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WEIGHTED MONTGOMERY'S IDENTITIES FOR HIGHER ORDER DIFFERENTIABLE FUNCTIONS OF TWO VARIABLES

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Abstract. We give weighted Montgomery's identities for higher order differentiable functions of two variables and by using these identities we obtain generalized Ostrowski-type and Grüss-type inequalities for double weighted integrals of higher order differentiable functions of two independent variables.

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1. INTRODUCTION

Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on $[a,b]$ and $f':[a,b]\to\mathbb{R}$ be an integrable function. Then, the Montgomery identity holds [15], (see also [20]),

$$f(x)=\tfrac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t+\int_a^b p(x,t)f'(t)\,\mathrm{d}t, \tag{1}$$

where $p(x,t)$ is the Peano kernel

$$p(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \frac{t-b}{b-a}, & x<t\le b.\end{cases}$$

Suppose now that $w:[a,b]\to[0,\infty)$ is some probability density function, i.e., it is a positive integrable function satisfying $\int_a^b w(t)\mathrm{d}t=1$ and

$$W(t)=\begin{cases}0, & t<a,\\ \int_a^t w(x)\mathrm{d}x, & t\in[a,b],\\ 1, & t>b.\end{cases}$$

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The following identity (Pečarić [17]) is a generalization of Montgomery's identity,

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \int_a^b p_w(x,t) f(t)\,\mathrm{d}t,$$

where the weighted Peano kernel is

$$p_w(x,t) = \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases}$$

In [3, 8] the authors obtained two identities which generalize (1) for functions of two variables. In fact, for a function $f : [a,b] \times [c,d] \to \mathbb{R}$ such that the partial derivatives $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ and $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ exist and are continuous on $[a,b] \times [c,d]$ and for all $(x,y) \in [a,b] \times [c,d]$ they obtained:

$$\begin{aligned} &(d-c)(b-a) f(x,y) = \\ &= -\int_a^b \int_c^d f(s,t)\,\mathrm{d}t\,\mathrm{d}s + (d-c)\int_a^b f(s,y)\,\mathrm{d}s \\ &\quad + (b-a)\int_c^d f(x,t)\,\mathrm{d}t + \int_a^b \int_c^d q(x,s) r(y,t) \tfrac{\partial^2 f(s,t)}{\partial s \partial t}\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} &(d-c)(b-a) f(x,y) = \\ &= \int_a^b \int_c^d f(s,t)\,\mathrm{d}t\,\mathrm{d}s + \int_a^b \int_c^d q(x,s) \tfrac{\partial f(s,t)}{\partial s}\,\mathrm{d}t\,\mathrm{d}s \\ &\quad + \int_a^b \int_c^d r(y,t) \tfrac{\partial f(s,t)}{\partial t}\,\mathrm{d}t\,\mathrm{d}s + \int_a^b \int_c^d q(x,s) r(y,t) \tfrac{\partial^2 f(s,t)}{\partial s \partial t}\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

where

$$q(x,s) = \begin{cases} s-a, & a \le s \le x, \\ s-b, & x < s \le b, \end{cases}$$

$$r(y,t) = \begin{cases} t-c, & a \le t \le x, \\ t-d, & x < t \le b. \end{cases}$$

We can also find weighted Montgomery's identities for functions of two variables in [20]. These identities may be summarized as:

Theorem 1.1. *Let $p : [a,b] \times [c,d] \to \mathbb{R}$ be an integrable function and $P(x,y)$ is defined as*

$$P(x,y) = \int_x^b \int_y^d p(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{2}$$

Let $f : [a,b] \times [c,d] \to \mathbb{R}$ *have continuous partial derivatives* $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ *and* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *on* $[a,b] \times [c,d]$. *Then, for all* $(x,y) \in [a,b] \times [c,d]$

$$
\begin{aligned}
(3) \qquad & f(x,y)P(a,c) = \\
&= \int_a^b \int_c^d p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s + \int_a^b \hat{P}(x,s) \tfrac{\partial f(s,y)}{\partial s} \, \mathrm{d}s \\
&\quad + \int_c^d \tilde{P}(y,t) \tfrac{\partial f(x,t)}{\partial t} \, \mathrm{d}t - \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) \tfrac{\partial^2 f(s,t)}{\partial s \partial t} \, \mathrm{d}t \, \mathrm{d}s,
\end{aligned}
$$

where

$$
\hat{P}(x,s) = \begin{cases} \displaystyle\int_a^s \int_c^d p(\xi,\eta) \mathrm{d}\eta \, \mathrm{d}\xi, & a \le s \le x, \\ -P(s,c), & x < s \le b, \end{cases}
$$

$$
\tilde{P}^{(i,M)}(x,y,t) = \begin{cases} \displaystyle\int_a^b \int_c^t p(\xi,\eta) \mathrm{d}\eta \, \mathrm{d}\xi, & c \le t \le y, \\ -P(a,t), & y < t \le d \end{cases} \quad \text{and}
$$

$$
\bar{P}(x,s,y,t) = \begin{cases} \displaystyle\int_a^s \int_c^t p(\xi,\eta) \mathrm{d}\eta \, \mathrm{d}\xi, & a \le s \le x, \; c \le t \le y, \\ \displaystyle -\int_s^b \int_c^t p(\xi,\eta) \mathrm{d}\eta \, \mathrm{d}\xi, & x < s \le b, \; c \le t \le y, \\ \displaystyle -\int_a^s \int_t^d p(\xi,\eta) \mathrm{d}\eta \, \mathrm{d}\xi, & a \le s \le x, \; y < t \le d, \\ P(s,t), & x < s \le b, \; y < t \le d. \end{cases}
$$

THEOREM 1.2. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *have continuous partial derivatives* $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ *and* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *on* $[a,b] \times [c,d]$. *Then, for all* $(x,y) \in [a,b] \times [c,d]$, *we have*

$$
\begin{aligned}
(4) \qquad & f(x,y)P(a,c) = \\
&= -\int_a^b \int_c^d p(s,t) f(s,t) \, \mathrm{d}t \, \mathrm{d}s + \int_a^b \int_c^d p(s,t) f(s,y) \, \mathrm{d}t \, \mathrm{d}s \\
&\quad + \int_a^b \int_c^d p(s,t) f(x,t) \, \mathrm{d}t \, \mathrm{d}s + \int_a^b \int_c^d \bar{P}(x,s,y,t) \tfrac{\partial^2 f(s,t)}{\partial s \partial t} \, \mathrm{d}t \, \mathrm{d}s,
\end{aligned}
$$

where $p(.,.)$, $P(a,c)$, $\bar{P}(x,s,y,t)$ *are as in Theorem* 1.1.

