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SIMULTANEOUS PROXIMINALITY IN $L^{\infty}(\mu, X)$

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Abstract. Let X be a Banach space and G be a closed subspace of X. Let us denote by $L^{\infty}(\mu, X)$ the Banach space of all X-valued essentially bounded functions on a σ-finite complete measure space (Ω, Σ, μ). In this paper we show that if G is separable, then $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$ if and only if G is simultaneously proximinal in X.

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1. INTRODUCTION

Let Z be a Banach space and Y be a closed subspace of Z. For a subset B of Z, define

$$d(B, Y) = \inf_{y \in Y} \sup_{b \in B} \|b - y\|.$$

An element $y^* \in Y$ is said to be a best simultaneous approximant to the subset B if

$$\sup_{b \in B} \|b - y^*\| = d(B, Y).$$

DEFINITION 1.1. *If every finite subset of Z admits a best simultaneous approximation in Y, then Y is said to be simultaneously proximinal in Z.*

The theory of best simultaneous approximation has been studied by many authors. Most of these works have dealt with the space of continuous functions with values in a Banach space e.g. [18, 10, 4]. Some recent results for best simultaneous approximation in the Banach space of P-Bochner integrable (essentially bounded) functions have been obtained in [5, 6, 9, 16, 19]. We consider here a problem of simultaneous approximation completing the work done in [6, 19]. In this paper (Ω, Σ, μ) stands for a complete σ-finite measure space and $L^{\infty}(\mu, X)$ the Banach space of all essentially bounded functions on (Ω, Σ, μ) with values in a Banach space X, endowed with the usual norm

$$\|f\|_{\infty} = \operatorname{ess\,sup} \|f(t)\|.$$

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In [19], it is shown that if G is a reflexive subspace of a Banach space X, then $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$. In [6], it is shown that if G is a separable w^*-closed subspace of a dual space X, then $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$. The aim of this paper is to show that if G is a closed separable subspace of a Banach space X, then $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$ if and only if G is simultaneously proximinal in X.

2. PRELIMINARY RESULTS

The following Theorem is a generalization of the distance formula given in [6]. Here χ_A is denoted to the characteristic function of A.

THEOREM 2.1. *Let X be a Banach space, G be a closed subspace of X, and $f_1, f_2, ..., f_m$ be any finite number of elements in $L^{\infty}(\mu, X)$. Then the function*

$$s \to d\left(\{f_i(s) : 1 \le i \le m\},\ G\right),$$

which we denote by $d(\{f_i(\cdot) : 1 \le i \le m\},\ G)$, is measurable, and

$$d\left(\{f_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\right) = \left\| d\left(\{f_i(s) : 1 \le i \le m\},\ G\right)\right\|_{\infty}$$

Proof. Let $f_1, f_2, ..., f_m \in L^{\infty}(\mu, X)$. Being strongly measurable functions, there exist sequences of simple functions $(f_{(i,n)})_{n=1}^{\infty}$, $i = 1, 2, ..., m$, such that

$$\lim \left\| f_{(i,n)}(t) - f_i(t)\right\| = 0,$$

for $i = 1, 2, ..., m$, and for almost all t's. We may write, [9],

$$f_{(i,n)} = \sum_{j=1}^{k(n)} \chi_{A(n,j)}(\cdot)\, x_{(i,n,j)}, \quad i = 1, 2, ..., m,$$

where $A(n,j)$ are disjoint and $\overset{k(n)}{\underset{j=1}{\cup}} A(n,j) = \Omega$. Define $d_n(\cdot) : \Omega \to \mathbb{R}$ by

$$d_n(s) = d\left(\left\{f_{(i,n)}(s) : 1 \le i \le m\right\},\ G\right).$$

Then

$$d_n(s) = \sum_{j=1}^{k(n)} \chi_{A(n,j)} d\left(\left\{x_{(i,n,j)} : 1 \le i \le m\right\},\ G\right)$$

and

$$\lim d_n(s) = d\left(\{f_i(s) : 1 \le i \le m\},\ G\right),$$

for almost all s. Thus $d(\{f_i(\cdot) : 1 \le i \le m\},\ G)$ is measurable.

