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ON AN ITERATIVE ALGORITHM OF UĽM-TYPE FOR SOLVING EQUATIONS

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Abstract. We provide a semilocal convergence analysis of an iterative algorithm for solving nonlinear operator equations in a Banach space setting. This algorithm is of order 1.839..., and has already been studied in [3, 8, 18, 20]. Using our new idea of recurrent functions we show that a finer analysis is possible with sufficient convergence conditions that can be weaker than before, and under the same computational cost. Numerical examples are also provided in this study.

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1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation:

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on an open subset D of a Banach space X with values in a Banach space Y. Many problems in computational mathematics can be written in the form (1) [8, 14, 16]. Potra in [18] used the Uľm-type method [20] (UTM):

$$x_{n+1} = x_n - A_n^{-1}F(x_n) \quad (n \geq 0) \quad (x_{-2}, x_{-1}, x_0 \in D), \tag{2}$$

where,

$$A_n = [x_n, x_{n-1}; F] + [x_{n-2}, x_n; F] - [x_{n-2}, x_{n-1}; F], \tag{3}$$

to provide a local as well as a semilocal convergence analysis under hypotheses on the first $[\cdot,\cdot;F]$ and second $[\cdot,\cdot,\cdot;F]$ order divided differences of operator F.

Here, an operator belonging to the space $L(X, Y)$ (the Banach space of linear and bounded operators from X into Y) is called a divided difference of

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order one for the operator $F : X \to Y$ on the points $x, y \in X$ if the following properties hold:

(4) $$[x, y; F](y - x) = F(y) - F(x) \quad \text{for} \quad x \neq y;$$

if F is Fréchet-differentiable at $x \in X$, then

(5) $$[x, x; F] = F'(x).$$

An operator belonging to the space $L(X, L(X, Y))$ denoted by $[x, y, z; F]$ is called a divided difference of order two for the operator $F : X \to Y$ on the points $x, y, z \in X$ if:

(6) $$[x, y, z; F](z - x) = [y, z; F] - [x, y; F],$$

for the distinct points x, y, z if F is twice Fréchet-differentiable at $x \in X$, then

(7) $$[x, x, x; F] = \tfrac{1}{2} F''(x).$$

Potra showed that the $R-$order of the method is given by the positive solution of the scalar equations:

(8) $$t^3 - t^2 - t - 1 = 0,$$

which is approximately $1.839\ldots$. Other methods using divided differences of order can be found in [1-21], and references therein.

Here, we are motivated by optimization considerations, and we show that it is possible to provide under the same computational cost an analysis with the following advantages:

Semilocal case:

(a) finer error bounds on he distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ $(n \geq 0)$,
(b) weaker sufficient convergence conditions and,
(c) an at least as precise information on the location of the solution $x^\star$.

Local case:

(a) finer error bounds on the distances involved,
(b) and at least as large radius of convergence.

The semilocal convergence is provided in §2 followed by local in §3. Numerical examples are also provided in §4.

2. SEMILOCAL CONVERGENCE ANALYSIS FOR (UTM)

We need the following result on majorizing sequence for (UTM).

LEMMA 1. *Let $\alpha, \phi, \gamma, a, b, c, p,$ and q be given non-negative constants. Assume:*

(9) $$\phi + \gamma < p + q\, c,$$

then, the polynomial g given by

$$g(s) = (qc + \alpha + \phi)s^2 + (p + \alpha + \gamma)s + \phi + \gamma - p - qc, \tag{10}$$

has a unique positive root $\delta \in (0, 1)$*;*

moreover, suppose

$$\alpha(b + c) + \phi(a + c) + \gamma(a + b) < 1; \tag{11}$$

$$\delta_0 \leq \delta; \tag{12}$$

$$f_2(\delta) \leq 0; \tag{13}$$

where,

$$\delta_0 = \frac{pc+qb(a+b)}{1-[\alpha(b+c)+\phi(a+c)+\gamma(a+b)]}, \tag{14}$$

and

$$\begin{aligned} f_2(s) = psc + q(s+1)s^2 + \alpha\left[(1+s+s^2)c + (1+s)c + b\right] \\ + \phi\left[(1+s+s^2)c + a + b + c\right] + \gamma\left[(1+s)c + a + 2b + c\right] - 1. \end{aligned}$$