THEOREM 1.3. *Let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *have continuous partial derivatives* $\frac{\partial f(s,t)}{\partial s}$, $\frac{\partial f(s,t)}{\partial t}$ *and* $\frac{\partial^2 f(s,t)}{\partial s \partial t}$ *on* $[a,b] \times [c,d]$. *Then, for all* $(x,y) \in [a,b] \times [c,d]$, *we have*

$$\begin{aligned} f(x,y)[P(a,c)]^2 = P(a,c)&\int_a^b\int_c^d p(s,t)f(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ &+\int_a^b\left(\int_a^b\int_c^d p(\xi,t)\hat{P}(x,s)\tfrac{\partial f(s,t)}{\partial s}\,\mathrm{d}t\,\mathrm{d}s\right)\mathrm{d}\xi \\ &+\int_c^d\left(\int_a^b\int_c^d p(s,\eta)\tilde{P}(y,t)\tfrac{\partial f(s,t)}{\partial t}\,\mathrm{d}t\,\mathrm{d}s\right)\mathrm{d}\eta \\ &+\int_a^b\int_c^d \check{P}(x,s,y,t)\tfrac{\partial^2 f(s,t)}{\partial s\partial t}\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

where $p(.,.)$, $P(a,c)$, $\hat{P}(x,s)$, $\tilde{P}(y,t)$, $\bar{P}(x,s,y,t)$ *are as in Theorem* 1.1 *and*

$$\check{P}(x,s,y,t) = 2\hat{P}(x,s)\tilde{P}(y,t) - P(a,c)\bar{P}(x,s,y,t).$$

Montgomery's identities have many applications and capture other well known and important identities and inequalities which includes Ostrowski-type, Čebyšev-type and Grüss-type inequalities, as we can see in this paper also. Ostrowski's inequalities have many applications in field of Numerical Integration (for an extensive reference see [10]) and Probability Theory (see [7, 14]). We can also obtain Special Means with the help of such inequalities (for example see [1, 2]). Very famous Čebyšev's inequality is also an special case of Ostrowski-type inequalities (see [18, 19]). Grüss-type inequalities have applications in Numerical Integration and other fields (for reference see [4, 5, 6]).

The structure of this paper based on four sections. In the second section, we give generalization of Montgomery's identities using higher order differentiable functions of two variables. In the third and the fourth sections respectively, we obtain some generalized Ostrowski-type and Grüss-type inequalities for higher order differentiable functions of two independent variables by using identities proved in the second section. These identities and inequalities generalize many results given in [3, 8, 9, 12, 16, 20] etc.

2. WEIGHTED MONTGOMERY'S IDENTITIES FOR HIGHER ORDER DIFFERENTIABLE FUNCTIONS OF TWO VARIABLES

In start of this section, we define some notation to reduce our lengthy expressions as follows:

$$P^{(i,j)}_{(a,c)\to(b,d)}(x,y) = \int_a^b\int_c^d p(\xi,\eta)\tfrac{(\xi-x)^i}{i!}\tfrac{(\eta-y)^j}{j!}\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{5}$$

$$P^{(0,j)}_{(a,c)\to(b,d)}(y) = \int_a^b\int_c^d p(\xi,\eta)\tfrac{(\eta-y)^j}{j!}\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{6}$$

$$P^{(i,0)}_{(a,c)\to(b,d)}(x) = \int_a^b \int_c^d p(\xi,\eta)\tfrac{(\xi-x)^i}{i!}\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{7}$$

$$\begin{aligned} R(x,y;f) &= \\ &= -\sum_{i=1}^{N}\sum_{j=1}^{M} f_{(i,j)}(x,y)P^{(i,j)}_{(a,c)\to(b,d)}(x,y) \\ &\quad -\sum_{j=1}^{M} f_{(0,j)}(x,y)P^{(0,j)}_{(a,c)\to(b,d)}(y) - \sum_{i=1}^{N} f_{(i,0)}(x,y)P^{(i,0)}_{(a,c)\to(b,d)}(x). \end{aligned} \tag{8}$$

For our next theorem, we give a lemma from [13] using our notations as follows.

Lemma 2.1. *Let $p, f : [a,b]\times[c,d] \to \mathbb{R}$ be integrable functions and $f \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then we have*

$$\begin{aligned} \int_a^b\int_c^d p(x,y)f(x,y)\mathrm{d}y\,\mathrm{d}x &= \sum_{i=0}^{N}\sum_{j=0}^{M} P^{(i,j)}_{(a,c)\to(b,d)}(a,c)f_{(i,j)}(a,c) \\ &\quad + \sum_{j=0}^{M}\int_a^b P^{(N,j)}_{(x,c)\to(b,d)}(x,c)f_{(N+1,j)}(x,c)\,\mathrm{d}x \\ &\quad + \sum_{i=0}^{N}\int_c^d P^{(i,M)}_{(a,y)\to(b,d)}(a,y)f_{(i,M+1)}(a,y)\,\mathrm{d}y \\ &\quad + \int_a^b\int_c^d P^{(N,M)}_{(x,y)\to(b,d)}(x,y)f_{(N+1,M+1)}(x,y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{9}$$

We give generalizations of Theorems 1.1, 1.2 and 1.3 respectively as follows:

Theorem 2.2. *Let $p, f : [a,b]\times[c,d] \to \mathbb{R}$ be integrable functions and $f \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then we have*

$$\begin{aligned} f(x,y)P(a,c) &= R(x,y;f) + \int_a^b\int_c^d p(s,t)f(s,t)\mathrm{d}t\,\mathrm{d}s \\ &\quad + \sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s \\ &\quad + \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,\mathrm{d}t \\ &\quad - \int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s, \end{aligned} \tag{10}$$

where

$$\hat{P}^{(N,j)}(x,s,y)=\begin{cases} P^{(N,j)}_{(a,c)\to(s,d)}(s,y), & a\le s\le x,\\ -P^{(N,j)}_{(s,c)\to(b,d)}(s,y), & x<s\le b,\end{cases}$$

$$\tilde{P}^{(i,M)}(x,y,t)=\begin{cases} P^{(i,M)}_{(a,c)\to(b,t)}(x,t), & c\le t\le y,\\ -P^{(i,M)}_{(a,t)\to(b,d)}(x,t), & y<t\le d,\end{cases} \quad \text{and}$$

$$\bar{P}^{(N,M)}(x,s,y,t)=\begin{cases} P^{(N,M)}_{(a,c)\to(s,t)}(s,t), & a\le s\le x,\ c\le t\le y,\\ -P^{(N,M)}_{(s,c)\to(b,t)}(s,t), & x<s\le b,\ c\le t\le y,\\ -P^{(N,M)}_{(a,t)\to(s,d)}(s,t), & a\le s\le x,\ y<t\le d,\\ P^{(N,M)}_{(s,t)\to(b,d)}(s,t), & x<s\le b,\ y<t\le d,\end{cases}$$

where $P^{(i,j)}_{(.,.)\to(.,.)}(.,.)$ *for* $i,j\in\{N,M\}$ *is defined in* (5), *and* $P(a,c)$ *and* $R(x,y;f)$ *are defined in* (2) *and* (8) *respectively.*