Now, for any $h \in L^{\infty}(\mu, G)$,

$$\begin{aligned} \operatorname{ess\,sup} d\left(\{f_i(s) : 1 \le i \le m\},\ G\right) &\le \operatorname{ess\,sup} \sup_{1 \le i \le m} \left\| f_i(s) - h(s)\right\| \\ &= \sup_{1 \le i \le m} \left\| f_i - h\right\|_{\infty} \end{aligned}$$

Hence

$$\operatorname{ess\,sup} d\left(\{f_i(s) : 1 \le i \le m\},\ G\right) \le d\left(\{f_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\right).$$

To prove the reverse inequality, let $\epsilon > 0$ be given and $w_i,\ i = 1, 2, ..., m$, be countably valued functions in $L^{\infty}(\mu, X)$ such that

$$\|f_i - w_i\|_{\infty} < \tfrac{\epsilon}{3}.$$

We may write $w_i = \sum_{k=1}^{\infty} \chi_{A_k}(\cdot)\ x_{(i,k)}$, where A_k are disjoint, $\overset{\infty}{\underset{k=1}{\cup}} A_k = \Omega$, and $\mu(A_k) > 0$, for all k. Let $h_k \in G$ be such that

$$\sup_{1 \le i \le m} \left\|x_{(i,k)} - h_k\right\| < d\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right) + \tfrac{\epsilon}{3},$$

for all k. Let $g = \sum_{k=1}^{\infty} \chi_{A_k}(\cdot)\, h_k$. It is clear that $g \in L^{\infty}(\mu, G)$. Further

$$\begin{aligned}
& d\left(\{f_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\right) \le \\
& \le \sup_{1 \le i \le m} \|f_i - w_i\|_{\infty} + d\left(\{w_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\right) \\
& \le \tfrac{\epsilon}{3} + \sup_{1 \le i \le m} \|g - w_i\|_{\infty} \\
& = \tfrac{\epsilon}{3} + \operatorname{ess\,sup} \sup_{1 \le i \le m} \|g(t) - w_i(t)\| \\
& = \tfrac{\epsilon}{3} + \operatorname{ess\,sup} \sup_{1 \le i \le m} \sum_{k=1}^{\infty} \chi_{A_k}(t) \left\|x_{(i,k)} - h_k\right\| \\
& = \tfrac{\epsilon}{3} + \operatorname{ess\,sup} \sum_{k=1}^{\infty} \chi_{A_k}(t) \sup_{1 \le i \le m} \left\|x_{(i,k)} - h_k\right\| \\
& < \tfrac{2\epsilon}{3} + \operatorname{ess\,sup} \sum_{k=1}^{\infty} \chi_{A_k}(t)\, d\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right) \\
& = \tfrac{2\epsilon}{3} + \operatorname{ess\,sup} d\left(\{w_i(t) : 1 \le i \le m\},\ G\right) \\
& \le \tfrac{2\epsilon}{3} + \operatorname{ess\,sup} \left[d\left(\{f_i(t) : 1 \le i \le m\},\ G\right) + \sup_{1 \le i \le m} \|f_i(t) - w_i(t)\| \right] \\
& \le \tfrac{2\epsilon}{3} + \operatorname{ess\,sup} d\left(\{f_i(t) : 1 \le i \le m\},\ G\right) + \sup_{1 \le i \le m} \|f_i - w_i\|_{\infty} \\
& < \epsilon + \operatorname{ess\,sup} d\left(\{f_i(t) : 1 \le i \le m\},\ G\right).
\end{aligned}$$

□

COROLLARY 2.2. *Let X be a Banach space, G be a closed subspace of X, and $f_1, f_2, ..., f_m$ be any finite number of elements in $L^{\infty}(\mu, X)$. Let $g : \Omega \to G$ be a measurable function such that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s. Then g is a best simultaneous approximation of $f_1, f_2, ..., f_n$ in $L^{\infty}(\mu, G)$ (and therefore $g \in L^{\infty}(\mu, G)$).*

