REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 42 (2013) no. 2, pp. 94–102
ictp.acad.ro/jnaat

BILATERAL INEQUALITIES FOR MEANS

MIRA-CRISTIANA ANISIU* and VALERIU ANISIU †

Abstract. Let (M_1, M_2, M_3) be three means in two variables chosen from H, G, L, I, A, Q, S, C so that

$$M_1(a,b) < M_2(a,b) < M_3(a,b), \quad 0 < a < b.$$

We consider the problem of finding $\alpha,\ \beta \in \mathbb{R}$ for which

$$\alpha M_1(a,b) + (1-\alpha)M_3(a,b) < M_2(a,b) < \beta M_1(a,b) + (1-\beta)M_3(a,b).$$

We solve the problem for the triplets (G,L,A), (G,A,Q), (G,A,C), (G,Q,C), (A,Q,C), (A,S,C), (A,Q,S) and (L,A,C). The Symbolic Algebra Program *Maple* is used to determine the range where some parameters can vary, or to find the minimal polynomial for an algebraic number.

MSC 2000. 26D15, 26E60; 26-04

Keywords. Two-variable means, weighted arithmetic mean, inequalities, symbolic computer algebra.

1. INTRODUCTION

We remind the definitions of the classical means, namely, for $0 < a < b$

- the *arithmetic*, *geometric* and *harmonic* ones

$$A = \tfrac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \tfrac{2ab}{a+b},$$

as well as

- the *Hölder* and the *anti-harmonic* mean $Q = \left(\tfrac{a^2+b^2}{2}\right)^{1/2}, \quad C = \tfrac{a^2+b^2}{a+b}$;
- the *Pólya & Szegő logarithmic* mean, the *exponential* (or *identric*), and the *weighted geometric* mean

$$L = \tfrac{b-a}{\ln b - \ln a}, \quad I = \tfrac{1}{e}\left(\tfrac{b^b}{a^a}\right)^{1/(b-a)}, \quad S = \left(a^a b^b\right)^{1/(a+b)}.$$

References on means and inequalities between them can be found in [5].
At first, the following inequalities between means were established

$$H < G < L < I < A < Q < S < C, \tag{1}$$

* "T. Popoviciu" Institute of Numerical Analysis, Romanian Academy, P.O. Box 68, 400110 Cluj-Napoca, Romania, e-mail: mira@math.ubbcluj.ro.

† "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 1 Kogălniceanu St., 400084 Cluj-Napoca, Romania, e-mail: anisiu@math.ubbcluj.ro.

followed by relations between some means and the arithmetic means of two others ([7], [3])

$$L < \frac{G+A}{2}, \quad \frac{G+Q}{2} < A < \frac{G+C}{2} < Q < \frac{A+C}{2} < S. \tag{2}$$

A more difficult problem is to obtain results of the type (2) for weighted arithmetic means and to determine the maximal interval for the parameter for which the inequalities hold.

We mention here an inequality proved by Alzer and Qiu for the means G, I and A.

Theorem 1. [1] *The double inequality*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{3}$$

holds true for all positive real numbers $a \neq b$, if and only if $\alpha \leq 2/3$ and $\beta \geq 2/\mathrm{e}$.

Results of this type continued to appear, recent ones are given in [9] for (H, L, A) and (H, I, A), and in [4] for (G, L, C).

Let M_1, M_2, M_3 be three means out of the eight listed in (1) so that

$$M_1(a,b) < M_2(a,b) < M_3(a,b). \tag{4}$$

We consider the problem of finding $\alpha,\ \beta \in \mathbb{R}$ for which

$$\alpha M_1(a,b) + (1-\alpha)M_3(a,b) < M_2(a,b) \tag{5}$$

and

$$M_2(a,b) < \beta M_1(a,b) + (1-\beta)M_3(a,b). \tag{6}$$

The inequalities (5) and (6) are equivalent to

$$\alpha > \frac{M_3(a,b) - M_2(a,b)}{M_3(a,b) - M_1(a,b)}, \tag{7}$$

respectively

$$\beta < \frac{M_3(a,b) - M_2(a,b)}{M_3(a,b) - M_1(a,b)}. \tag{8}$$

Basically, denoting by $t = b/a$, $t > 1$, the problem reduces to find $\inf f$ and $\sup f$, where

