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SCHUR CONVEXITY PROPERTIES OF THE WEIGHTED ARITHMETIC INTEGRAL MEAN AND CHEBYSHEV FUNCTIONAL*

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Abstract. In this paper, we discuss the Schur convexity, Schur geometrical convexity and Schur harmonic convexity of the weighted arithmetic integral mean and Chebyshev functional. Several sufficient conditions, and necessary and sufficient conditions are established.

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Keywords. Weighted arithmetic integral mean, Weighted Chebyshev functional, Schur convex, Schur geometrically convex, Schur harmonic convex.

1. INTRODUCTION

Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space $(n \geq 2)$ and $\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}$. In particular, we use $\mathbb{R}$ to denote $\mathbb{R}^1$.

For the sake of convenience, we use the following notation system.

For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n_+$ and $\alpha \in \mathbb{R}_+$, we denote by

$$\begin{aligned} x \pm y &= (x_1 \pm y_1, x_2 \pm y_2, \cdots, x_n \pm y_n), \\ xy &= (x_1 y_1, x_2 y_2, \cdots, x_n y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), \\ x^\alpha &= (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \\ \log x &= (\log x_1, \log x_2, \cdots, \log x_n) \end{aligned}$$

and

$$\tfrac{1}{x} = \left(\tfrac{1}{x_1}, \tfrac{1}{x_2}, \cdots, \tfrac{1}{x_n}\right).$$

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Definition 1. *A real-valued function F on $E \subseteq \mathbb{R}^n$ is said to be Schur convex if*

$$F(x_1, x_2, \cdots, x_n) \leq F(y_1, y_2, \cdots, y_n) \tag{1}$$

for each pair of n-tuples $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in E$, such that $x \prec y$, i.e.

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component of x. F is called Schur concave if $-F$ is Schur convex.

Definition 2. *A real-valued function F on $E \subseteq \mathbb{R}^n_+$ is said to be Schur geometrically convex if* (1) *holds for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in E$, such that $\log x \prec \log y$. F is called Schur geometrically concave if $\frac{1}{F}$ is Schur geometrically convex.*

Definition 3. *A real-valued function F on $E \subseteq \mathbb{R}^n_+$ is said to be Schur harmonic convex if* (1) *holds for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in E$, such that $\frac{1}{x} \prec \frac{1}{y}$. F is called Schur harmonic concave on E if inequality* (1) *is reversed.*

The Schur convexity was introduced by I. Schur [1] in 1923 and it has many important applications in analytic inequalities [2-7], extended mean values [8-11] and other related fields. Recently, the Schur geometrical and harmonic convexities were investigated in [4, 5, 12, 13, 14].

Lemma 4. (see [2]). *Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior* int$E$ *and $f : E \to \mathbb{R}$ be a continuous symmetric function. If f is differentiable on* int$E$*, then f is Schur convex on E if and only if*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2}$$

for all $x = (x_1, \cdots, x_n) \in$ int$E$. f is Schur concave on E if and only if inequality (2) *is reversed. Here E is a symmetric set means that $x \in E$ implies $Px \in E$ for any $n \times n$ permutation matrix P.*

Lemma 5. (see [13]). *Let $E \subseteq \mathbb{R}^n_+$ be a symmetric geometrically convex set with nonempty interior* int$E$ *and $f : E \to \mathbb{R}_+$ be a continuous symmetric function. If f is differentiable on* int$E$*, then f is Schur geometrically convex on E if and only if*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0$$

for all $x = (x_1, x_2, \cdots, x_n) \in intE$*. Here* E *is a geometrically convex set means that* $x^{\frac{1}{2}} y^{\frac{1}{2}} \in E$ *whenever* $x, y \in E$*.*

Lemma 6. (see [14]). *Let* $E \subseteq \mathbb{R}_+^n$ *be a symmetric harmonic convex set with nonempty interior* $\mathrm{int}E$ *and* $f : E \to \mathbb{R}_+$ *be a continuous symmetric function. If* f *is differentiable on* $\mathrm{int}E$*, then* f *is Schur harmonic convex on* E *if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial f}{\partial x_1} - x_2^2 \frac{\partial f}{\partial x_1}\right) \geq 0$$

for all $x = (x_1, x_2, \cdots, x_n) \in \mathrm{int}E$*. Here* E *is a harmonic convex set means that* $\frac{2xy}{x+y} \in E$ *whenever* $x, y \in E$*.*