Then, scalar sequence $\{t_n\}$ $(n \geq -2)$ *given by*

$$t_{-2} = 0, \quad t_{-1} = a, \quad t_0 = a + b, \quad t_1 = a + b + c,$$
$$t_{n+2} = t_{n+1} + M_{n+1}\left(t_{n+1} - t_n\right),$$

where

$$M_{n+1} = \frac{p(t_{n+1}-t_n)+q(t_n-t_{n-2})(t_n-t_{n-1})/\mu_n}{\mu_{n+1}}, \tag{15}$$

is non-decreasing, bounded from above by

$$t^{\star\star} = \frac{c}{1-\delta} + a + b, \tag{16}$$

and converges to its unique least upper bound $t^\star$ *such that* $t \in [0, t^{\star\star}]$*. Moreover, the following estimates hold for all* $n \geq 0$ *:*

$$0 \leq t_{n+2} - t_{n+1} \leq \delta\left(t_{n+1} - t_n\right) \leq \delta^{n+1}c \quad (n \geq 0) \tag{17}$$

and

$$0 \leq t^\star - t_n \leq \frac{\delta^n c}{1-\delta} \quad (n \geq 1), \tag{18}$$

where,

$$\begin{aligned} \mu_{n+1} = 1 - [\alpha\left(t_{n+1} + t_n - 2a - b\right) + \phi\left(t_{n+1} + t_{n-1} - a - b\right) \\ + \gamma\left(t_n + t_{n-1} - a - b - c\right)]. \end{aligned} \tag{19}$$

Proof. We shall show using induction that

$$M_{n+1} \leq \delta \tag{20}$$

and

$$\mu_{n+1} < 1, \tag{21}$$

hold for all n. It will then follow that (2.9) also holds. Estimates (20) and (21) hold true for $n = 0$, by (2.3) and (2.4), respectively. Let us assume (17), (20) and (21) hold for all $k \leq n$. It then follows from the induction hypotheses:

$$\begin{aligned} t_{k+2} &\leq t_{k+1} + \delta\left(t_{k+1} - t_k\right) \leq t_k + \delta\left(t_k - t_{k-1}\right) + \delta\left(t_{k+1} - t_k\right) \\ &\leq t_1 + \delta\left(t_1 - t_0\right) + \cdots + \delta\left(t_{k+1} - t_k\right) \\ &\leq a + b + c + \delta c + \cdots + \delta^{k+1} c \\ &= a + b + \tfrac{1-\delta^{k+2}}{1-\delta} c < a + b + \tfrac{c}{1-\delta} = t^{\star\star}. \end{aligned}$$

Estimates (20) and (21) will be true if

$$\begin{aligned} p\delta^k c + q\delta^{k-1} c\left(\delta^{k-1} + \delta^{k-2}\right) c &\leq \delta - \delta\left[\alpha\left(t_{k+1} + t_k - t_0 - t_{-1}\right)\right. \\ &\left. + \phi\left(t_{k+1} + t_{k-1} - t_0 - t_{-2}\right) + \gamma\left(t_k + t_{k-1} - t_{-1} - t_{-2}\right)\right] \end{aligned}$$

or

$$\begin{aligned} & p\delta^k c + q\delta^{k-1}\left(\delta^{k-1} + \delta^{k-2}\right) c^2 + \delta\left[\alpha\left(\tfrac{1-\delta^{k+1}}{1-\delta} c + \tfrac{1-\delta^k}{1-\delta} c + b\right) + \right. \\ & \left. + \phi\left(\tfrac{1-\delta^{k+1}}{1-\delta} c + \tfrac{1-\delta^{k-1}}{1-\delta} + a + b\right) + \gamma\left(\tfrac{1-\delta}{1-\delta} c + \tfrac{1-\delta^{k-1}}{1-\delta} c + a + 2b\right)\right] - \delta \leq 0. \end{aligned} \tag{22}$$

Estimate (22) motivates us to introduce functions f_k $(k \geq 2)$ on $[0, \infty)$ (for $\delta = s$) by:

$$\begin{aligned} f_k(s) = {} & pcs^{k-1} + qc^2 s^{k-1}(s+1) + \alpha\Big[(1 + s + \cdots + s^k)c \\ & + (1 + s + \cdots + s^{k-1})c + b\Big] + \phi\Big[(1 + s + \cdots + s^k)c \\ & + (1 + s + \cdots + s^{k-2})c + a + b\Big] \\ & + \gamma\Big[(1 + s + \cdots + s^{k-1})c + (1 + s + \cdots + s^{k-2})c + a + 2b\Big] - 1. \end{aligned} \tag{23}$$