$$f(t) = \frac{M_3(1,t) - M_2(1,t)}{M_3(1,t) - M_1(1,t)}. \tag{9}$$

The function f is obviously bounded, $0 \leq f(t) \leq 1$. If $\sup f$ is attained at some $t \in (1, \infty)$, then $\alpha \in (\sup f, \infty)$; otherwise $\alpha \in [\sup f, \infty)$. Similarly, $\beta \in (-\infty, \inf f)$ if $\inf f$ is attained in $(1, \infty)$, and $\beta \in (-\infty, \inf f]$ otherwise. Symbolic Algebra Programs can be of great help to determine the range where the parameters can vary. *Maple* was used in [3] to find the interval for α in Theorem 9 below. We also use it to simplify the polynomials in the proof of Theorem 5 and to obtain the optimal value β_0 of β.

Starting from the means listed in (1), we can formulate $\binom{8}{3} = 56$ bilateral inequalities of the type (3). We shall choose seven of them, for which one of

(5) and (6) was already proved in [3], and we shall find the possible values of the parameter for the remaining one. Then, for $L < A < C$ we find the optimal intervals for α and β in order that both inequalities (5) and (6) hold. To this aim *Maple* is again very useful.

2. BILATERAL INEQUALITIES

We consider means in two variables, but we prefer to use a simpler (and shorter) notation.

Let us denote for $0 < a < b$, $t = b/a$, $t > 1$. It it obvious, due to the homogeneity, that, if $M(a, b)$ is any mean from (1), it suffices to prove the inequalities for $M(1, t)$. We shall write from now on $M(t)$ instead of $M(1, t)$.

THEOREM 2. *The double inequality*

$$\alpha G(t) + (1-\alpha)A(t) < L(t) < \beta G(t) + (1-\beta)A(t), \quad \forall t > 1,$$

holds if and only if $\alpha \geq 1$ *and* $\beta \leq 2/3$.

Proof. We denote, for $t > 1$,

$$f_1(t) = \frac{A(t)-L(t)}{A(t)-G(t)} = \frac{(t+1)\ln t - 2(t-1)}{(t+1-2\sqrt{t})\ln t}. \tag{10}$$

Let us suppose that the first inequality in the theorem holds. From $\lim_{t\to\infty} f_1(t) = 1$ it follows obviously that $\alpha \geq 1$. Conversely, if $\alpha \geq 1$ it suffices to have

$$\frac{A(t)-L(t)}{A(t)-G(t)} < 1,$$

which is true because $L(t) > G(t)$. We evaluate $f_1(t) - 2/3$, where $2/3 = \lim_{t\to 1} f_1(t)$ and show that it is positive. The denominator is obviously positive; we substitute $u = \sqrt{t}$ in the numerator and obtain

$$f(u) = (u^2 + 4u + 1)\ln u - 3u^2 + 3.$$

We have $f(1) = f'(1) = f''(1) = 0$ and $f'''(u) = 2(u-1)^2/u^3 > 0$ for $u > 1$, hence $f_1(t) > 2/3$ for $t > 1$. It follows that $L(t) < \beta G(t) + (1-\beta)A(t)$, $\forall t > 1$ if and only if $\beta \leq 2/3$. □

THEOREM 3. *The double inequality*

$$\alpha G(t) + (1-\alpha)Q(t) < A(t) < \beta G(t) + (1-\beta)Q(t), \quad \forall t > 1,$$

holds if and only if $\alpha \geq 1/2$ *and* $\beta \leq 1 - \sqrt{2}/2$.