Definition 7. *Let* f *be a continuous function on* $I \subseteq \mathbb{R}$ *and* p *be a positive continuous weight on* I*. Then the well-known weighted arithmetic integral mean* $F_p(x, y)$ *is defined by*

$$(3)\qquad F_p(x, y) = \begin{cases} \frac{1}{\int_x^y p(t)\mathrm{d}t} \displaystyle\int_x^y p(t) f(t)\mathrm{d}t, & x, y \in I, x \neq y, \\ f(x), & x = y. \end{cases}$$

If $p(t) = 1$*, then* (3) *reduces to the arithmetic integral mean*

$$(4)\qquad F(x, y) = \begin{cases} \frac{1}{y-x} \displaystyle\int_x^y f(t)\mathrm{d}t, & x, y \in I, x \neq y, \\ f(x), & x = y. \end{cases}$$

Recently, the weighted arithmetic integral mean $F_p(x, y)$ has been the subject of intensive research [8, 15]. In particular, the following Theorems 8-10 can be found in the literature [16-18].

Theorem 8. *Let* f *be a continuous function on* I*, let* p *be a positive continuous weight on* I*. Then the weighted arithmetic integral mean* $F_p(x, y)$ *is Schur convex (concave) on* I^2 *if and only if the inequality*

$$(5)\qquad \frac{1}{\int_x^y p(t)\mathrm{d}t} \int_x^y p(t) f(t)\mathrm{d}t \leq \frac{p(x)f(x) + p(y)f(y)}{p(x) + p(y)}$$

holds (reverses) for all $x, y \in I$*.*

Theorem 9. *Let* f *be a continuous function on* I*. Then the arithmetic integral mean* $F(x, y)$ *defined as in* (4) *is Schur convex (concave) on* I^2 *if and only if* f *is convex (concave) on* I*.*

Theorem 10. *Let* f *be a second order differentiable function on* I *with* $3f'(x) + xf''(x) \geq 0 (\leq 0)$*. Then the arithmetic integral mean* $F(x, y)$ *is Schur geometrically convex (concave) on* I^2*.*

Definition 11. *The weighted Chebyshev functional* $T(f, g, p)$ *is defined by*

$$(6)\qquad \begin{aligned} T(f, g, p) &= T(f, g, p; x, y) \\ &= \frac{\int_x^y p(t)f(t)g(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} - \frac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} \cdot \frac{\int_x^y p(t)g(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}, \end{aligned}$$

where $p(t)$ is a positive Lebesgue measurable function on $[x,y]$ such that $0 < \int_x^y p(t)\mathrm{d}t < +\infty$, f, $g : [x,y] \to \mathbb{R}$ are two Lebesgue measurable functions on $[x,y]$ and the integrals in (6) *are assumed to exist.*

If $p(t) = 1$, then (6) *becomes the Chebyshev functional*

$$T(f,g) = T(f,g;x,y) = \frac{\int_x^y f(t)g(t)\mathrm{d}t}{y-x} - \frac{\int_x^y f(t)\mathrm{d}t}{y-x} \cdot \frac{\int_x^y g(t)\mathrm{d}t}{y-x}. \tag{7}$$

The weighted Chebyshev functional has a long history and an extensive repertoire of applications in many fields in including numerical quadrature, transform theory, probability and statistical problems, and special functions [19-24]. Čuljak [25] proved that

Theorem 12. *If f and g are monotonic in the same sense (in the opposite sense) on I, then the Chebyshev functional $T(f,g)$ is Schur convex (concave) on I^2.*

Theorem 13. *Let p be a positive continuous weight on I. Then the weighted Chebyshev functional $T(f,g,p)$ is Schur convex (concave) on I^2 if and only if the inequality*

$$T(f,g,p;x,y) \le \frac{p(x)\big(f(x)-\overline{f_p}(x,y)\big)\big(g(x)-\overline{g_p}(x,y)\big)+p(y)\big(f(y)-\overline{f_p}(x,y)\big)\big(g(y)-\overline{g_p}(x,y)\big)}{p(x)+p(y)}$$

holds (reverses) for all $x,y \in I$. Here,

$$\begin{aligned} P(x,y) &= \int_x^y p(t)\mathrm{d}t, \\ \overline{f_p} = \overline{f_p}(x,y) &= \tfrac{1}{P(x,y)} \int_x^y p(t)f(t)\mathrm{d}t, \\ \overline{g_p} = \overline{g_p}(x,y) &= \tfrac{1}{P(x,y)} \int_x^y p(t)g(t)\mathrm{d}t. \end{aligned} \tag{8}$$

Our main purpose of this paper is to discuss the Schur convexity, Schur geometrical convexity and Schur harmonic convexity of the weighted arithmetic integral mean and Chebyshev functional.

