\lambda) \coloneqq \int_{x_0-\delta}^{x_0+\delta} |(t-x_0)^r| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t-x) \right| \mathrm{d}t = \int_{x_0-x-\delta}^{x_0-x+\delta} |(t+x-x_0)^r| \left| \tfrac{\partial^r}{\partial t^r} L_\lambda(t) \right| \mathrm{d}t.
$$

It can be rewrite $I_{2,1}(x,\lambda)$ as follows:

$$\begin{aligned} I_{2,1}(x,\lambda) &= \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^r - t^r + t^r\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &\le \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^r - t^r\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t + \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^r \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &=: I_{2,1,1}(x,\lambda) + I_{2,1,2}(x,\lambda). \end{aligned}$$

From (5) $I_{2,1,2}(x,\lambda)$ is finite. So it is sufficient to show that $I_{2,1,1}(x,\lambda)$ is finite. Using the obvious identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + ... + ab^{n-2} + b^{n-1}) \tag{11}$$

yields

$$\begin{aligned} I_{2,1,1}(x,\lambda) &= |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-1} + ... + t^{r-1}\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &\le |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-1}\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &\quad + |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)^{r-2}\, t\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &\quad \vdots \\ &\quad + |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} \left|(t+x-x_0)\, t^{r-2}\right| \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t \\ &\quad + |x-x_0| \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-1} \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t. \end{aligned} \tag{12}$$

Applying the formula (11) successively to the right-hand side of (12), we can see that $I_{2,1,1}(x,\lambda)$ is less than or equal to the linear combinations of

$$|x-x_0|^\nu \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r-\nu} \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t)\right| \mathrm{d}t, \qquad (\nu = 1,2,...,r).$$

Taking into account (5) and (4), we show that $I_{2,1}(x,\lambda)$ is finite on any planar set D_r.

Hence

$$|I_\lambda(x)| \le I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda)$$
$$\le \varepsilon I_2(x,\lambda) + 2 \sup_{0<\delta\le|t|} \left|\tfrac{\partial^r}{\partial t^r} L_\lambda(t-x)\right| \int_a^b |f(t)-g(t)|\,\mathrm{d}t.$$

Since $f(t)-g(t)$ belongs to $L_1(a,b)$, in view of (5), (4) and the condition c), we obtain

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |I_\lambda(x)| = 0,$$

i.e.,

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \tfrac{\partial^r}{\partial x^r}\left(I_\lambda f\right)(x) = \lim_{(x,\lambda)\to(x_0,\lambda_0)} \tfrac{\partial^r}{\partial x^r}\left(I_\lambda g\right)(x).$$

This completes the proof of the Theorem. □

Secondly, we will investigate the approximation for finite $(r+1)-th$ derivatives of the operator $(I_\lambda f)$ in $L_1(I)$.

Theorem 3. *Let the function $L_\lambda(t)$ and its derivatives $\frac{\partial^\nu}{\partial t^\nu}L_\lambda(t)$, $(\nu = 1,2,...,r,r+1)$ be continuous with respect to t on $(-\infty,\infty)$ and $L_\lambda(t)$ be integrable with respect to t for each fixed $\lambda\in\Lambda$. Suppose that conditions* c) *and* d) *are satisfied. Also we assume that the relations*

$$\sup_{\lambda\in\Lambda} \int_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r+1} \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t)\right| \mathrm{d}t < \infty \quad \textit{and} \quad \lim_{\lambda\to\lambda_0} \sup_{0<\delta\le|t|} \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t)\right| = 0$$

hold for every $\delta>0$. Suppose that the function $f\in L_1(I)$ has at x_0 finite derivatives $f_+^{(r+1)}(x_0)$ and $f_-^{(r+1)}(x_0)$.
Then

$$\lim \tfrac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda f\right)(x) = Bf_+^{(r+1)}(x_0) + (1-B)f_-^{(r+1)}(x_0)$$

as $(x,\lambda)\to(x_0,\lambda_0)$ and $(x,\lambda)\in D_{r+1}$, where

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \int_{x_0}^{\infty} L_\lambda(t-x)\mathrm{d}t = B, \qquad 0\le B\le 1. \tag{13}$$