Proof. Let us consider for $t > 1$, the function

$$f_2(t) = \frac{Q(t)-A(t)}{Q(t)-G(t)} = 1 - \frac{\sqrt{t^2+1}+\sqrt{2t}}{\sqrt{2}(\sqrt{t}+1)^2}. \tag{11}$$

We have $f_2(t) < 1/2$, since $\sqrt{t^2+1} + \sqrt{2t} > \sqrt{2}/2\ (\sqrt{t}+1)^2 \Leftrightarrow \sqrt{t^2+1} > \sqrt{2}/2\ (t+1) \Leftrightarrow (t-1)^2 > 0$. Since $\lim_{t\to 1} f_2(t) = 1/2$, it follows that $\alpha G(t) + (1-\alpha)Q(t) < A(t)$, $\forall t > 1$ if and only if $\alpha \geq 1/2$.

Let us suppose that $A(t) < \beta G(t) + (1-\beta)Q(t)$, $\forall t > 1$. Since $\lim_{t\to\infty} f_2(t) = 1 - \sqrt{2}/2$, it follows that $\beta \leq 1 - \sqrt{2}/2$. Conversely, if

$\beta \le 1-\sqrt{2}/2$, it suffices to prove that $f_2(t) > 1-\sqrt{2}/2$. This is equivalent with

$$\frac{\sqrt{t^2+1}+\sqrt{2t}}{\left(\sqrt{t}+1\right)^2} < 1,$$

i. e. $\sqrt{t^2+1}+\sqrt{2t} < \left(\sqrt{t}+1\right)^2$ and this is true because $\sqrt{t^2+1}+\sqrt{2t} < t+1+\sqrt{2t} < \left(\sqrt{t}+1\right)^2$. □

THEOREM 4. *The double inequality*

$$\alpha G(t)+(1-\alpha)C(t) < A(t) < \beta G(t)+(1-\beta)C(t), \quad \forall t>1,$$

holds if and only if $\alpha \ge 2/3$ *and* $\beta \le 1/2$.

Proof. For $t>1$ we define

$$f_3(t)=\frac{C(t)-A(t)}{C(t)-G(t)}=\frac{(t-1)^2}{2(t^2+1-\sqrt{t}(t+1))}. \tag{12}$$

Since $\lim_{t\to 1} f_3(t)=2/3$, from $\alpha G(t)+(1-\alpha)C(t) < A(t)$, $\forall t>1$ it follows that $\alpha \ge 2/3$. If $\alpha \ge 2/3$, it is true that $f_3(t) < \alpha$, because $f_3(t) < 2/3$ is equivalent with

$$\frac{\sqrt{t}}{t+\sqrt{t}+1} < \frac{1}{3},$$

or $(\sqrt{t}-1)^2 > 0$.

Similarly, it follows that $f_3(t) > 1/2$, since

$$f_3(t)-\frac{1}{2}=\frac{\sqrt{t}(\sqrt{t}-1)^2}{2(t^2+1-\sqrt{t}(t+1))}=\frac{\sqrt{t}}{2(t+\sqrt{t}+1)} > 0.$$

The infimum of f_3 on $(1,\infty)$ is precisely $1/2$, because $\lim_{t\to\infty} f_3(t)=1/2$ □

THEOREM 5. *The double inequality*

$$\alpha G(t)+(1-\alpha)C(t) < Q(t) < \beta G(t)+(1-\beta)C(t), \quad \forall t>1,$$

holds if and only if $\alpha \ge 1-\sqrt{2}/2$ *and* $\beta < \beta_0$, *where* $\beta_0 \approxeq 0.3471574308...$ *is the unique positive root of the polynomial*

$$9x^4-26x^3+22x^2-2x-1.$$

Proof. We have to find, for $t>1$, the extreme values of

$$f_4(t)=\frac{C(t)-Q(t)}{C(t)-G(t)}=\frac{2t^2-\sqrt{2}\sqrt{t^2+1}-\sqrt{2}t\sqrt{t^2+1}+2}{2(t^2-t^{\frac{3}{2}}-\sqrt{t}+1)}. \tag{13}$$

Denoting by $u=\sqrt{t}$, we compute the derivative of

$$h(u)=f_4(u^2)$$

and we obtain

$$h'(u)=\frac{(u+1)\left(\sqrt{2(u^4+1)}\left(-u^4-4\,u^2-1\right)+u^6+2\,u^5+3\,u^4+3\,u^2+2\,u+1\right)}{\sqrt{2(u^4+1)}(u-1)^3(u^2+u+1)^2}.$$