We shall show instead of (20) and (21) that

$$f_k(\delta) \leq 0. \tag{24}$$

We need a relationship between two consecutive functions f_k :

$$\begin{aligned}
f_{k+1}(s) &= pcs^k + pcs^{k-1} - pcs^{k-1} + qc^2 s^k (s+1) \\
&\quad + qc^2 s^{k-1} (s+1) - qc^2 s^{k-1} (s+1) \\
&\quad + \alpha \Big[(1 + s + \cdots + s^k + s^{k+1})c + (1 + s + \cdots + s^{k-1} + s^k)c + b \Big] \\
&\quad + \phi \Big[(1 + s + \cdots + s^k + s^{k+1})c \\
&\quad + (1 + s + \cdots + s^{k-2} + s^{k-1})c + a + b \Big] \\
&\quad + \gamma \Big[(1 + s + \cdots + s^k + s^{k+1})c \\
&\quad + (1 + s + \cdots + s^{k-2} + s^{k-1})c + a + 2b \Big] - 1 \\
&= f_k(s) + g(s) s^{k-1} c, \qquad (25)
\end{aligned}$$

where, function g is given by (10). In view of (10) and (25) we get

$$f_k(\delta) = f_2(\delta) \quad (k \geq 2). \qquad (26)$$

Hence, (24) is true if $f_2(\delta) \leq 0$, which is true by (13).

Define function f_∞ on $[0,1)$ by

$$f_\infty(s) = \lim_{n \to \infty} f_n(s). \qquad (27)$$

Then, we have by (24) and (27):

$$f_\infty(\infty) = \lim_{n \to \infty} f_n(\delta) \leq \lim_{n \to \infty} 0 = 0. \qquad (28)$$

The induction for (17), (20) and (21) is completed. Hence, we showed sequence $\{t_n\}$ is non-decreasing, bounded above by $t^{\star\star}$ and as such it converges to $t^\star$. Finally, estimate (18) follows from

$$\begin{aligned}
0 \leq t_{k+m} - t_n &= (t_{k+m} - t_{k+m-1}) \\
&\quad + (t_{k+m-1} - t_{k+m-2}) + \cdots + (t_{k+1} - t_k) \\
&\leq \left(\delta^{k+m-1} + \delta^{k+m-2} + \cdots + \delta^k \right) c = \tfrac{1-\delta^{k+m}}{1-\delta} \delta^k c, \qquad (29)
\end{aligned}$$

by letting $m \to \infty$. That completes the proof of the lemma. □

We can show the main semilocal convergence result for (UTM).

Theorem 2. *Let F be a nonlinear operator defined on an open subset D of a Banach space X with values in a Banach space Y. Let $[\cdot,\cdot;F]$, $[\cdot,\cdot,\cdot;F]$ be divided differences of first and second order of F on D, respectively. Let $x_{-2}, x_{-1}, x_0 \in D$ be three given points from D, and assume A_0 is invertible. Let*

$a,b,c,p,q,\alpha,\phi,\gamma$ be non-negative numbers such that for all $x,y,z,u,v \in D$:

$$\|x_{-1}-x_0\| \leq a, \quad \|x_{-2}-x_{-1}\| \leq b, \quad \left\|A_0^{-1}F(x_0)\right\| \leq c, \tag{30}$$

$$\left\|A_0^{-1}\left([x,y;F]-[u,v;F]\right)\right\| \leq p\left(\|x-u\|+\|y-v\|\right), \tag{31}$$

$$\left\|A_0^{-1}\left([x,y,z;F]-[u,v,z;F]\right)\right\| \leq q\,\|x-u\|, \tag{32}$$

$$\left\|A_0^{-1}\left([x,y;F]-[x_0,x_{-1},;F]\right)\right\| \leq \alpha\left(\|x-x_0\|+\|y-x_{-1}\|\right), \tag{33}$$

$$\left\|A_0^{-1}\left([x,y;F]-[x_{-2},x_0;F]\right)\right\| \leq \phi\left(\|x-x_{-2}\|+\|y-x_0\|\right), \tag{34}$$

$$\left\|A_0^{-1}\left([x,y;F]-[x_{-2},x_{-1};F]\right)\right\| \leq \gamma\left(\|x-x_{-2}\|+\|y-x_{-1}\|\right), \tag{35}$$

hypotheses of Lemma 1 hold and

$$\overline{U}(x_0,t^\star) := \{x \in X : \|x-x^\star\| \subseteq t^\star\} \subseteq D. \tag{36}$$