2. SCHUR CONVEXITY OF THE WEIGHTED ARITHMETIC INTEGRAL MEAN

Theorem 14. *Let f be a continuous function on I and p be a positive continuous weight on I. Then the weighted arithmetic integral mean $F_p(x,y)$ defined by* (3) *on I^2 is* (1) *Schur geometrically convex if and only if*

$$\overline{f_p}(x,y) \le \tfrac{xp(x)f(x)+yp(y)f(y)}{xp(x)+yp(y)} \tag{9}$$

for all $x,y \in I$; (2) *Schur harmonic convex if and only if*

$$\overline{f_p}(x,y) \le \tfrac{x^2p(x)f(x)+y^2p(y)f(y)}{x^2p(x)+y^2p(y)} \tag{10}$$

for all $x,y \in I$.

Proof. From (3) and (8) one has

$$\frac{\partial F_p(x,y)}{\partial x} = \frac{-p(x)f(x)P(x,y)+p(x)\int_x^y p(t)f(t)\mathrm{d}t}{P^2(x,y)},$$
$$\frac{\partial F_p(x,y)}{\partial y} = \frac{-p(y)f(y)P(x,y)+p(y)\int_x^y p(t)f(t)\mathrm{d}t}{P^2(x,y)}.$$

(1) Simple computations lead to

$$(\log y - \log x)\left(y\frac{\partial F_p(x,y)}{\partial y} - x\frac{\partial F_p(x,y)}{\partial x}\right) =$$
$$= \frac{\log y-\log x}{P(x,y)}\left[xp(x)f(x) + yp(y)f(y) - xp(x)\overline{f_p}(x,y) - yp(y)\overline{f_p}(x,y)\right].$$

We clearly see that $\frac{\log y-\log x}{P(x,y)} \geq 0$, then by Lemma 5 we know that $F_p(x,y)$ is Schur geometrically convex if and only if

$$xp(x)f(x) + yp(y)f(y) - xp(x)\overline{f_p}(x,y) - yp(y)\overline{f_p}(x,y) \geq 0$$

for all $x, y \in I$. That is to say for all $x, y \in I$,

$$\overline{f_p}(x,y) \leq \frac{xp(x)f(x)+yp(y)f(y)}{xp(x)+yp(y)}.$$

(2) Direct computations yield

$$(y-x)\left(y^2\frac{\partial F_p(x,y)}{\partial y} - x^2\frac{\partial F_p(x,y)}{\partial x}\right) =$$
$$= \frac{y-x}{P(x,y)}[x^2p(x)f(x) + y^2p(y)f(y) - x^2p(x)\overline{f_p}(x,y) - y^2p(y)\overline{f_p}(x,y)].$$

Since $\frac{y-x}{P(x,y)} \geq 0$, by Lemma 6 we know that $F_p(x,y)$ is Schur harmonic convex if and only if for all $x, y \in I$ it holds

$$x^2p(x)f(x) + y^2p(y)f(y) - x^2p(x)\overline{f_p}(x,y) - y^2p(y)\overline{f_p}(x,y) \geq 0,$$

i.e.

$$\overline{f_p}(x,y) \leq \frac{x^2p(x)f(x)+y^2p(y)f(y)}{x^2p(x)+y^2p(y)}$$

for all $x, y \in I$. □

Let

$$\overline{f} = \overline{f}(x,y) = \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t, \quad \overline{g} = \overline{g}(x,y) = \frac{1}{y-x}\int_x^y g(t)\mathrm{d}t, \tag{11}$$

and take $p(t) = 1$. Then Theorem 14 leads to the following Corollary 15.

Corollary 15. *Let f be a continuous function on I, then the arithmetic integral mean $F(x,y)$ on I^2 is*

(1) *Schur geometrically convex if and only if*

$$\overline{f}(x,y) \leq \frac{xf(x)+yf(y)}{x+y} \tag{12}$$

for all $x, y \in I$;

(2) *Schur harmonic convex if and only if*

$$\overline{f}(x,y) \leq \frac{x^2f(x)+y^2f(y)}{x^2+y^2} \tag{13}$$

for all $x, y \in I$.

Theorem 16. *Let p be a positive continuous weight on I, f be a differentiable function on I with $f'(y) \geq \frac{p(y)}{P(x,y)} \cdot \frac{f(y)-f(x)}{y-x}$ for any $x, y \in I$, $F_p(x,y)$ be the weighted arithmetic integral mean. Then the following statements are true.*

(1) If f and p have the same monotonicity on I, then $F_p(x,y)$ is Schur convex on I^2;

(2) If $f(t)$ and $tp(t)$ have the same monotonicity on I, then $F_p(x,y)$ is Schur geometrically convex on I^2;

(3) If $f(t)$ and $t^2p(t)$ have the same monotonicity on I, then $F_p(x,y)$ is Schur harmonic convex on I^2.