So, the roots of the derivative satisfy the algebraic equation

$$2(u^4+1)\left(u^4+4\,u^2+1\right)^2=\left(u^6+2\,u^5+3\,u^4+3\,u^2+2\,u+1\right)^2.$$

After the simplification of a quartic polynomial whose roots are not in the interval $(1, \infty)$, we obtain the equation

$$u^8 - 8u^5 - 10u^4 - 8u^3 + 1 = 0, \tag{14}$$

which has a unique root u_0 in the interval $(1, \infty)$. This can be easily proved by using the Sturm sequence. Then u_0 will be the unique root of h' in $(1, \infty)$.

Now $h'(2) > 0$, $h'(3) < 0$, so $2 < u_0 < 3$ and h is strictly increasing in the interval $(1, u_0)$ and strictly decreasing in the interval (u_0, ∞). We also have $\lim_{u\to 1} h(u) = 1/3$, $\lim_{u\to\infty} h(u) = 1 - \sqrt{2}/2$ and therefore $\inf f_4 = \inf h = 1 - \sqrt{2}/2$, $\sup f_4 = \sup h = h(u_0) = \beta_0$.

Since $h(u_0)$ is an algebraic number, we can easily find its minimal polynomial by performing the following commands in *Maple*:

```
>theta:=RootOf(u^8-8*u^5-10*u^4-8*u^3+1, u):
> M:= g(theta):
> sqrfree(evala(Norm(convert(Z-M,RootOf))),Z)[2][1][1];
```

$$9x^4 - 26x^3 + 22x^2 - 2x - 1$$

Notice that *Maple* is of course able to express the maximum $h(u_0)$ in terms of radicals by executing the command:

```
> select(u-> is(u>0),[solve(9*x^4-26*x^3+22*x^2-2*x-1,Explicit)]);
```

but the resulting expression is cumbersome and we will not print it here. □

THEOREM 6. *The double inequality*

$$\alpha A(t) + (1 - \alpha)C(t) < Q(t) < \beta A(t) + (1 - \beta)C(t), \quad \forall t > 1,$$

holds if and only if $\alpha \geq 2 - \sqrt{2}$ *and* $\beta \leq 1/2$.

Proof. Let us consider, for $t > 1$

$$f_5(t) = \frac{C(t)-Q(t)}{C(t)-A(t)} = \frac{2(t^2+1)-(t+1)\sqrt{2(t^2+1)}}{(t-1)^2}. \tag{15}$$

From $\lim_{t\to\infty} f_5(t) = 2-\sqrt{2}$ it follows that $\alpha A(t)+(1-\alpha)C(t) < Q(t)$, $\forall t > 1$ implies $\alpha \geq 2-\sqrt{2}$. Now if $\alpha \geq 2-\sqrt{2}$ we have to prove that $f_5(t) < 2-\sqrt{2}$, which can be written as $\sqrt{2(t^2+1)}\,(t+1) > -\sqrt{2}t^2 + \left(4 + 2\sqrt{2}\right) t - \sqrt{2}$. If $-\sqrt{2}t^2 + \left(4 + 2\sqrt{2}\right) t - \sqrt{2} \leq 0$, the inequality holds. Otherwise, squaring both sides it reduces to $4\left(3 + 2\sqrt{2}\right) t\,(t-1)^2 > 0$.

We obtain also

$$f_5(t) - \tfrac{1}{2} = \frac{3(t^2+1)+2t-2(t+1)\sqrt{2(t^2+1)}}{2(t-1)^2} > 0,$$

because $(3(t^2+1) + 2t)^2 - 8(t+1)^2(t^2+1) > 0 \Leftrightarrow (t-1)^4 > 0$. We have $\lim_{t\to 1} f_5(t) = 1/2$, hence this is the infimum of f_5 on $(1, \infty)$ and the second part of the theorem is also true. □