Then, sequence $\{x_n\}$ $(n \geq -2)$ generated by (UTM) is well defined, remains in $\overline{U}(x_0,t^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0,t^\star)$ of equation $F(x)=0$. Moreover, the following estimates hold for all $n \geq 0$:

$$\|x_{n+2}-x_{n+1}\| \leq L_{n+1}\|x_{n+1}-x_n\| \leq t_{n+2}-t_{n+1}, \tag{37}$$

$$\|x_n-x^\star\| \leq t^\star-t_n, \tag{38}$$

where

$$L_{n+1} = \frac{p\|x_{n+1}-x_n\|+q\|x_n-x_{n-2}\|\ \|x_n-x_{n-1}\|}{d_{n+1}},$$

$$d_{n+1} = 1-[\alpha\left(\|x_{n+1}-x_0\|+\|x_n-x_{-1}\|\right)+\phi\left(\|x_{n-1}-x_{-2}\|+\|x_{n+1}-x_0\|\right)$$
$$+\gamma\left(\|x_{n-1}-x_{-2}\|+\|x_n-x_{-1}\|\right)]. \tag{39}$$

Furthermore, if there exists $r \geq t^\star$ such that

$$U(x_0,r) \subseteq D \tag{40}$$

and

$$\phi(t^\star+r)+(\phi+\gamma)(a+b) \leq 1, \tag{41}$$

then, the solution $x^\star$ is unique in $U(x_0,r)$.

Proof. We shall show using induction on $k \geq 0$:

$$\|t_{k+1}-t_k\| \leq t_{k+1}-t_k. \tag{42}$$

Estimate (42) holds for $k=-2,-1,0$ by (14), and (30). We also have have $x_{-2},x_{-1},x_0 \in \overline{U}(x_0,t^\star)$. Let us assume (42), and $x_k \in \overline{U}(x_0,t^\star)$ hold for all

$n \le k+1$. We have using (33)-(35):

$$\begin{aligned}
&\left\|A_0{}^{-1}\left(A_{k+1}-A_0\right)\right\| = \\
&= \big\|A_0^{-1}\left([x_{k+1},x_k;F]-[x_0,x_{-1};F]\right) \\
&\quad +A_0^{-1}\left([x_{k-1},x_{k+1};F]-[x_{-2},x_0;F]\right)+A_0^{-1}\left([x_{k-1},x_k;F]-[x_{-2},x_{-1};F]\right)\big\| \\
&\le \alpha\left(\|x_{k+1}-x_0\|+\|x_k-x_{-1}\|\right)+\phi\left(\|x_{k-1}-x_{-2}\|+\|x_{k+1}-x_0\|\right) \\
&\quad +\gamma\left(\|x_{k-1}-x_{-2}\|+\|x_k-x_{-1}\|\right) \\
&\le \alpha\left(t_{k+1}+t_k-2a-b\right)+\phi\left(t_{k+1}+t_{k-1}-a-b\right)+ \\
&\qquad +\gamma\left(t_{k+1}+t_{k-1}-a-b-c\right)<1 \qquad \text{(by(22))}.
\end{aligned} \tag{43}$$

It follows from (43) and the Banach lemma on invertible operators [8, 14] that A_{k+1}^{-1} exists and

$$\begin{aligned}
\left\|A_{k+1}^{-1}A_0\right\| &\le d_{k+1}^{-1} \\
&\le \big\{1-[\alpha\left(t_{k+1}+t_k-2a-b\right)+\phi\left(t_{k+1}+t_{k-1}-a-b\right) \\
&\qquad +\gamma\left(t_{k+1}+t_{k-1}-a-b-c\right)]\big\}^{-1}
\end{aligned} \tag{44}$$

In view of (2), (44), (45), we have:

$$\begin{aligned}
\|x_{k+2}-x_{k+1}\| &= \left\|\left(A_{k+1}^{-1}A_0\right)\left(A_0^{-1}F(x_{k+1})\right)\right\| \\
&\le \left\|A_{k+1}^{-1}A_0\right\|\left\|A_0^{-1}F(x_{k+1})\right\| \\
&\le L_{k+1}\|x_{k+1}-x_k\| \le t_{k+2}-t_{k+1},
\end{aligned}$$

which shows (37) and (42) for all n. By Lemma 2.1, sequence $\{x_n\}$ is Cauchy in a Banach space X and as such it converges to some $x^\star \in \overline{U}(x_0,t^\star)$ (since $\overline{U}(x_0,t^\star)$ is a closed set). Estimate (38) follows from (37) by using standard majorization techniques [8, 14, 16].