Proof. Let $a<x_0<b$ and $0<x_0-x<\frac{\delta}{2}$ $(\delta>0)$. Setting

(14)
$$g(t) := \begin{cases} f(x_0)+...+\frac{(t-x_0)^r}{r!}f^{(r)}(x_0)+\frac{(t-x_0)^{r+1}}{(r+1)!}f_-^{(r+1)}(x_0), & a<t<x_0,\\[2ex] f(x_0)+...+\frac{(t-x_0)^r}{r!}f^{(r)}(x_0)+\frac{(t-x_0)^{r+1}}{(r+1)!}f_+^{(r+1)}(x_0), & x_0\le t<b.\end{cases}$$

Note that

$$\left(I_\lambda g\right)(x) = \int_a^b K_\lambda(t-x,g(t))\mathrm{d}t = \int_R K_\lambda(t-x,\widetilde{g}(t))\mathrm{d}t.$$

Differentiating both sides of the last equality $(r+1)$ times with respect to x, one has

$$\frac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda g\right)(x) = \int_a^b g^{(r+1)}(t) L_\lambda(t-x)\mathrm{d}t.$$

By (14), the last equality can be rewritten in the form

$$\frac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda g\right)(x) = f_-^{(r+1)}(x_0)\int_a^{x_0} L_\lambda(t-x)\mathrm{d}t + f_+^{(r+1)}(x_0)\int_{x_0}^b L_\lambda(t-x)\mathrm{d}t.$$

The hypothesis (13) yields

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda g\right)(x) = B\, f_+^{(r+1)}(x_0) + (1-B) f_-^{(r+1)}(x_0).$$

Define

$$\left|I_\lambda(x)\right| := \left|\frac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda g\right)(x) - \frac{\partial^{r+1}}{\partial x^{r+1}}\left(I_\lambda f\right)(x)\right|.$$

In order to complete the proof of the theorem, it is sufficient to show

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \left|I_\lambda(x)\right| = 0.$$

One has

$$\begin{aligned}
\left|I_\lambda(x)\right| &= \left|\int_a^b \left[f(t)-g(t)\right] \frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\mathrm{d}t\right| \\
&\leq \int_a^{x_0-\delta} \left|f(t)-g(t)\right| \left|\frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t \\
&\quad + \int_{x_0-\delta}^{x_0} \left|\frac{f(t)-g(t)}{(t-x_0)^{r+1}}\right| \left|(t-x_0)^{r+1}\right| \left|\frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t \\
&\quad + \int_{x_0+\delta}^{b} \left|f(t)-g(t)\right| \left|\frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t \\
&\quad + \int_{x_0}^{x_0+\delta} \left|\frac{f(t)-g(t)}{(t-x_0)^{r+1}}\right| \left|(t-x_0)^{r+1}\right| \left|\frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t \\
&=: I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda). \qquad (15)
\end{aligned}$$

Since f has at x_0 a finite $(r+1)-th$ right and left derivatives, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$I_2(x,\lambda) = \int\limits_{x_0-\delta}^{x_0} \left|\tfrac{f(t)-g(t)}{(t-x_0)^{r+1}}\right| \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t$$

$$\leq \varepsilon \int\limits_{x_0-\delta}^{x_0} \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t \tag{16}$$

and

$$I_4(x,\lambda) = \int\limits_{x_0}^{x_0+\delta} \left|\tfrac{f(t)-g(t)}{(t-x_0)^{r+1}}\right| \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t$$

$$\leq \varepsilon \int\limits_{x_0}^{x_0+\delta} \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t. \tag{17}$$

Thus we get

$$I_2(x,\lambda) + I_4(x,\lambda) \leq \varepsilon \int\limits_{x_0-\delta}^{x_0+\delta} \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t.$$

Set

$$I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda) := \int\limits_{x_0-\delta}^{x_0+\delta} \left|(t-x_0)^{r+1}\right| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t.$$

Using the same method as in the proof of Theorem 2 we deduce $I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda)$ is less than or equal to a linear combination of

$$|x-x_0|^\nu \int\limits_{x_0-x-\delta}^{x_0-x+\delta} |t|^{r+1-\nu} \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} K(t,\lambda)\right| \mathrm{d}t, \ (\nu = 1,...,r+1).$$

By virtue of (4), the term $I_2(x,\lambda) + I_4(x,\lambda)$ is bounded.