Proof. Assume that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s. Then

$$\sup_{1\leq i\leq m} \|f_i(s) - g(s)\| \leq \sup_{1\leq i\leq m} \|f_i(s) - z\|,$$

for almost all s, and for all $z \in G$. Then

$$\|g(s)\| \leq 2 \sup_{1\leq i\leq m} \|f_i(s)\| \leq 2 \sup_{1\leq i\leq m} \|f_i\|_\infty,$$

for almost all s, therefore $g \in L^\infty(\mu, G)$. By Theorem 2.1,

$$\begin{aligned} d(\{f_i : 1 \leq i \leq m\},\ L^\infty(\mu, G)) &= \operatorname{ess\,sup} d(\{f_i(s) : 1 \leq i \leq m\},\ G) \\ &= \operatorname{ess\,sup} \sup_{1\leq i\leq m} \|f_i(s) - g(s)\| \\ &= \sup_{1\leq i\leq m} \|f_i - g\|_\infty. \end{aligned}$$

Therefore g is a best simultaneous approximation for $f_1, f_2, .., f_m$ in $L^\infty(\mu, G)$. $\square$

The condition in Corollary 2.2 is sufficient; $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s in G, implies g is a best simultaneous approximation of $f_1, f_2, ..., f_m$ in $L^\infty(\mu, G)$. For the converse, we need the following easy lemma.

LEMMA 2.3. *Let X be a Banach space, G be a closed subspace of X, $A \subset \Omega$ be such that $\mu(A) > 0$, and $f_1, f_2, ..., f_m \in L^\infty(\mu, X)$ be such that*

$$d(\{f_i(s) : 1 \leq i \leq m\},\ G) = \begin{cases} 1, & \text{if } s \in A \\ 0, & \text{if } s \in \Omega \setminus A. \end{cases}$$

Then $d(\{f_i : 1 \leq i \leq m\},\ L^\infty(\mu, G)) = 1$.

Proof. Let $g \in L^\infty(\mu, G)$, then

$$\sup_{1\leq i\leq m} \|f_i(s) - g(s)\| \geq d(\{f_i(s) : 1 \leq i \leq m\},\ G),$$

for all $s \in \Omega$.

$$\begin{aligned} \operatorname{ess\,sup} \sup_{1\leq i\leq m} \|f_i(s) - g(s)\| &\geq \operatorname{ess\,sup} d(\{f_i(s) : 1 \leq i \leq m\},\ G) \\ &= 1. \end{aligned}$$

Thus $\sup_{1\leq i\leq m} \|f_i - g\|_\infty \geq 1$. Since $g \in L^\infty(\mu, G)$ was arbitrary, then

$$d(\{f_i : 1 \leq i \leq m\},\ L^\infty(\mu, G)) \geq 1.$$

To prove the converse inequality. Let $\epsilon > 0$ be given. Let $f_1', f_2', ..., f_m' \in L^\infty(\mu, X)$ be countably valued functions such that $f_i'(\Omega) \subset f_i(\Omega)$, $i = 1, 2, \ldots, m$, and

$$\left\| f_i' - f_i \right\| < \tfrac{\epsilon}{3}.$$

We may write $f_i' = \sum_{k=1}^{\infty} \chi_{A_k}(\cdot)\ x_{(i,k)}$, with the subsets A_k disjoint and measurable, and $x_{(i,k)} \in f_i(\Omega)$. For each k take $h_k \in G$ such that

$$\begin{aligned}\sup_{1\le i\le m} \left\| x_{(i,k)} - h_k \right\| &< d\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right) + \tfrac{\epsilon}{3} \\ &\le 1 + \tfrac{\epsilon}{3}.\end{aligned}$$

It is clear that g defined by

$$g = \sum_{k=1}^{\infty} \chi_{A_k}(\cdot)\, h_k$$

belongs to $L^{\infty}(\mu, G)$ and

$$\begin{aligned}& d\big(\{f_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\big) \le \\ & \le \sup_{1\le i\le m} \left\| f_i - f_i' \right\|_{\infty} + d\left(\left\{f_i' : 1 \le i \le m\right\},\ L^{\infty}(\mu, G)\right) \\ & \le \tfrac{\epsilon}{3} + \sup_{1\le i\le m} \left\| g - f_i' \right\|_{\infty} \\ & = \tfrac{\epsilon}{3} + \operatorname{ess\,sup} \sup_{1\le i\le m} \left\| g(t) - f_i'(t) \right\| \\ & = \tfrac{\epsilon}{3} + \operatorname{ess\,sup} \sup_{1\le i\le m} \sum_{k=1}^{\infty} \chi_{A_k}(t) \left\| x_{(i,k)} - h_k \right\| \\ & < \tfrac{2\epsilon}{3} + \operatorname{ess\,sup} \sum_{k=1}^{\infty} \chi_{A_k}(t)\, d\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right) \\ & \le \epsilon + \operatorname{ess\,sup} d\left(\{f_i(t) : 1 \le i \le m\},\ G\right) \\ & = \epsilon + 1.\end{aligned}$$