Proof. Using Lemma 2.1 for $[a,x]\times[c,y]$, we obtain the equality

$$\int_a^x\int_c^y p(s,t)f(s,t)\mathrm{d}t\,\mathrm{d}s=\int_x^a\int_y^c p(s,t)f(s,t)\mathrm{d}t\,\mathrm{d}s=$$

$$=\sum_{i=0}^{N}\sum_{j=0}^{M}P^{(i,j)}_{(x,y)\to(a,c)}(x,y)f_{(i,j)}(x,y)+\sum_{j=0}^{M}\int_x^a P^{(N,j)}_{(s,y)\to(a,c)}(s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s$$

$$+\sum_{i=0}^{N}\int_y^c P^{(i,M)}_{(x,t)\to(a,c)}(x,t)f_{(i,M+1)}(x,t)\,\mathrm{d}t$$

$$+\int_x^a\int_y^c P^{(N,M)}_{(s,t)\to(a,c)}(s,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s$$

$$=\sum_{i=0}^{N}\sum_{j=0}^{M}f_{(i,j)}(x,y)\Big[P^{(i,j)}_{(x,y)\to(b,d)}(x,y)P^{(i,j)}_{(x,c)\to(b,d)}(x,y)-P^{(i,j)}_{(a,y)\to(b,d)}(x,y)$$

$$+P^{(i,j)}_{(a,c)\to(b,d)}(x,y)\Big]-\sum_{j=0}^{M}\int_a^x f_{(N+1,j)}(s,y)\Big[P^{(N,j)}_{(s,y)\to(b,d)}(s,y)$$

$$-P^{(N,j)}_{(s,c)\to(b,d)}(s,y)-P^{(N,j)}_{(a,y)\to(b,d)}(s,y)+P^{(N,j)}_{(a,c)\to(b,d)}(s,y)\Big]\,\mathrm{d}s$$

$$-\sum_{i=0}^{N}\int_c^y f_{(i,M+1)}(x,t)\Big[P^{(i,M)}_{(x,t)\to(b,d)}(x,t)-P^{(i,M)}_{(x,c)\to(b,d)}(x,t)-P^{(i,M)}_{(a,t)\to(b,d)}(x,t)$$

$$+P^{(i,M)}_{(a,c)\to(b,d)}(x,t)\Big]\,\mathrm{d}t+\int_a^x\int_c^y f_{(N+1,M+1)}(s,t)\Big[P^{(N,M)}_{(s,t)\to(b,d)}(s,t)$$

$$-P^{(N,M)}_{(s,c)\to(b,d)}(s,t)-P^{(N,M)}_{(a,t)\to(b,d)}(s,t)+P^{(N,M)}_{(a,c)\to(b,d)}(s,t)\Big]\,\mathrm{d}t\,\mathrm{d}s.$$

Similarly, for $[x,b] \times [c,y]$, we get

$$\int_x^b \int_c^y p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s = -\int_x^b \int_y^c p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s =$$

$$= -\sum_{i=0}^{N} \sum_{j=0}^{M} f_{(i,j)}(x,y) \left[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) P^{(i,j)}_{(x,c)\to(b,d)}(x,y) \right]$$

$$- \sum_{j=0}^{M} \int_x^b f_{(N+1,j)}(s,y) \left[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(s,c)\to(b,d)}(s,y) \right] \mathrm{d}s$$

$$+ \sum_{i=0}^{N} \int_c^y f_{(i,M+1)}(x,t) \left[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(x,c)\to(b,d)}(x,t) \right] \mathrm{d}t$$

$$+ \int_x^b \int_c^y f_{(N+1,M+1)}(s,t) \left[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(s,c)\to(b,d)}(s,t) \right] \mathrm{d}t \, \mathrm{d}s.$$

For $[a,x] \times [y,d]$, we have

$$\int_a^x \int_y^d p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s = -\int_x^a \int_y^d p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s =$$

$$= -\sum_{i=0}^{N} \sum_{j=0}^{M} f_{(i,j)}(x,y) \left[P^{(i,j)}_{(x,y)\to(b,d)}(x,y) - P^{(i,j)}_{(a,y)\to(b,d)}(x,y) \right]$$

$$+ \sum_{j=0}^{M} \int_a^x f_{(N+1,j)}(s,y) \left[P^{(N,j)}_{(s,y)\to(b,d)}(s,y) - P^{(N,j)}_{(a,y)\to(b,d)}(s,y) \right] \mathrm{d}s$$

$$- \sum_{i=0}^{N} \int_y^d f_{(i,M+1)}(x,t) \left[P^{(i,M)}_{(x,t)\to(b,d)}(x,t) - P^{(i,M)}_{(a,t)\to(b,d)}(x,t) \right] \mathrm{d}t$$

$$+ \int_a^x \int_y^d f_{(N+1,M+1)}(s,t) \left[P^{(N,M)}_{(s,t)\to(b,d)}(s,t) - P^{(N,M)}_{(a,t)\to(b,d)}(s,t) \right] \mathrm{d}t \, \mathrm{d}s.$$

Finally, for $[x,b] \times [y,d]$ we obtain

$$\int_x^b \int_y^d p(s,t) f(s,t) \mathrm{d}t \, \mathrm{d}s = \sum_{i=0}^{N} \sum_{j=0}^{M} f_{(i,j)}(x,y) P^{(i,j)}_{(x,y)\to(b,d)}(x,y) =$$

$$+ \sum_{j=0}^{M} \int_x^b f_{(N+1,j)}(s,y) P^{(N,j)}_{(s,y)\to(b,d)}(s,y) \, \mathrm{d}s +$$

$$+\sum_{i=0}^{N}\int_y^d f_{(i,M+1)}(x,t)P^{(i,M)}_{(x,t)\to(b,d)}(x,t)\,\mathrm{d}t$$
$$+\int_x^b\int_y^d f_{(N+1,M+1)}(s,t)P^{(N,M)}_{(s,t)\to(b,d)}(s,t)\,\mathrm{d}t\,\mathrm{d}s.$$

Adding the four expressions, we get our required result. □

THEOREM 2.3. *Let $p, f : [a,b]\times[c,d]\to\mathbb{R}$ be integrable functions and $f\in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then we have*

$$\begin{aligned}(11)\quad & f(x,y)P(a,c)=\\ &=R(x,y;f)+\sum_{j=1}^{M}\int_a^b\int_c^d p(s,\eta)\tfrac{(\eta-y)^j}{j!}f_{(0,j)}(s,y)\,\mathrm{d}\eta\,\mathrm{d}s\\ &+\sum_{i=1}^{N}\int_a^b\int_c^d p(\xi,t)\tfrac{(\xi-x)^i}{i!}f_{(i,0)}(x,t)\,\mathrm{d}t\,\mathrm{d}\xi-\int_a^b\int_c^d p(s,t)f(s,t)\mathrm{d}t\,\mathrm{d}s\\ &+\int_a^b\int_c^d p(s,t)f(s,y)\,\mathrm{d}t\,\mathrm{d}s+\int_a^b\int_c^d p(s,t)f(x,t)\,\mathrm{d}t\,\mathrm{d}s\\ &+\int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s,\end{aligned}$$

where $\bar{P}^{(N,M)}(x,s,y,t)$ is as in Theorem 2.2*, and $P(a,c)$ and $R(x,y,f)$ are defined in* (2) *and* (8) *respectively.*