Proof. For any $x, y \in I$, let

$$G(x,y) = 2\int_x^y p(t)f(t)\mathrm{d}t - [f(x)+f(y)]\int_x^y p(t)\mathrm{d}t.$$

Then

$$G(x,x) = 0, \tag{14}$$

$$\frac{\partial G(x,y)}{\partial y} = p(y)[f(y)-f(x)] - f'(y)\int_x^y p(t)\mathrm{d}t. \tag{15}$$

If

$$f'(y) \geq \frac{p(y)}{P(x,y)} \cdot \frac{f(y)-f(x)}{y-x}, \tag{16}$$

then equation (15) and inequality (16) lead to the conclusion that

$$\frac{\partial G(x,y)}{\partial y} \leq 0. \tag{17}$$

Equation (14) and inequality (17) imply

$$G(x,y) \leq 0. \tag{18}$$

Inequality (18) leads to

$$\overline{f_p}(x,y) \leq \frac{f(x)+f(y)}{2}. \tag{19}$$

(1) If f and p have the same monotonicity on I, then

$$(f(y)-f(x))[p(y)-p(x)] \geq 0$$

for any $x, y \in I$. It follows that

$$\frac{f(x)+f(y)}{2} \leq \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)}. \tag{20}$$

From inequalities (19) and (20) together with Theorem 8 we clearly see that $F_p(x,y)$ is Schur convex on I^2.

(2) If $f(t)$ and $tp(t)$ have the same monotonicity on I, then

$$(f(y)-f(x))[yp(y)-xp(x)] \geq 0$$

for any $x, y \in I$. So, we have

$$\frac{f(x)+f(y)}{2} \leq \frac{xp(x)f(x)+yp(y)f(y)}{xp(x)+yp(y)}. \tag{21}$$

From inequalities (19) and (21) together with Theorem 14(1) we known that $F_p(x,y)$ is Schur geometrically convex on I^2.

(3) If $f(t)$ and $t^2p(t)$ have the same monotonicity on I, then

$$(f(y)-f(x))[y^2p(y)-x^2p(x)]\geq 0$$

for any $x,y\in I$. Hence, we get

$$\frac{f(x)+f(y)}{2}\leq\frac{x^2p(x)f(x)+y^2p(y)f(y)}{x^2p(x)+y^2p(y)} \tag{22}$$

for any $x,y\in I$.

It follows from inequalities (19) and (22) together with Theorem 14(2) that $F_p(x,y)$ is Schur harmonic convex on I^2. □

Theorem 17. *Let f be a continuous, increasing (decreasing) and convex (concave) function on I. Then the arithmetic integral mean $F(x,y)$ is Schur geometrically and harmonic convex (concave) on I^2.*

Proof. If f is convex on I, then by the well known Hermite-Hadamard inequality we have

$$\overline{f}(x,y)=\frac{1}{y-x}\int_x^y f(t)\mathrm{d}t\leq\frac{f(x)+f(y)}{2}. \tag{23}$$

If f is increasing on I, then

$$(f(y)-f(x))(y-x)\geq 0$$

for any $x,y\in I$. Therefore,

$$\frac{f(x)+f(y)}{2}\leq\frac{xf(x)+yf(y)}{x+y}. \tag{24}$$

From inequalities (23) and (24) together with Corollary 15(1) we clearly see that $F(x,y)$ is Schur geometrically convex on I^2.

If f is increasing on I, then

$$(f(y)-f(x))(y^2-x^2)\geq 0$$

for any $x,y\in I$. Hence,

$$\frac{f(x)+f(y)}{2}\leq\frac{x^2f(x)+y^2f(y)}{x^2+y^2}. \tag{25}$$

From inequalities (23) and (25) together with Corollary 15(2) we known that $F(x,y)$ is Schur harmonic convex on I^2. □

3. SCHUR CONVEXITY OF THE WEIGHTED CHEBYSHEV FUNCTIONAL

It is not difficult to verify that the weighted Chebyshev functional $T(f,g,p)$ satisfy the identity:

$$T(f,g,p;x,y)=\frac{1}{\int_x^y p(t)\mathrm{d}t}\int_x^y p(t)(f(t)-\overline{f_p}(x,y))(g(t)-\overline{g_p}(x,y))\mathrm{d}t. \tag{26}$$

For fixed x and y with $x\neq y$, let

$$G_p(t)=(f(t)-\overline{f_p}(x,y))(g(t)-\overline{g_p}(x,y)) \tag{27}$$

for all $t \in [x, y]$. Then (26) can be rewritten as

$$T(f,g,p,x,y) = \frac{1}{\int_x^y p(t)dt} \int_x^y p(t) G_p(t) \mathrm{d}t. \tag{28}$$

Equation (28) shows that the weighted Chebyshev functional $T(f, g, p)$ can be expressed by the weighted arithmetic integral mean of $G_p(t)$.