Therefore,

$$d\left(\{f_i : 1 \le i \le m\},\ L^{\infty}(\mu, G)\right) \le \epsilon + 1.$$

□

THEOREM 2.4. *Let X be a Banach space and G be a closed subspace of X. Then $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$ if and only if for any finite number of elements $f_1, f_2, ..., f_m$ in $L^{\infty}(\mu, X)$, there exists $g \in L^{\infty}(\mu, G)$ such that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s.*

Proof. Sufficiency of the condition is an immediate consequence of Corollary 2.2. We will show the necessity. Assume that $L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$ and take $f_1, f_2, ..., f_m$ in $L^{\infty}(\mu, X)$. Consider the non-negative measurable function

$$\begin{aligned}& h : \Omega \to [0, \infty) \\ & s \to d\left(\{f_i(s) : 1 \le i \le m\},\ G\right).\end{aligned}$$

Take $\Omega_0 = \{s \in \Omega : h(s) = 0\}$, and for each $n = 1, 2, ...,$ take $\Omega_n = \{s \in \Omega : n - 1 < h(s) \leq n\}$. Of course, we may forget those Ω_n which are $\mu-$null sets, so, without loss of generality, we will assume that $\mu(\Omega_n) > 0$ for all n. Now for each $n = 1, 2, ...,$ we define $f_{in} : \Omega \to X$ by

$$f_{in}(s) = \begin{cases} \frac{1}{h(s)} f_i(s), & \text{if } s \in \Omega_n \\ 0, & \text{if } s \in \Omega \setminus \Omega_n. \end{cases}$$

It is clear that $f_{in} \in L^\infty(\mu, X)$, $i = 1, 2, ..., m$, and also that

$$\begin{aligned} d(\{f_{in}(s) : 1 \leq i \leq m\}, G) &= d\left(\left\{\tfrac{1}{h(s)} f_i(s) : 1 \leq i \leq m\right\}, G\right) \\ &= \tfrac{1}{h(s)} d(\{f_i(s) : 1 \leq i \leq m\}, G) \\ &= 1, \end{aligned}$$

for all $s \in \Omega_n$. So, it follows from proceeding lemma that

$$d(\{f_{in} : 1 \leq i \leq m\}, L^\infty(\mu, G)) = 1.$$

On the other hand, using simultaneous proximinality of $L^\infty(\mu, G)$, we deduce that there exists $g_n \in L^\infty(\mu, G)$ such that

$$\begin{aligned} \sup_{1 \leq i \leq m} \|f_{in} - g_n\|_\infty &= d(\{f_{in} : 1 \leq i \leq m\}, L^\infty(\mu, G)) \\ &= 1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} 1 &= d(\{f_{in}(s) : 1 \leq i \leq m\}, G) \\ &\leq \sup_{1 \leq i \leq m} \|f_{in}(s) - g_n(s)\| \\ &\leq \sup_{1 \leq i \leq m} \|f_{in} - g_n\|_\infty \\ &= 1, \end{aligned}$$

for almost all $s \in \Omega_n$. Then,

$$\sup_{1 \leq i \leq m} \|f_{in}(s) - g_n(s)\| = 1,$$

for almost all $s \in \Omega_n$. Thus

$$\begin{aligned} d(\{f_i(s) : 1 \leq i \leq m\}, G) &= h(s) \\ &= h(s) \sup_{1 \leq i \leq m} \|f_{in}(s) - g_n(s)\| \\ &= \sup_{1 \leq i \leq m} \|h(s) f_{in}(s) - h(s) g_n(s)\| \\ &= \sup_{1 \leq i \leq m} \|f_i(s) - h(s) g_n(s)\|, \end{aligned}$$

for almost all $s \in \Omega_n$. We do notice that, if $s \in \Omega_0$, then $f_i(s) = f_j(s)$, for $1 \leq i \leq j \leq m$. Hence, it is clear that g defined by

$$g(s) = \chi_{\Omega_0}(s) f_1(s) + \sum_{n=1}^{\infty} \chi_{\Omega_n}(s) h(s) g_n(s),$$

for all $s \in \Omega$ enjoys the required property.