LEMMA 7. [3] *For* $t > 1$*, the following inequality holds*

$$t^{\frac{t}{t+1}} > t - \ln t. \tag{16}$$

Proof. The inequality (16) is equivalent to

$$\tfrac{t}{t+1}\ln t > \ln(t-\ln t).$$

We consider the function

$$k(t) = \ln(t-\ln t) - \tfrac{t-1}{t}\ln t, \quad t>1,$$

with

$$k'(t) = \tfrac{(\ln t-1)\ln t}{t^2(t-\ln t)}.$$

It has $\lim_{t\to 1} k(t)=0$, $\lim_{t\to\infty} k(t)=0$ and a minimum at $t_0=e$. It follows that $k(t)<0$ on $(1,\infty)$, hence $((t-1)/t)\ln t > \ln(t-\ln t)$. It follows that

$$\tfrac{t}{t+1}\ln t > \tfrac{t-1}{t}\ln t > \ln(t-\ln t).$$

□

THEOREM 8. *The double inequality*

$$\alpha A(t) + (1-\alpha)C(t) < S(t) < \beta A(t) + (1-\beta)C(t), \quad \forall t>1,$$

holds if and only if $\alpha \geq 1/2$ *and* $\beta \leq 0$.

Proof. We define

$$f_6(t) = \tfrac{C(t)-S(t)}{C(t)-A(t)} = 2\tfrac{t^2+1-(t+1)t^{\frac{t}{t+1}}}{(t-1)^2}. \tag{17}$$

We have

$$f_6(t) - \tfrac{1}{2} = \tfrac{3(t^2+1)+2t-4(t+1)t^{\frac{t}{t+1}}}{2(t-1)^2} = \tfrac{2(t+1)}{(t-1)^2}\cdot g(t),$$

where

$$g(t) = \tfrac{3(t^2+1)+2t}{4(t+1)} - t^{\frac{t}{t+1}}.$$

Then

$$g'(t) = -\tfrac{g_1(t)}{4(t+1)^2},$$

where

$$g_1(t) = 4t^{\frac{t}{t+1}}(t+1+\ln t) + 1 - 3t^2 - 6t. \tag{18}$$

Using the fact that $S>Q$, i. e. $t^{t/(t+1)} > \sqrt{(t^2+1)/2}$, we obtain that $g_1(t) > \sqrt{2(t^2+1)}g_2(t)$, where

$$g_2(t) = 2(t+1+\ln t) - (3t^2+6t-1)/\sqrt{2(t^2+1)}.$$

The derivative of g_2 is

$$g_2'(t) = \tfrac{(t+1)\left(\sqrt{2(t^2+1)}\right)^3 - \left(3t^4+7t^2+6t\right)}{t(t^2+1)\sqrt{2(t^2+1)}}.$$

In order to establish its sign we consider the polynomial

$$P(t) = (t+1)^2 \left(\sqrt{2(t^2+1)}\right)^6 - \left(3t^4 + 7t^2 + 6t\right)^2$$
$$= t^6 (t-1)^2 \left(\tfrac{8}{t^6} + \tfrac{32}{t^5} + \tfrac{52}{t^4} + \tfrac{36}{t^3} + \tfrac{19}{t^2} + \tfrac{14}{t} - 1\right).$$

The expression from the last parenthesis is obviously decreasing for $t \geq 1$ and it is positive for $t = 10$. It follows that it is positive on $(1, 10)$, hence on this interval P is also positive. Therefore $g_2'(t) > 0$, $g_2(t) > g_2(1) = 0$, so g_1 is positive too for $1 < t < 10$.

Let us consider now that $t \geq 10$. Using (16) in (18) we obtain that $g_1(t) > g_3(t)$, where

$$g_3(t) = t^2 - 2t + 2 - (2\ln t + 1)^2.$$

For

$$g_4(t) = \sqrt{t^2 - 2t + 2} - 2\ln t - 1,$$

the sign of g_4' is given by $t^2 - t - 2\sqrt{t^2 - 2t + 2}$; but $(t^2 - t)^2 - 4(t^2 - 2t + 2) = (t-10)^4 + 38(t-10)^3 + 537(t-10)^2 + 3348(t-10) + 7772 > 0$ for $t \geq 10$. It follows that $g_3(t) \geq g_3(10) = 3.45... > 0$, hence g_1 is positive for $t \geq 10$ too.