Using (31), (32) and the induction hypotheses, we obtain in turn

$$\begin{aligned}
&\left\|A_0^{-1}\left([x_k,x_{k+1};F]-A_k\right)\right\| = \\
&= \left\|A_0^{-1}\left([x_k,x_{k+1};F]-[x_k,x_k;F]+[x_k,x_k;F]-A_k\right)\right\| \\
&= \Big\|A_0^{-1}\left\{[x_k,x_{k+1};F]-[x_k,x_k;F]-([x_k,x_k,x_{k-1};F]\right. \\
&\qquad \left.-[x_{k-2},x_k,x_{k-1};F])\left(x_k-x_{k-1}\right)\right\}\Big\| \\
&\le p\|x_{k+1}-x_k\|+q\|x_k-x_{k-2}\|\|x_k-x_{k-1}\| \\
&\le p\left(t_{k+1}-t_k\right)+q\left(t_k-t_{k-2}\right)\left(t_k-t_{k-1}\right).
\end{aligned} \tag{45}$$

We also need the estimate:

$$\begin{aligned} \|x_{k+2}-x_{k+1}\| &= \left\|\left(A_{k+1}^{-1}A_0\right)\left(A_0^{-1}F\left(x_{k+1}\right)\right)\right\| \\ &= \left\|\left(A_{k+1}^{-1}A_0\right)A_0^{-1}\left(F\left(x_{k+1}\right)-F\left(x_k\right)-A_k\left(x_{k+1}-x_k\right)\right)\right\| \\ &\leq \left\|A_{k+1}^{-1}A_0\right\|\left\|A_0^{-1}\left(\left[x_k,x_{k+1};F\right]-A_k\right)\right\|\left\|x_{k+1}-x_k\right\|. \end{aligned} \tag{46}$$

The fact that $x^\star$ is a solution of equation $F(x)=0$ follows by letting $k\to\infty$ in the estimate:

$$\begin{aligned} \left\|A_0^{-1}F\left(x_{k+1}\right)\right\| &= \left\|A_0^{-1}\left(\left[x_{k,k+1};F\right]-A_k\right)\left(x_{k+1}-x_k\right)\right\| \\ &\leq p\left\|x_{k+1}-x_k\right\|^2+q\left\|x_k-x_{k-2}\right\|\left\|x_k-x_{k-1}\right\|\left\|x_{k+1}-x_k\right\|. \end{aligned} \tag{47}$$

Finally, to show the uniqueness part, let $y^\star\in U(x_0,r)$ be a solution of equation $F(x)=0$. We can write for $L=[y^\star,x^\star;F]$:

$$F(y^\star)-F(x^\star)=L\left(y^\star-x^\star\right). \tag{48}$$

We shall show linear operator L is invertible. Using (33)-(35), (40) and (41), we have:

$$\begin{aligned} &\left\|A_0^{-1}\left(\left[x_0,x_{-2};F\right]+\left[x_{-1},x_0;F\right]-\left[y^\star,x^\star;F\right]-\left[x_{-2},x_{-1};F\right]\right)\right\| \leq \\ &\leq \left\|A_0^{-1}\left(\left[x_0,x_{-2};F\right]-\left[y^\star,x^\star;F\right]\right)\right\| \\ &\quad+\left\|A_0^{-1}\left(\left[x_{-1},x_0;F\right]-\left[x_{-2},x_{-1};F\right]\right)\right\| \\ &\leq \phi\left(\|x_0-y^\star\|+\|x_{-2}-x^\star\|\right)+\gamma\left(\|x_{-1}-x_{-2}\|+\|x_0-x_{-1}\|\right) \\ &< \phi\left(r+t^\star+b+a\right)+\gamma\left(b+a\right)\leq 1. \end{aligned} \tag{49}$$

In view of (49) and the Banach lemma on invertible operators, L^{-1} exists. We deduce from (48) that $x^\star=y^\star$. That completes the proof of the Theorem. □

Remark 3. (*a*) A similar existence Theorem (without a uniqueness result) was provided in [18, p.91] using conditions (30)-(31), a decreasing majorizing sequence, and some different sufficient convergence conditions. Therefore a direct comparison is not possible. However, in §4, we show that the results obtained in Theorem 2.2 can be weaker than the corresponding ones of Theorem 5.1 in [18, p.91].