If $p(t) = 1$, then (27) and (28) can be rewritten as

$$G(t) = (f(t) - \overline{f}(x,y))(g(t) - \overline{g}(x,y)), \tag{29}$$

$$T(f,g) = T(f,g;x,y) = \frac{1}{y-x} \int_x^y G(t) \mathrm{d}t, \tag{30}$$

respectively. From (27), (28) and Theorem 14, we have

THEOREM 18. *Let p be a positive continuous weight on I. Then the weighted Chebyshev functional $T(f, g, p)$ defined as in (6) is*

(1) *Schur geometrically convex on I^2 if and only if the inequality*

$$T(f,g,p;x,y) \le \frac{xp(x)(f(x)-\overline{f_p}(x,y))(g(x)-\overline{g_p}(x,y))+yp(y)(f(y)-\overline{f_p}(x,y))(g(y)-\overline{g_p}(x,y))}{xp(x)+yp(y)}$$

holds for all $x, y \in I$;

(2) *Schur harmonic convex on I^2 if and only if the inequality*

$$T(f,g,p;x,y) \le \frac{x^2p(x)(f(x)-\overline{f_p}(x,y))(g(x)-\overline{g_p}(x,y))+y^2p(y)(f(y)-\overline{f_p}(x,y))(g(y)-\overline{g_p}(x,y))}{x^2p(x)+y^2p(y)}$$

holds for all $x, y \in I$.

From Theorems 13 and 18, we get the following Corollary 19.

COROLLARY 19. *The Chebyshev functional $T(f, g)$ defined as in (7) is*

(1) *Schur convex on I^2 if and only if*

$$T(f,g;x,y) \le \frac{(f(x)-\overline{f}(x,y))(g(x)-\overline{g}(x,y))+(f(y)-\overline{f}(x,y))(g(y)-\overline{g}(x,y))}{2}$$

for all $x, y \in I$;

(2) *Schur geometrically convex on I^2 if and only if*

$$T(f,g;x,y) \le \frac{x(f(x)-\overline{f}(x,y))(g(x)-\overline{g}(x,y))+y(f(y)-\overline{f}(x,y))(g(y)-\overline{g}(x,y))}{x+y}$$

for all $x, y \in I$;

(3) *Schur harmonic convex on I^2 if and only if*

$$T(f,g;x,y) \le \frac{x^2(f(x)-\overline{f}(x,y))(g(x)-\overline{g}(x,y))+y^2(f(y)-\overline{f}(x,y))(g(y)-\overline{g}(x,y))}{x^2+y^2}$$

for all $x, y \in I$.

From (28) and Theorem 16, we have the following Theorem 20.

THEOREM 20. *Let p be a positive continuous weight on I, $G_p(t) = (f(t) - \overline{f_p}(x,y))(g(t) - \overline{g_p}(x,y))$ be a differentiable function on I with $G'_p(y) \geq \frac{p(y)}{P(x,y)} \cdot \frac{G(y)-G(x)}{y-x}$ for any $x, y \in I$. Then the following statements are true.*

(1) *If $G_p(t)$ and $p(t)$ have the same monotonicity on I, then $T(f,g,p)$ is Schur convex on I^2.*

(2) *If $G_p(t)$ and $tp(t)$ have the same monotonicity on I, then $T(f,g,p)$ is Schur geometrically convex on I^2.*

(3) *If $G_p(t)$ and $t^2p(t)$ have the same monotonicity on I, then $T(f,g,p)$ is Schur harmonic convex on I^2.*

THEOREM 21. *Let $G(t) = (f(t) - \overline{f}(x,y))(g(t) - \overline{g}(x,y))$ be a continuous function on $[x,y] \subseteq I$ for any $x, y \in I$. Then the following statements are true.*

(1) *$T(f,g)$ is Schur convex (concave) on I^2 if and only if $G(t)$ is convex (concave) on I.*

(2) *If $G(t)$ is increasing (decreasing) and convex (concave) on I, then $T(f,g)$ is Schur geometrically and harmonic convex (concave) on I^2.*

Proof. Theorem 21(1) follows from (30) and Theorem 9, and Theorem 21(2) follows form (30) and Theorem 17. □

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