In conclusion, $g_1(t) > 0$ on $(1, \infty)$, therefore $g'(t) < 0$ on $(1, \infty)$. The function g being decreasing, $g(t) < g(1) = 0$ for $t > 1$ and $f_6(t) < 1/2$ for $t > 1$.

The second part of the theorem follows from $\lim_{t\to\infty} f_6(t) = 0$ and $f_6(t) > 0$, $\forall t > 1$. □

THEOREM 9. *The double inequality*

$$\alpha A(t) + (1-\alpha)S(t) < Q(t) < \beta A(t) + (1-\beta)S(t), \quad \forall t > 1$$

holds if and only if $\alpha \geq 2 - \sqrt{2}$ *and* $\beta \leq 0$.

Proof. We shall prove that the first inequality holds for $\alpha = 2 - \sqrt{2}$ (hence *a fortiori* for $\alpha \geq 2 - \sqrt{2}$).

Let us denote

(19)

$$H(t,\alpha) = Q(t) - \alpha A(t) - (1-\alpha)S(t) = \tfrac{1}{2}\sqrt{2 + 2\,t^2} - \tfrac{1}{2}\alpha\,(1+t) - (1-\alpha)\,t^{\frac{t}{1+t}}.$$

and

$$h_1(t) = (\sqrt{2} + 1)H(t, 2 - \sqrt{2}),$$

where H is given in (19). We have to prove that $h_1(t) > 0$ for $t > 1$. It follows that

$$h_1(t) = \tfrac{(\sqrt{2}+1)\sqrt{2(t^2+1)}}{2} - \tfrac{\sqrt{2}(t+1)}{2} - t^{\frac{t}{t+1}}.$$

We put in the inequality $(1+x)^q < 1 + qx$, which holds for $x > 0$, $0 < q < 1$, $x = t - 1$ and $q = t/(t+1)$. It follows that

$$t^{\frac{t}{t+1}} < \tfrac{t^2+1}{t+1},$$

and

$$h_1(t) > \tfrac{(1+\sqrt{2})}{2(t+1)} \left((t+1)\sqrt{2+2t^2} - \sqrt{2}(t^2 + 2(\sqrt{2}-1)t + 1) \right).$$

Let us denote the positive expressions

$$h_2(t) = (t+1)\sqrt{2+2t^2}, \quad h_3(t) = \sqrt{2}(t^2 + 2(\sqrt{2}-1)t + 1);$$

it follows easily that $h_2^2(t) - h_3^2(t) = 4t(t-1)^2$, therefore $h_1(t) > 0$.

The second part of the theorem is obvious, since

$$f_7(t) = \tfrac{S(t)-Q(t)}{S(t)-A(t)}$$

satisfies $f_7(t) > 0$, $\forall t > 1$ and $\lim_{t\to 1} f_7(t) = 0$. □

Theorem 10. *The double inequality*

$$\alpha L(t) + (1-\alpha)C(t) < A(t) < \beta L(t) + (1-\beta)C(t), \quad \forall t > 1,$$

holds if and only if $\alpha \geq 3/4$ *and* $\beta \leq 1/2$.

Proof. We have to find the extreme values of

$$f_8(t) = \tfrac{C(t)-A(t)}{C(t)-L(t)}$$

for $t > 1$, where f_8 is given by

$$f_8(t) = \tfrac{1}{2} \cdot \tfrac{(t-1)^2 \ln t}{\ln t + t^2 \ln t - t^2 + 1}.$$

We obtain

$$f_8'(t) = -\tfrac{(t-1)h_3(t)}{2t(\ln t + t^2 \ln t - t^2 + 1)^2},$$

where

$$h_4(t) = t^3 - 2\left(t^2 \ln t\right)^2 + 2\,t^2 \ln t - t^2 - 2\,t \ln t - 2\,t\,(\ln t)^2 - t + 1.$$