Proof. First, we find an expression for

$$\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s$$

by using integration by parts as follows

$$\begin{aligned}&\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s=\\ &=\int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s-\int_x^b P^{(N,j)}_{(s,c)\to(b,d)}(s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s\\ &=\int_a^x P^{(N,j)}_{(a,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s+\int_x^b P^{(N,j)}_{(b,c)\to(s,d)}(s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= P^{(N,j)}_{(a,c)\to(x,d)}(x,y) f_{(N,j)}(x,y) + \int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y) f_{(N,j)}(s,y)\,\mathrm{d}s \\
&\quad + P^{(N,j)}_{(x,c)\to(b,d)}(x,y) f_{(N,j)}(x,y) + \int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y) f_{(N,j)}(s,y)\,\mathrm{d}s \\
&= P^{(N,j)}_{(a,c)\to(b,d)}(x,y) f_{(N,j)}(x,y) + \int_a^x P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y) f_{(N,j)}(s,y)\,\mathrm{d}s \\
&\quad + \int_x^b P^{(N-1,j)}_{(b,c)\to(s,d)}(s,y) f_{(N,j)}(s,y)\,\mathrm{d}s \\
&= P^{(N,j)}_{(a,c)\to(b,d)}(x,y) f_{(N,j)}(x,y) + \int_a^b P^{(N-1,j)}_{(a,c)\to(s,d)}(s,y) f_{(N,j)}(s,y)\,\mathrm{d}s,
\end{aligned}
$$

continuing in this way, we finally get

$$
\begin{aligned}
&\int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\,\mathrm{d}s = \\
&= \int_a^b \int_c^d p(\xi,\eta) \tfrac{(\eta-y)^j}{j!} \left[\sum_{k=0}^{N} \tfrac{(\xi-x)^k}{k!} f_{(k,j)}(x,y) \right] \mathrm{d}\eta\,\mathrm{d}\xi \\
&\quad - \int_a^b \int_c^d p(s,\eta) \tfrac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\,\mathrm{d}\eta\,\mathrm{d}s.
\end{aligned} \tag{12}
$$

Similarly

$$
\begin{aligned}
&\int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\,\mathrm{d}t = \\
&= \int_a^b \int_c^d p(\xi,\eta) \tfrac{(\xi-x)^i}{i!} \left[\sum_{l=0}^{M} \tfrac{(\eta-y)^l}{l!} f_{(i,l)}(x,y) \right] \mathrm{d}\eta\,\mathrm{d}\xi \\
&\quad - \int_a^b \int_c^d p(\xi,t) \tfrac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\,\mathrm{d}\xi\,\mathrm{d}t.
\end{aligned} \tag{13}
$$

If we put all these values in (10), then after some cancelation and some rearrangements, we get our required identity. □

Theorem 2.4. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a function such that $f \in C^{(2N+1,2M+1)}([a,b] \times [c,d])$. Then for all $(x,y) \in [a,b] \times [c,d]$ we have*

$$
\begin{aligned}
&f(x,y)[P(a,c)]^2 = \\
&= P(a,c) R(x,y;f) + P(a,c) \int_a^b \int_c^d p(s,t) f(s,t) \mathrm{d}t\,\mathrm{d}s \\
&\quad + \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) R(x,t; f_{(i,M+1)})\,\mathrm{d}t +
\end{aligned} \tag{14}
$$

$$+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) R(s,y;f_{(N+1,j)})\,\mathrm{d}s$$

$$+ \sum_{i=0}^{N}\sum_{j=0}^{M} \int_a^b \int_a^b \int_c^d \hat{P}^{(N,j)}(x,s,y) p(\xi,t) \tfrac{(\xi-x)^i}{i!} f_{(N+1+i,j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\xi$$

$$+ \sum_{i=0}^{N}\sum_{j=0}^{M} \int_c^d \int_a^b \int_c^d \tilde{P}^{(i,M)}(x,y,t) p(s,\eta) \tfrac{(\eta-y)^j}{j!} f_{(i,M+1+j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\eta$$

$$+ 2\int_a^b \int_c^d \sum_{i=0}^{N}\sum_{j=0}^{M} \hat{P}^{(N,j)}(x,s,y)\tilde{P}^{(i,M)}(x,y,t) f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s$$

$$- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

where $\hat{P}^{(N,j)}(x,s,y)$ $\tilde{P}^{(i,M)}(x,y,t)$, $\bar{P}^{(N,M)}(x,s,y,t)$ *are as in Theorem* 2.2 *and* $P(a,c)$ *is defined in* (2).

Proof. Summing (12) for $j = 0,\dots,M$ and (13) for $i = 0,\dots,N$, we get respectively

$$f(x,y)P(a,c) = R(x,y;f) + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \tfrac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\,\mathrm{d}\eta\,\mathrm{d}s + \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) f_{(N+1,j)}(s,y)\,\mathrm{d}s, \tag{15}$$

and

$$f(x,y)P(a,c) = R(x,y;f) + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t) \tfrac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\,\mathrm{d}t\,\mathrm{d}\xi + \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) f_{(i,M+1)}(x,t)\,\mathrm{d}t, \tag{16}$$

for all $(x,y) \in [a,b]\times[c,d]$.
Formula (15) applied for partial derivatives $f_{(i,M+1)}$ for $i = 0,1,\dots,N$, gives

$$\begin{aligned} f_{(i,M+1)}(x,t)P(a,c) &= \\ &= R(x,t;f_{(i,M+1)}) \\ &\quad + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta) \tfrac{(\eta-t)^j}{j!} f_{(i,M+1+j)}(s,t)\,\mathrm{d}\eta\,\mathrm{d}s \\ &\quad + \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}s. \end{aligned} \tag{17}$$

Formula (16) applied for partial derivatives $f_{(N+1,j)}$ for $j = 0, 1, \dots, M$, gives

$$
\begin{aligned}
&f_{(N+1,j)}(s,y)P(a,c) = \\
&= R(s,y;f_{(N+1,j)}) \\
&\quad + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t)\tfrac{(\xi-s)^i}{i!} f_{(N+1+i,j)}(s,t)\,\mathrm{d}t\,\mathrm{d}\xi \\
&\quad + \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}t.
\end{aligned} \tag{18}
$$

Substituting (17) and (18) into (10), we get

$$
\begin{aligned}
&f(x,y)P(a,c) = \\
&= R(x,y;f) + \int_a^b \int_c^d p(s,t) f(s,t)\mathrm{d}t\,\mathrm{d}s \\
&\quad + \tfrac{1}{P(a,c)} \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y) \Bigg[R(s,y;f_{(N+1,j)}) \\
&\quad + \sum_{i=0}^{N} \int_a^b \int_c^d p(\xi,t)\tfrac{(\xi-s)^i}{i!} f_{(N+1+i,j)}(s,t)\,\mathrm{d}t\,\mathrm{d}\xi \\
&\quad + \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(s,y,t) f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}t \Bigg]\,\mathrm{d}s \\
&\quad + \tfrac{1}{P(a,c)} \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t) \Bigg[R(x,t;f_{(i,M+1)}) \\
&\quad + \sum_{j=0}^{M} \int_a^b \int_c^d p(s,\eta)\tfrac{(\eta-t)^j}{j!} f_{(i,M+1+j)}(s,t)\,\mathrm{d}\eta\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,t) f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}s \Bigg]\,\mathrm{d}t \\
&\quad - \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

After some rearrangements and using Fubini's Theorem, we get our required result. □

REMARK 2.5. For $N = M = 0$, Theorems 1.1, 1.2 and 1.3 become special cases of Theorem 2.2, 2.3 and 2.4 respectively (see also [20]).