(*b*) Note that $t^{\star\star}$ given by (16) can replace $t^\star$ in hypotheses (36) and (41) of Theorem 2.2. □

3. LOCAL CONVERGENCE OF (UTM)

We can show the local convergence result for (UTM).

Theorem 4. *Let $F\colon D\subseteq X\to Y$ and let $x^\star\in D$ be such that $F'(x^\star)^{-1}$ exists. Assume that for all $x,y,u,v,z\in D$:*

$$\left\|F'(x^\star)^{-1}\left(\left[x^\star,x^\star;F\right]-\left[x,x^\star;F\right]\right)\right\|\leq p_0\left\|x^\star-x\right\|, \tag{50}$$

$$\left\|F'(x^\star)^{-1}\left(\left[z,x^\star;F\right]-\left[z,x;F\right]\right)\right\|\leq p_1\left\|x^\star-x\right\|, \tag{51}$$

$$\left\| F'(x^\star)^{-1}\left([x,x^\star,y;F]-[z,x^\star,y;F]\right)\right\| \le q_0 \left\| x-z \right\|, \tag{52}$$

$$\left\| F'(x^\star)^{-1}\left([u,x,y;F]-[v,x,y;F]\right)\right\| \le q_\star \left\| u-v \right\| \tag{53}$$

and

$$U\left(x^\star,r^\star\right)\subseteq D, \tag{54}$$

where

$$r^\star=\frac{2}{p_0+2p_1+\sqrt{(p_0+2p_1)^2+16(q_0+q_\star)}}. \tag{55}$$

Then, sequence $\{x_n\}$ *generated by (UTM) is well defined, remains in* $U\left(x^\star,r^\star\right)$ *for all* $n\ge 0$ *and converges to* $x^\star$, *provided that* $x_0\in U(x^\star,r^\star)$. *Moreover, the following estimates hold:*

$$\left\| x_{n+1}-x^\star\right\| \le \frac{e_n}{h_n}\left\| x_n-x^\star\right\|, \tag{56}$$

where

$$e_n=p_1\left\| x_n-x^\star\right\| + q_\star\left\| x_n-x_{n-2}\right\| \left\| x_n-x_{n-1}\right\| \tag{57}$$

and

$$h_n=1-(p_0+p_1)\left\| x_n-x^\star\right\| - q_0\left\| x_n-x_{n-2}\right\| \left\| x_{n-1}-x^\star\right\|. \tag{58}$$

Proof. It follows as the proof of Theorem 4.1 in [8, p.87] but uses the needed conditions (50)-(53) instead of:

$$\left\| F'(x^\star)^{-1}\left([x,y;F]-[u,v;F]\right)\right\| \le p_\star\left(\left\| x-u\right\| + \left\| y-v\right\|\right) \tag{59}$$

and

$$\left\| F'(x^\star)^{-1}\left([u,x,y;F]-[v,x,y;F]\right)\right\| \le q_\star \left\| u-v \right\|. \tag{60}$$

□

Remark 5. (a) Clearly

$$p_0\le p_\star, \tag{61}$$

$$p_1\le p_\star, \tag{62}$$

$$q_0\le q_\star, \tag{63}$$

hold in general, and $p_\star/p_0$, $p_\star/p_1$ and $q_\star/q_0$ can be arbitrarily large [7, 8]. If equality holds in (61)-(63), then our results reduce to the ones in [18]. Otherwise they constitute an improvement with advantages as noted in the Introduction of this study. (b) The radius of convergence $r^\star$ obtained in Theorem 3.1 is smaller in general than the corresponding one of Newton's method. Indeed from the hypotheses (59) it follows that F is Fréchet-differentiable on D and its Fréchet derivative satisfies

$$\left\| F'(x^\star)^{-1}\left(F'(x)-F'(y)\right)\right\| \le 2p_\star\left\| x-y\right\| \tag{64}$$

and

$$\left\| F'(x^\star)^{-1}\left(F'(x) - F'(x^\star)\right)\right\| \leq 2p_2 \left\| x - x^\star \right\|. \tag{65}$$