Now, $h_4(1) = h_4'(1) = h_4''(1) = h_4'''(1) = 0$ and

$$h_4^{(4)}(t) = \tfrac{8(t-1)\ln t}{t^3} > 0 \text{ for } t > 1.$$

Therefore $h_4 > 0$ in $(1,\infty)$ and $f_8' < 0$ in $(1,\infty)$. The function f_8 being strictly decreasing on $(1,\infty)$, $\inf f_8 = \lim_{t\to\infty} f_8(t) = 1/2$ and $\sup f_8 = \lim_{t\to 1} f_8(t) = 3/4$. □

3. FINAL REMARKS

From the eight means considered in this paper, two enter the class of Gini means [6] defined for $a, b > 0$, $u, v \in \mathbb{R}$,

$$G_{u,v}(a,b) = \begin{cases} \left(\tfrac{a^u+b^u}{a^v+b^v}\right)^{\frac{1}{u-v}}, & u \neq v \\ \exp\left(\tfrac{a^u \log a + b^u \log b}{a^u+b^u}\right), & u = v \end{cases}$$

namely $S = G_{1,1}$, $C = G_{2,1}$; two belong to the class of Stolarsky means [8] defined for $a, b > 0$, $a \neq b$, $u, v \in \mathbb{R}$,

$$E_{r,s}(a,b) = \begin{cases} \left(\frac{s}{r}\frac{a^r-b^r}{a^s-b^s}\right)^{\frac{1}{r-s}}, & rs(r-s) \neq 0 \\ \exp\left(-\frac{1}{r} + \frac{a^r \log a - b^r \log b}{a^r - b^r}\right), & r = s \neq 0 \\ \left(\frac{1}{s}\frac{a^s-b^s}{\log a - \log b}\right)^{\frac{1}{s}}, & r = 0,\ s \neq 0 \\ \sqrt{ab}, & r = s \neq 0, \end{cases}$$

namely $L = E_{1,0}$, $I = E_{1,1}$. The other four are in both classes, namely $H = G_{-1,0} = E_{-2,-1}$, $G = G_{0,0} = E_{0,0}$, $A = G_{1,0} = E_{2,1}$ and $Q = G_{2,0} = E_{4,2}$. As it was shown in [2], the families of Gini means $G_{u,v}$ and Stolarsky means $E_{r,s}$ have in common only the power means. So even if general results will be proved for these two classes of means, not all the inequalities from this paper will be consequences; for example, in the last theorem L is a Stolarsky mean, while C is a Gini one.

REFERENCES

[1] H. Alzer and S. L. Qiu, *Inequalities for means in two variables*, Arch. Math. (Basel), **80** (2003), pp. 201–215.

[2] H. Alzer and S. Ruscheweyh, *On the intersection of two-parameter mean value families*, Proc. A. M. S., **129**(9) (2001), pp. 2655–2662.

[3] M. C. Anisiu and V. Anisiu, *Refinement of some inequalities for means*, Rev. Anal. Numér. Théor. Approx., **35** (2006) no. 1, pp. 5–10.

[4] M. C. Anisiu and V. Anisiu, *Logarithmic mean and weighted sum of geometric and anti-harmonic means*, Rev. Anal. Numér. Théor. Approx., **41** (2012) no. 2, pp. 95–98.

[5] P. S. Bullen, *Handbook of Means and Their Inequalities*, Series: Mathematics and Its Applications, vol. 560, 2nd ed., Kluwer Academic Publishers Group, Dordrecht, 2003.

[6] C. Gini, *Di una formula comprensiva delle medie*, Metron, **13** (1938), pp. 3–22.

[7] M. Ivan and I. Raşa, *Some inequalities for means*, Tiberiu Popoviciu Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, May 23-29, 2000, pp. 99–102.

[8] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag., **48** (1975), pp. 87–92.

[9] W. F. Xia and Y. M. Chu, *Optimal inequalities related to the logarithmic, identric, arithmetic and harmonic means*, Rev. Anal. Numér. Théor. Approx., **39** (2010) no. 2, pp. 176–183.

Received by the editors: October 24, 2013.