If $p(s,t) = q(s)r(t)$ in identities (10), (11) and (14), then we get respectively the following special cases:

$$\begin{aligned}
&f(x,y)P_{a\to b}(q)P_{c\to d}(r) = \\
&= Q(x,y;f) + \int_a^b\int_c^d q(s)r(t)f(s,t)\mathrm{d}t\,\mathrm{d}s \\
&\quad + \sum_{j=0}^{M}\int_a^b \hat{Q}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s \\
&\quad + \sum_{i=0}^{N}\int_c^d \tilde{Q}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,\mathrm{d}t \\
&\quad - \int_a^b\int_c^d \bar{Q}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
&f(x,y)P_{a\to b}(q)P_{c\to d}(r) = \\
&= Q(x,y;f) + \sum_{j=1}^{M}\int_a^b q(s)f_{(0,j)}(s,y)\,\mathrm{d}s\,Q^{(j)}_{c\to d}(r,y) \\
&\quad + \sum_{i=1}^{N} Q^{(i)}_{a\to b}(q,x)\int_c^d r(t)f_{(i,0)}(x,t)\,\mathrm{d}t - \int_a^b\int_c^d q(s)r(t)f(s,t)\mathrm{d}t\,\mathrm{d}s \\
&\quad + \int_a^b\int_c^d q(s)r(t)f(s,y)\,\mathrm{d}t\,\mathrm{d}s + \int_a^b\int_c^d q(s)r(t)f(x,t)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad - \int_a^b\int_c^d \bar{Q}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
&f(x,y)[P_{a\to b}(q)P_{c\to d}(r)]^2 = \\
&= P_{a\to b}(q)P_{c\to d}(r)Q(x,y;f) \\
&\quad + \sum_{j=0}^{M}\int_a^b \hat{Q}^{(N,j)}(x,s,y)Q(s,y;f_{(N+1,j)})\,\mathrm{d}s \\
&\quad + \sum_{i=0}^{N}\int_c^d \tilde{Q}^{(i,M)}(x,y,t)Q(x,t;f_{(i,M+1)})\,\mathrm{d}t \\
&\quad + P_{a\to b}(q)P_{c\to d}(r)\int_a^b\int_c^d q(s)r(t)f(s,t)\mathrm{d}t\,\mathrm{d}s \\
&\quad + \sum_{i=0}^{N}\sum_{j=0}^{M} Q^{(i)}_{a\to b}(q,x)\int_a^b\int_c^d \hat{Q}^{(N,j)}(x,s,y)r(t)f_{(N+1+i,j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + \sum_{i=0}^{N}\sum_{j=0}^{M} Q^{(j)}_{c\to d}(r,y)\int_a^b\int_c^d \tilde{Q}^{(i,M)}(x,y,t)q(s)\,f_{(i,M+1+j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad + 2\int_a^b\int_c^d \sum_{i=0}^{N}\sum_{j=0}^{M}\hat{Q}^{(N,j)}(x,s,y)\tilde{Q}^{(i,M)}(x,y,t)f_{(N+1+i,M+1+j)}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad - \int_a^b\int_c^d \bar{Q}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s,
\end{aligned}$$

where

$$
\begin{aligned}
&P_{a\to b}(q)=\int_a^b q(s)\,\mathrm{d}s,\quad Q^{(i)}_{a\to b}(q,x)=\int_a^b q(\xi)\tfrac{(\xi-x)^i}{i!}\,\mathrm{d}\xi,\\
&Q^{(i,j)}_{(a,c)\to(b,d)}(x,y)=Q^{(i)}_{a\to b}(q,x)Q^{(j)}_{c\to d}(r,y),\\
&Q^{(0,j)}_{(a,c)\to(b,d)}(y)=P_{a\to b}(q)\;Q^{(j)}_{c\to d}(r,y),\\
&Q^{(i,0)}_{(a,c)\to(b,d)}(x)=Q^{(i)}_{a\to b}(q,x)\;P_{c\to d}(r),
\end{aligned}
$$

$$
\begin{aligned}
Q(x,y;f)=&-\sum_{i=1}^{N}\sum_{j=1}^{M}f_{(i,j)}(x,y)Q^{(i,j)}_{(a,c)\to(b,d)}(x,y)\\
&-\sum_{j=1}^{M}f_{(0,j)}(x,y)Q^{(0,j)}_{(a,c)\to(b,d)}(y)-\sum_{i=1}^{N}f_{(i,0)}(x,y)Q^{(i,0)}_{(a,c)\to(b,d)}(x),
\end{aligned}
$$

$$
\begin{aligned}
\hat{Q}^{(N,j)}(x,s,y)&=\begin{cases} Q^{(N,j)}_{(a,c)\to(s,d)}(s,y), & a\le s\le x,\\ -Q^{(N,j)}_{(s,c)\to(b,d)}(s,y), & x<s\le b,\end{cases}\\
\tilde{Q}^{(i,M)}(x,y,t)&=\begin{cases} Q^{(i,M)}_{(a,c)\to(b,t)}(x,t), & c\le t\le y,\\ -Q^{(i,M)}_{(a,t)\to(b,d)}(x,t), & y<t\le d,\end{cases}\quad\text{and}\\
\bar{Q}^{(N,M)}(x,s,y,t)&=\begin{cases} Q^{(N,M)}_{(a,c)\to(s,t)}(s,t), & a\le s\le x,\ c\le t\le y,\\ -Q^{(N,M)}_{(s,c)\to(b,t)}(s,t), & x<s\le b,\ c\le t\le y,\\ -Q^{(N,M)}_{(a,t)\to(s,d)}(s,t), & a\le s\le x,\ y<t\le d,\\ Q^{(N,M)}_{(s,t)\to(b,d)}(s,t), & x<s\le b,\ y<t\le d.\end{cases}
\end{aligned}
$$

Particularly, if $p(.,.)=1$ in identities (10), (11) and (14), then the expressions look like

$$
\begin{aligned}
&P_{a\to b}=b-a,\quad Q^{(i)}_{a\to b}(x)=\tfrac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!},\\
Q(x,y;f)=&-\sum_{i=1}^{N}\sum_{j=1}^{M}\tfrac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}\times\tfrac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}f_{(i,j)}(x,y)\\
&-(b-a)\sum_{j=1}^{M}\tfrac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}\;f_{(0,j)}(x,y)\\
&-(d-c)\sum_{i=1}^{N}\tfrac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}\;f_{(i,0)}(x,y),
\end{aligned}
$$

$$\hat{Q}^{(N,j)}(x,s,y) = \begin{cases} -\frac{(a-s)^{N+1}}{(N+1)!}\frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}, & a \le s \le x, \\ -\frac{(b-s)^{N+1}}{(N+1)!}\frac{(d-y)^{j+1}-(c-y)^{j+1}}{(j+1)!}, & x < s \le b, \end{cases}$$

$$\tilde{Q}^{(i,M)}(x,y,t) = \begin{cases} -\frac{(c-t)^{M+1}}{(M+1)!}\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}, & c \le t \le y, \\ -\frac{(d-t)^{M+1}}{(M+1)!}\frac{(b-x)^{i+1}-(a-x)^{i+1}}{(i+1)!}, & y < t \le d \end{cases} \quad \text{and}$$

$$\bar{Q}^{(N,M)}(x,s,y,t) = \begin{cases} \frac{(a-s)^{N+1}}{(N+1)!}\frac{(c-t)^{M+1}}{(M+1)!}, & a \le s \le x,\ c \le t \le y, \\ \frac{(b-s)^{N+1}}{(N+1)!}\frac{(c-t)^{M+1}}{(M+1)!}, & x < s \le b,\ c \le t \le y, \\ \frac{(a-s)^{N+1}}{(N+1)!}\frac{(d-t)^{M+1}}{(M+1)!}, & a \le s \le x,\ y < t \le d, \\ \frac{(b-s)^{N+1}}{(N+1)!}\frac{(d-t)^{M+1}}{(M+1)!}, & x < s \le b,\ y < t \le d. \quad \square \end{cases}$$