The radius of convergence $r^\star$ is then given for $q_0 = q_\star = 0$:

$$r_A^\star = \tfrac{1}{2p_2 + p_\star} \tag{66}$$

for

$$p_2 \leq p_\star, \tag{67}$$

whereas the one obtained by Theorem 4.1 in [18] is given by

$$r_R^\star = \tfrac{1}{3p_\star}, \tag{68}$$

found by Rheinboldt in [16]. Note that

$$r_R^\star \leq r_A^\star. \tag{69}$$

If strict inequality holds in (67), then so does in (69). □

4. A NUMERICAL EXAMPLE

We provide a numerical example to show that Theorem 2.2 can be used to solve equation (1) but not corresponding Theorem 5.1 in [18]. Let $X = Y = R^2$ be equipped with the max-norm, $x_0 = (1, 1)^T$, $D = U(x_0, 1 - \lambda)$, $\lambda \in [0, 1/2)$ and define function F on D for $x = (\mu_1, \mu_2)$ by

$$F(x) = \left(\mu_1^3 - \lambda, \mu_2^3 - \lambda\right)^T. \tag{70}$$

Using (70) we obtain the Fréchet-derivative

$$F'(x) = \begin{bmatrix} 3\mu_1^2 & 0 \\ 0 & 3\mu_2^2 \end{bmatrix}. \tag{71}$$

Define

$$[x, y; F] = \int_0^1 F'(y + t(x - y))\mathrm{d}t. \tag{72}$$

Let $x_{-2} = (1.02, 1.02)^T$, $x_{-1} = (1.01, 1.01)$, $\lambda = 0.49$. Using (30)-(35) and (72), we get

$$a = b = 0.1, \quad c = 0.170011334, \quad \left\| {A_0}^{-1} \right\| = 0.33335557,$$
$$p = 3\left\| A_0^{-1} \right\|(2 - \lambda), \quad q = 3\left\| A_0^{-1} \right\|,$$
$$\alpha = \tfrac{1}{2}\left(1 - \lambda + a + \left\| x_0 + x_{-1} \right\|\right)\left\| A_0^{-1} \right\|,$$
$$\phi = \tfrac{1}{2}\left(1 - \lambda + 2\left\| x_0 \right\| + \left\| x_{-2} - x_0 \right\|\right)\left\| A_0^{-1} \right\|,$$
$$\gamma = \tfrac{1}{2}\left(1 - \lambda + b + \left\| x_0 + x_{-1} \right\|\right)\left\| A_0^{-1} \right\|,$$

so

$$p = 1.510100673, \quad q = 1.000066671, \quad \alpha = \phi = \gamma = 0.42169478.$$

Moreover, using (10)-(13) and (16), we obtain (9), (11) become

$$0.84338956 < 1.680123342,$$
$$0.160253576 < 1,$$

respectively,

$$\delta_0 = 0.305966463, \quad \delta = 0.310470973 > \delta_0, \quad f_2(\delta) = -0.310669379 < 0$$

and

$$t^{\star\star} = 0.267566675.$$

That is, the hypotheses of Theorem 2.2 are satisfied. Moreover, using Remark 2.3(b) and (41), we see that we can set $r = 1 - \lambda = 0.51$. Hence, there exists a unique solution:

$$x^\star = \left(\sqrt[3]{0.49}, \sqrt[3]{0.49}\right)^T = (0.788373516, 0.78373516)^T$$

of equation $F(x) = 0$ in $\overline{U}(x_0, r)$, which can be obtained as the limit of (UTM). However, hypotheses of Theorem 5.1 in [18] do not hold. The sufficient convergence condition corresponding to (11) is given by

$$c \leq \lambda = \tfrac{1}{3} \tfrac{p+qa+2\lambda_0}{(p+qa+\lambda_0)^2} \left[1 - qa(a+b)\right]^2, \tag{73}$$

where

$$\lambda_0 = \left\{(p+qa)^2 + 3q\left(1 - qa(a+b)\right)\right\}^{1/2}.$$

We have

$$\lambda_0 = 1.520496025, \quad \lambda = 0.08533979$$

and, so (73) is violated, since

$$c = 0.170011334 > \lambda = 0.08533979.$$

Hence, there is no guarantee that (UTM) starting at x_0 converges to $x^\star$.

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