Now, we consider the integrals $I_1(x,\lambda)$ and $I_3(x,\lambda)$, respectively.

$$I_1(x,\lambda) = \int\limits_{a}^{x_0-\delta} |f(t)-g(t)| \left|\tfrac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t$$

$$\leq M \sup_{\frac{\delta}{2}<|u|} \left|\tfrac{\partial^{r+1}}{\partial u^{r+1}} L_\lambda(u)\right| \tag{18}$$

and

$$I_3(x,\lambda) = \int_{x_0+\delta}^{b} |f(t)-g(t)| \left|\frac{\partial^{r+1}}{\partial t^{r+1}} L_\lambda(t-x)\right| \mathrm{d}t$$

$$\le M \sup_{\frac{\delta}{2}<|u|} \left|\frac{\partial^{r+1}}{\partial u^{r+1}} L_\lambda(u)\right|. \tag{19}$$

Using (16), (19) in (15) one has

$$|I_\lambda(x)| \le I_1(x,\lambda) + I_2(x,\lambda) + I_3(x,\lambda) + I_4(x,\lambda)$$

$$\le \varepsilon \left[I_{2,1}(x,\lambda) + I_{4,1}(x,\lambda)\right] \& \quad + 2M \sup_{\frac{\delta}{2}<|u|} \left|\frac{\partial^{r+1}}{\partial u^{r+1}} L_\lambda(u)\right|. \tag{20}$$

Under the hypotheses of the theorem, (20) yields

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |I_\lambda(x)| = \lim_{(x,\lambda)\to(x_0,\lambda_0)} \left|\frac{\partial^{r+1}}{\partial x^{r+1}} \left(I_\lambda g\right)(x) - \frac{\partial^{r+1}}{\partial x^{r+1}} \left(I_\lambda f\right)(x)\right| = 0.$$

This completes the proof. □

Remark 4. If $B = \frac{1}{2}$, then we have

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} \frac{\partial^{r+1}}{\partial x^{r+1}} \left(I_\lambda f\right)(x) = \frac{f_+^{(r+1)}(x_0) + f_-^{(r+1)}(x_0)}{2}. \quad \square$$

Remark 5. Let $I = (a,b)$ be an arbitrary bounded interval in $\mathbb{R}$ and $f \in L_1(a,b)$ be a $(b-a)$-periodic function. In this case the proofs of the Theorems are similar. □

3. EXAMPLES

Example 6. A special case of the function $K_\lambda(t,u)$ satisfying the conditions, is the linear case with respect to the second variable, i.e.,

$$K_\lambda(t,u) = Z_\lambda(t)\, u.$$

This case is widely used in Approximation Theory [4]. □

Example 7. We introduce the function

$$K_\lambda(t,u) = \begin{cases} (r+1)\lambda^{r+1} t^r\, u + u, & t \in [0,\frac{1}{\lambda}], \\ 0, & t \notin [0,\frac{1}{\lambda}], \end{cases}$$

where $\Lambda = [1,\infty)$ is a set of indices with natural topology and $\lambda_0 = \infty$ is an accumulation point of Λ in this topology.

First of all $K_\lambda(t,u)$ is a kernel, i.e., $K_\lambda(t,0) = 0$.

It is seen that for every $u \in \mathbb{R}$,

$$\frac{\partial^r}{\partial x^r} K_\lambda(t-x,u) = \begin{cases} (-1)^r (r+1)!\lambda^{r+1}\, u, & t-x \in [0,\frac{1}{\lambda}], \\ 0, & t-x \notin [0,\frac{1}{\lambda}]. \end{cases}$$

According to (2), we obtain

$$\frac{\partial^r}{\partial x^r} L_\lambda(t-x) = \begin{cases} (-1)^r (r+1)! \lambda^{r+1}, & t-x \in [0, \frac{1}{\lambda}], \\ 0, & t-x \notin [0, \frac{1}{\lambda}]. \end{cases}$$

This implies

$$L_\lambda(t-x) = \begin{cases} (r+1)\lambda^{r+1}(t-x)^r, & t-x \in [0, \frac{1}{\lambda}], \\ 0, & t-x \notin [0, \frac{1}{\lambda}]. \end{cases}$$

Moreover

$$\int_{\mathbb{R}} L_\lambda(t) \mathrm{d}t = \int_{[0, \frac{1}{\lambda}]} (r+1)\lambda^{r+1} t^r \mathrm{d}t = 1 < \infty.$$

It is easy to see that

$$\lim_{\lambda \to \infty} \int_{\mathbb{R} \setminus U} L_\lambda(t) \mathrm{d}t = 0,$$

for every $U \in \mathcal{U}(0)$ and

$$\lim_{\lambda \to \infty} \left[\sup_{|t| \geq \delta} L_\lambda(t) \right] = 0,$$

for every $\delta > 0$. □

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