3. OSTROWSKI-TYPE INEQUALITIES FOR DOUBLE WEIGHTED INTEGRALS FOR HIGHER ORDER DIFFERENTIABLE FUNCTIONS

The following well known Ostrowski's inequality is extracted from [16].

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)M, \qquad x \in [a,b], \tag{19}$$

where $f : [a,b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \le M$ for every $x \in [a,b]$. This inequality undergoes many generalizations and in [20] Pečarić and Vukelić provided additional ones for two independent variables using identities (3) and (4). By using identities (10) and (11), we can give generalized results of Ostrowski-type for higher order differentiable functions of two independent variables as follows.

THEOREM 3.1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be such that $f \in C^{(N+1,M+1)}([a,b] \times [c,d])$. Then $\forall\,(x,y) \in [a,b] \times [c,d]$ we have*

$$\left| f(x,y) - \frac{1}{P(a,c)}\int_a^b \int_c^d p(s,t)f(s,t)\,\mathrm{d}t\,\mathrm{d}s \right| \le$$
$$\le D(x,y) + \sum_{j=0}^{M} \hat{D}^{(0,j)}(x,y) + \sum_{i=0}^{N} \tilde{D}^{(i,0)}(x,y) + \bar{D}(x,y),$$

where

$$D(x,y) = \frac{1}{|P(a,c)|}|R(x,y;f)|,$$
$$\hat{D}^{(0,j)}(x,y) = \frac{1}{|P(a,c)|}\left(\sum_{j=0}^{M} \int_a^b |\hat{P}^{(N,j)}(x,s,y)|^{\hat{q}_j}\,\mathrm{d}s \right)^{1/\hat{q}_j} \cdot \|f_{(N+1,j)}\|_{\hat{p}_j},$$

provided that

$$f_{(N+1,j)} \in L_{\hat{p}_j}([a,b]\times[c,d]),\ 1/\hat{p}_j + 1/\hat{q}_j = 1,$$

$$\tilde{D}^{(i,0)}(x,y) = \tfrac{1}{|P(a,c)|}\left(\sum_{i=0}^{N}\int_c^d |\tilde{P}^{(i,M)}(x,y,t)|^{\tilde{q}_i} dt\right)^{1/\tilde{q}_i} . \|f_{(i,M+1)}\|_{\tilde{p}_i},$$

provided that

$$f_{(i,M+1)} \in L_{\tilde{p}_i}([a,b]\times[c,d]),\ 1/\tilde{p}_i + 1/\tilde{q}_i = 1,$$

$$\bar{D}(x,y) = \tfrac{1}{|P(a,c)|}\left(\int_a^b\int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|^{\bar{q}} dt\, ds\right)^{1/\bar{q}} . \|f_{(N+1,M+1)}\|_{\bar{p}},$$

provided that

$$f_{(N+1,M+1)} \in L_{\bar{p}}([a,b]\times[c,d]), \quad 1/\bar{p} + 1/\bar{q} = 1,$$

where $\hat{P}^{(N,j)}(x,s,y)$ $\tilde{P}^{(i,M)}(x,y,t)$, $\bar{P}^{(N,M)}(x,s,y,t)$ *are as in Theorem* 2.2 *whereas* $P(a,c)$ *and* $R(x,y,f)$ *are defined in* (2) *and* (8) *respectively.*

Proof. Identity (10) may be written as

$$\begin{aligned} f(x,y) - \tfrac{1}{P(a,c)}\int_a^b\int_c^d p(s,t)f(s,t)\,\mathrm{d}t\,\mathrm{d}s &= \\ = \tfrac{1}{P(a,c)}\Bigg[R(x,y;f) + \sum_{j=0}^{M}\int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s \\ + \sum_{i=0}^{N}\int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,\mathrm{d}t \\ - \int_a^b\int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s\Bigg]. \end{aligned}$$

Now, taking absolute value and applying Hölder's inequality for double integrals, we easily obtain our required inequality. □

REMARK 3.2. For $N = M = 0$, Theorem 4 of [20] becomes special case of Theorem 3.1 and we also retrieve results of [9] by simply putting $p(.,.) = 1$. □

THEOREM 3.3. *Let* $f : [a,b]\times[c,d] \to \mathbb{R}$ *be a continuous function on* $[a,b]\times[c,d]$ *such that* $f \in C^{(N+1,M+1)}((a,b)\times(c,d))$ *and* $|f_{(N+1,M+1)}|^p$ *be an integrable function, i.e.,*

$$\left\|f_{(N+1,M+1)}\right\|_p := \left(\int_a^b\int_c^d |f_{(N+1,M+1)}(s,t)|^p\,\mathrm{d}t\,\mathrm{d}s\right)^{1/p} < \infty,$$

$1/p + 1/q = 1$. *Then, it follows that*

$$\begin{aligned}
&\Bigg| \int_a^b \int_c^d p(s,t) f(x,t)\, \mathrm{d}t\, \mathrm{d}s - \Bigg[R(x,y;f) \\
&\quad + \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta) \tfrac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\, \mathrm{d}\eta\, \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t) \tfrac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\, \mathrm{d}t\, \mathrm{d}\xi \\
&\quad + \int_a^b \int_c^d p(s,t) f(x,t)\, \mathrm{d}t\, \mathrm{d}s \\
&\quad + \int_a^b \int_c^d p(s,t) f(s,y) \mathrm{d}t\, \mathrm{d}s - f(x,y) P(a,c) \Bigg] \Bigg| \leq \\
&\leq \left(\int_a^b \int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|\, \mathrm{d}t\, \mathrm{d}s \right)^{1/q} \| f_{(N+1,M+1)} \|_p .
\end{aligned}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Proof. Identity (11) may be written as

$$\begin{aligned}
&\int_a^b \int_c^d p(s,t) f(s,t)\, \mathrm{d}t\, \mathrm{d}s - \Bigg[R(x,y;f) \\
&\quad + \int_a^b \int_c^d p(s,t) f(s,y)\, \mathrm{d}t\, \mathrm{d}s + \int_a^b \int_c^d p(s,t) f(x,t) \mathrm{d}t\, \mathrm{d}s \\
&\quad + \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta) \tfrac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\, \mathrm{d}\eta\, \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t) \tfrac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\, \mathrm{d}t\, \mathrm{d}\xi - f(x,y) P(a,c) \Bigg] = \\
&= \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\, \mathrm{d}t\, \mathrm{d}s.
\end{aligned}$$