□

3. MAIN RESULT

The following lemmas will be used to prove our main result.

LEMMA 3.1. [21] *Let (Ω, Σ, μ) be a complete measure space, X a Banach space, and f a strongly measurable function from Ω to X. Then f is measurable in the classical sense; $f^{-1}(O)$ is measurable for every open set $O \subset X$.*

LEMMA 3.2. [21] *Let (Ω, Σ, μ) be a complete measure space, X a Banach space. If $f : \Omega \to X$ is measurable in the classical sense and has essentially separable range, then f is strongly measurable.*

Let Φ be a set-valued mapping, taking each point of a measurable space Ω into a subset of a metric space X. We say that Φ is weakly measurable if $\Phi^{-1}(O)$ is measurable in Ω whenever O is open in X. Here we have put, for any $A \subset X$,

$$\Phi^{-1}(A) = \{s \in \Omega : \phi(s) \cap A \neq \phi\}.$$

The following theorem is due to Kuratowski [17], it is known as Measurable Selection Theorem.

THEOREM 3.3. [17] *Let Φ be a weakly measurable set-valued map which carries each point of measurable space Ω to a closed nonvoid subset of a complete separable metric space X. Then Φ has a measurable selection; i.e., there exists a function $f : \Omega \to X$ such that $f(s) \in \phi(s)$ for each $s \in \Omega$ and $f^{-1}(O)$ is measurable in Ω whenever O is open in X.*

THEOREM 3.4. *Let X be a Banach space and G be a closed separable subspace of X. Then the following are equivalent:*

(1) *G is simultaneously proximinal in X.*
(2) *$L^{\infty}(\mu, G)$ is simultaneously proximinal in $L^{\infty}(\mu, X)$.*

Proof. $(2) \Rightarrow (1)$: Let $x_1, x_2, ..., x_m$ be any finite number of elements in X. Define $f_i : \Omega \to X$, $i = 1, 2, ..., m$, by

$$f_i(s) = x_i \ .$$

Using simultaneous proximinality of $L^{\infty}(\mu, G)$ and Theorem 2.4, we get $g \in L^{\infty}(\mu, G)$ such that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s. Choose $s_0 \in \Omega$ so that $g(s_0)$ is a best simultaneous approximation of $x_1, x_2, ..., x_m$ in G.

$(1) \Rightarrow (2)$: Let $f_1, f_2, ..., f_m$ be any finite number of elements in $L^{\infty}(\mu, X)$. For each $s \in \Omega$ define

$$\Phi(s) = \left\{ g \in G : \sup_{1 \le i \le m} \|f_i(s) - g\| = d\left(\{f_i(s) : 1 \le i \le m\},\ G\right) \right\}.$$

For each $s \in \Omega$, $\Phi(s)$ is closed, bounded, and nonvoid subset of G. We shall show that Φ is weakly measurable. Let O be an open set in X, the set

$$\Phi^{-1}(O) = \{s \in \Omega : \Phi(s) \cap O \neq \phi\}$$

can be also be described as

$$\Phi^{-1}(O) = \left\{ s \in \Omega : \inf_{g \in G} \sup_{1 \le i \le m} \|f_i(s) - g\| = \inf_{g \in O} \sup_{1 \le i \le m} \|f_i(s) - g\| \right\}.$$

Since (Ω, Σ, μ) is complete, f_i is measurable in the classical sense for $i = 1, 2, ..., m$, by Lemma 3.1. Since subtraction in X, sup, and the norm in X are continuous, then the map

$$s \to \inf_{g \in A} \sup_{1 \le i \le m} \|f_i(s) - g\|$$

is measurable for any set A. It follows that $\Phi^{-1}(O)$ is measurable. By Theorem 3.3, Φ has a measurable selection; i.e., there exists a function $f : \Omega \to G$ such that $f(s) \in \phi(s)$ for each $s\epsilon\Omega$ and f is measurable in the classical sense. By Lemma 3.2, f is strongly measurable. Hence f is a best simultaneous approximation for $f_1, f_2, ..., f_m$ in $L^{\infty}(\mu, G)$ by Theorem 2.4.

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