Now taking absolute value and applying Hölder's inequality for double integrals, we easily obtain our required inequality. □

REMARK 3.4. For $N = M = 0$, Theorem 5 of [20] becomes special case of Theorem 3.3 and we also retrieve results of [3] and [8] by simply putting $p(.,.) = 1$. □

4. GRÜSS-TYPE INEQUALITIES FOR DOUBLE WEIGHTED INTEGRALS FOR HIGHER ORDER DIFFERENTIABLE FUNCTIONS

A celebrated integral inequality proved by Grüss [11] in 1935, can be stated as follows (see [15, p. 296]),

$$\left|\tfrac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x-\left(\tfrac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(\tfrac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right)\right|\le$$
$$\le\tfrac{1}{4}(M-m)(N-n)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the conditions

$$m\le f(x)\le M,\quad n\le g(x)\le N,$$

for all $x\in[a,b]$, where m,M,n,N are real constants.

In [20] Pečarić and Vukelić gave new Grüss-type inequalities for double weighted integrals by using identities (3) and (4). Now, we give more generalized results by using higher order differentiable functions of two independent variables but in order to simplify the details of the presentations we define the following notations.

$$A^{(i,j)}(x,y)=p(x,y)[f_{(i,j)}(x,y)g(x,y)+g_{(i,j)}(x,y)f(x,y)]\times \times P^{(i,j)}_{(a,c)\to(b,d)}(x,y), \tag{20}$$

$$A(x,y)=p(x,y)\int_a^b\int_c^d p(s,t)[f(s,t)g(x,y)+g(s,t)f(x,y)]\,\mathrm{d}t\,\mathrm{d}s, \tag{21}$$

$$\hat{A}^{(N,j)}(x,y)=p(x,y)\int_a^b[f_{(N+1,j)}(s,y)g(x,y)+g_{(N+1,j)}(s,y)f(x,y)]\times \times\hat{P}^{(N,j)}(x,s,y)\,\mathrm{d}s, \tag{22}$$

$$\tilde{A}^{(i,M)}(x,y)=p(x,y)\int_c^d[f_{(i,M+1)}(x,t)g(x,y)+g_{(i,M+1)}(x,t)f(x,y)]\times \times\tilde{P}^{(i,M)}(x,y,t)\,\mathrm{d}t, \tag{23}$$

$$\bar{A}^{(N,M)}(x,y)=p(x,y)\int_a^b\int_c^d[f_{(N+1,M+1)}(s,t)g(x,y) + g_{(N+1,M+1)}(s,t)f(x,y)]\bar{P}^{(N,M)}(x,s,y,t)\,\mathrm{d}t\,\mathrm{d}s, \tag{24}$$

$$B^{(i,j)}(x,y)=|p(x,y)g(x,y)|\,\|f_{(i,j)}(x,y)\|_\infty+|p(x,y)f(x,y)|\times \times\|g_{(i,j)}(x,y)\|_\infty, \tag{25}$$

$$C^{(i,j)}(x,y) = \tfrac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!}\ \tfrac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!} \times \times \int_a^b \int_c^d |p(\xi,\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{26}$$

$$C^{(0,j)}(y) = (b-a)\tfrac{(\max\{d-y,y-c\})^{j+1}}{(j+1)!} \int_a^b \int_c^d |p(\xi,\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{27}$$

$$C^{(i,0)}(x) = (d-c)\tfrac{(\max\{b-x,x-a\})^{i+1}}{(i+1)!} \int_a^b \int_c^d |p(\xi,\eta)|\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{28}$$

$$\hat{C}^{(N,j)}(x,y) = \int_a^b |\hat{P}^{(N,j)}(x,s,y)|\,\mathrm{d}s, \tag{29}$$

$$\tilde{C}^{(i,M)}(x,y) = \int_c^d |\tilde{P}^{(i,M)}(x,y,t)|\,\mathrm{d}t, \tag{30}$$

$$\bar{C}^{(N,M)}(x,y) = \int_a^b \int_c^d |\bar{P}^{(N,M)}(x,s,y,t)|\,\mathrm{d}t\,\mathrm{d}s, \tag{31}$$

$$\begin{aligned} F(x,y) = R(x,y;f) &+ \int_a^b \int_c^d p(s,t)f(s,y)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \int_a^b \int_c^d p(s,t)f(x,t)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\tfrac{(\eta-y)^j}{j!} f_{(0,j)}(s,y)\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\tfrac{(\xi-x)^i}{i!} f_{(i,0)}(x,t)\,\mathrm{d}t\,\mathrm{d}\xi, \end{aligned} \tag{32}$$

$$\begin{aligned} G(x,y) = R(x,y;g) &+ \int_a^b \int_c^d p(s,t)g(s,y)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \int_a^b \int_c^d p(s,t)g(x,t)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \sum_{j=1}^{M} \int_a^b \int_c^d p(s,\eta)\tfrac{(\eta-y)^j}{j!} g_{(0,j)}(s,y)\,\mathrm{d}\eta\,\mathrm{d}s \\ &+ \sum_{i=1}^{N} \int_a^b \int_c^d p(\xi,t)\tfrac{(\xi-x)^i}{i!} g_{(i,0)}(x,t)\,\mathrm{d}t\,\mathrm{d}\xi, \end{aligned} \tag{33}$$

where $\hat{P}^{(N,j)}(x,s,y)$ $\tilde{P}^{(i,M)}(x,y,t)$, $\bar{P}^{(N,M)}(x,s,y,t)$ are as in Theorem 2.2 whereas $P(a,c)$ and $R(x,y,f)$ are defined in (2) and (8) respectively.

Now, we are ready to present our main results of this section by using notations defined above, which are as follows.

Theorem 4.1. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ be two functions such that $f,\, g \in C^{(N+1,M+1)}([a,b] \times [c,d])$. Then*

$$\begin{aligned}&\Bigg|\tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)f(x,y)g(x,y)\,\mathrm{d}y\,\mathrm{d}x\\&\quad - \left(\tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)\\&\quad \times \left(\tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y)g(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)\Bigg| \le\\&\le \tfrac{1}{2[P(a,c)]^2} \int_a^b \int_c^d \Bigg[\sum_{i=1}^{N}\sum_{j=1}^{M} B^{(i,j)}(x,y)C^{(i,j)}(x,y)\\&\quad + \sum_{j=1}^{M} B^{(0,j)}(y)C^{(0,j)}(y) + \sum_{i=1}^{N} B^{(i,0)}(x)C^{(i,0)}(x)\\&\quad + B^{(N+1,j)}(x,y)\hat{C}^{(N,j)}(x,y) + B^{(i,M+1)}(x,y)\tilde{C}^{(i,M)}(x,y)\\&\quad + B^{(N+1,M+1)}(x,y)\bar{C}^{(N,M)}(x,y)\Bigg]\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Proof. From (10), we have the following identities:

$$\begin{aligned}f(x,y)P(a,c) = R(x,y;f) &+ \int_a^b \int_c^d p(s,t)f(s,t)\mathrm{d}t\,\mathrm{d}s\\&+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y)f_{(N+1,j)}(s,y)\,\mathrm{d}s\\&+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t)f_{(i,M+1)}(x,t)\,\mathrm{d}t\\&- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t)f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s\end{aligned} \tag{34}$$

$$\begin{aligned}g(x,y)P(a,c) = R(x,y;g) &+ \int_a^b \int_c^d p(s,t)g(s,t)\mathrm{d}t\,\mathrm{d}s\\&+ \sum_{j=0}^{M} \int_a^b \hat{P}^{(N,j)}(x,s,y)g_{(N+1,j)}(s,y)\,\mathrm{d}s\\&+ \sum_{i=0}^{N} \int_c^d \tilde{P}^{(i,M)}(x,y,t)g_{(i,M+1)}(x,t)\,\mathrm{d}t\\&- \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t)g_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s\end{aligned} \tag{35}$$

for $(x,y) \in [a,b]\times[c,d]$. Multiplying (34) by $p(x,y)g(x,y)$ and (35) by $p(x,y)f(x,y)$ and adding the resulting identities, we obtain

$$(36)\quad 2P(a,c)p(x,y)f(x,y)g(x,y) = -\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) - \sum_{i=1}^{N} A^{(i,0)}(x) + A(x,y) + \hat{A}^{(N,j)}(x,y) + \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y)$$

Integrating (36) over $[a,b]\times[c,d]$, we get

$$\int_a^b\int_c^d p(x,y)f(x,y)g(x,y)\,\mathrm{d}y\,\mathrm{d}x = \tfrac{1}{2P(a,c)}\int_a^b\int_c^d\Big[-\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) - \sum_{i=1}^{N} A^{(i,0)}(x + A(x,y) + \hat{A}^{(N,j)}(x,y) + \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y)\Big]\,\mathrm{d}y\,\mathrm{d}x$$

It may be written as

$$(37)\quad \tfrac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)g(x,y)\,\mathrm{d}y\,\mathrm{d}x - \left(\tfrac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right) \times \left(\tfrac{1}{P(a,c)}\int_a^b\int_c^d p(x,y)g(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)$$
$$= \tfrac{1}{2[P(a,c)]^2}\int_a^b\int_c^d\Big[-\sum_{i=1}^{N}\sum_{j=1}^{M} A^{(i,j)}(x,y) - \sum_{j=1}^{M} A^{(0,j)}(y) - \sum_{i=1}^{N} A^{(i,0)}(x) + \hat{A}^{(N,j)}(x,y) + \tilde{A}^{(i,M)}(x,y) - \bar{A}^{(N,M)}(x,y)\Big]\,\mathrm{d}y\,\mathrm{d}x.$$

Using (20),…,(31) we have the following inequalities

$$|A^{(i,j)}(x,y)| \le B^{(i,j)}(x,y)\; C^{(i,j)}(x,y),$$
$$|A^{(0,j)}(y)| \le B^{(0,j)}(y)\; C^{(0,j)}(y),$$
$$|A^{(i,0)}(x)| \le B^{(i,0)}(x)\; C^{(i,0)}(x),$$
$$|\hat{A}^{(N,j)}(x,y)| \le B^{(N+1,j)}(x,y)\; \hat{C}^{(N,j)}(x,y),$$
$$|\tilde{A}^{(i,M)}(x,y)| \le B^{(i,M+1)}(x,y)\; \tilde{C}^{(i,M)}(x,y),$$
$$|\bar{A}^{(N,M)}(x,y)| \le B^{(N+1,M+1)}(x,y)\; \bar{C}^{(N,M)}(x,y),$$

$\forall (x,y) \in [a,b]\times[c,d]$. Taking absolute value on both sides in (37) and using all these inequalities in it, we get our required result. $\square$

THEOREM 4.2. *Let $f,g:[a,b]\times[c,d]\to\mathbb{R}$ be two continuous functions on $[a,b]\times[c,d]$, such that $f,\ g \in C^{(N+1,M+1)}([a,b]\times[c,d])$. Then*

$$\Bigg| \tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y) g(x,y)\,\mathrm{d}y\,\mathrm{d}x + \left(\tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)\left(\tfrac{1}{P(a,c)} \int_a^b \int_c^d p(x,y) g(x,y)\,\mathrm{d}y\,\mathrm{d}x\right) - \tfrac{1}{2[P(a,c)]^2} \int_a^b \int_c^d p(x,y)[g(x,y)F(x,y)+f(x,y)G(x,y)]\,\mathrm{d}y\,\mathrm{d}x \Bigg| \le$$

$$\le \tfrac{1}{2[P(a,c)]^2} \int_a^b \int_c^d B^{(N+1,M+1)}(x,y)\ \bar{C}^{(N,M)}(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

Proof. From (11), we have the following identities:

$$f(x,y)P(a,c) = F(x,y) - \int_a^b \int_c^d p(s,t) f(s,t)\mathrm{d}t\,\mathrm{d}s + \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) f_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s, \tag{38}$$

$$g(x,y)P(a,c) = G(x,y) - \int_a^b \int_c^d p(s,t) g(s,t)\mathrm{d}t\,\mathrm{d}s + \int_a^b \int_c^d \bar{P}^{(N,M)}(x,s,y,t) g_{(N+1,M+1)}(s,t)\,\mathrm{d}t\,\mathrm{d}s, \tag{39}$$

for $(x,y) \in [a,b]\times[c,d]$. Multiplying (38) by $p(x,y)g(x,y)$ and (39) by $p(x,y)f(x,y)$ and adding the resulting identities, we obtain

$$2P(a,c)p(x,y)f(x,y)g(x,y) = p(x,y)g(x,y)F(x,y) + p(x,y)f(x,y)G(x,y) - A(x,y) + \bar{A}^{(N,M)}(x,y) \tag{40}$$

Integrating (40) over $[a,b]\times[c,d]$, we get

$$\int_a^b \int_c^d p(x,y) f(x,y) g(x,y)\,\mathrm{d}y\,\mathrm{d}x = \tfrac{1}{2P(a,c)} \int_a^b \int_c^d p(x,y)[g(x,y)F(x,y)+f(x,y)G(x,y)]\,\mathrm{d}y\,\mathrm{d}x - \tfrac{1}{P(a,c)}\left(\int_a^b \int_c^d p(x,y) f(x,y)\,\mathrm{d}y\,\mathrm{d}x\right)\left(\int_a^b \int_c^d p(x,y) g(x,y)\,\mathrm{d}y\,\mathrm{d}x\right) + \tfrac{1}{2P(a,c)} \int_a^b \int_c^d \bar{A}^{(N,M)}(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{41}$$

$$|\bar{A}^{(N,M)}(x,y)| \le B^{(N+1,M+1)}(x,y)\ \bar{C}^{(N,M)}(x,y) \tag{42}$$

From (41) and (42), we obtain our required inequality. □

Remark 4.3. For $N = M = 0$ Theorems 6 and 7 of [20] become special cases of Theorems 4.1 and 4.2 respectively and we also retrieve results of [16] by simply putting $p(.,.) = 1$. For $N = M = 0$, we can also find similar results in [12]. □

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