SHARP INEQUALITIES FOR THE NEUMAN-SÁNDOR MEAN IN TERMS OF ARITHMETIC AND CONTRA-HARMONIC MEANS

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Abstract. In this paper, we find the greatest values \( \alpha \) and \( \lambda \), and the least values \( \beta \) and \( \mu \) such that the double inequalities

\[
C^\alpha(a,b)A^{1-\alpha}(a,b) < M(a,b) < C^\beta(a,b)A^{1-\beta}(a,b)
\]

and

\[
[C(a,b)/6 + 5A(a,b)/6]^\lambda \left[ C^{1/6}(a,b)A^{5/6}(a,b) \right]^{1-\lambda} < M(a,b) <
\]

\[
[C(a,b)/6 + 5A(a,b)/6]^{\mu} \left[ C^{1/6}(a,b)A^{5/6}(a,b) \right]^{1-\mu}
\]

hold for all \( a, b > 0 \) with \( a \neq b \), where \( M(a,b) \), \( A(a,b) \) and \( C(a,b) \) denote the Neuman-Sándor, arithmetic, and contra-harmonic means of \( a \) and \( b \), respectively.

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1. INTRODUCTION

For \( a, b > 0 \) with \( a \neq b \) the Neuman-Sándor mean \( M(a,b) \) [1] is defined by

\[
M(a,b) = \frac{a-b}{2\arcsinh|(a-b)/(a+b)|},
\]

where \( \arcsinh(x) = \log(x + \sqrt{1 + x^2}) \) is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean \( M(a,b) \) can be found in the literature [1]–[4].

Let \( A(a,b) = (a+b)/2 \) and \( C(a,b) = (a^2+b^2)/(a+b) \) be the arithmetic and contra-harmonic means of \( a \) and \( b \), respectively. Then from [1], [2] we clearly see that the double inequality

\[
A(a,b) < M(a,b) < C(a,b)
\]
also strict. If 
\[ f_n(x) = \frac{a_n x^n}{g(x)^n} \]
for all \( n \geq 0 \) with \( a 
eq b \).

In [4], Neuman proved that the double inequality
\[
(1.1) \quad \alpha C(a, b) + (1 - \alpha) A(a, b) < M(a, b) < \beta C(a, b) + (1 - \beta) A(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \leq \frac{1}{1 + \sqrt{2}} \) and \( \beta \geq \frac{1}{1 + \sqrt{2}} \).

The main purpose of this paper is to give some refinements and improvements for inequalities (1.1) and (1.2). Our main results are the following Theorems 1.1 and 1.2.

**Theorem 1.1.** The double inequality
\[
C^\alpha(a, b) A^{1-\alpha}(a, b) < M(a, b) < C^\beta(a, b) A^{1-\beta}(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha \leq \alpha_0 \) and \( \beta \geq \beta_0 \).

**Theorem 1.2.** The double inequality
\[
[C(a, b)/6 + 5A(a, b)/6]^\lambda \left[ C^{1/6}(a, b) A^{5/6}(a, b) \right]^{1-\lambda} < M(a, b)
\]
holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \lambda \leq -\frac{6 \log(1 + \sqrt{2}) + \log 2}{6 \log(7/6) - \log 2} \).

### 2. LEMMAS

In order to prove our main results we need three lemmas, which we present in this section.

**Lemma 2.1.** (See [5] Theorem 1.25). For \( -\infty < a < b < \infty \), let \( f, g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), and be differentiable on \((a, b)\), let \( g'(x) \neq 0 \) on \((a, b)\). If \( f'(x) / g'(x) \) is increasing (decreasing) on \((a, b)\), then so are
\[
\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.
\]
If \( f'(x) / g'(x) \) is strictly monotone, then the monotonicity in the conclusion is also strict.

**Lemma 2.2.** (See [6] Lemma 1.1). Suppose that the power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) have the radius of convergence \( r > 0 \) and \( b_n > 0 \) for all \( n \in \{0, 1, 2, \cdots\} \). Let \( h(x) = f(x) / g(x) \), then the following statements are true:
(1) If the sequence \( \{a_n/b_n\}_{n=0}^\infty \) is (strictly) increasing (decreasing), then \( h(x) \) is also (strictly) increasing (decreasing) on \((0, r)\);

(2) If the sequence \( \{a_n/b_n\} \) is (strictly) increasing (decreasing) for \( 0 < n \leq n_0 \) and (strictly) decreasing (increasing) for \( n > n_0 \), then there exists \( x_0 \in (0, r) \) such that \( h(x) \) is (strictly) increasing (decreasing) on \((0, x_0)\) and (strictly) decreasing (increasing) on \((x_0, r)\).

**Lemma 2.3.** The function

\[
(2.1) \\
h(t) = \frac{90t + 52t \cosh(2t) - 66 \sinh(2t) + 2t \cosh(4t) - 3 \sinh(4t)}{15t - 26t \cosh(2t) + 5t \cosh(4t)}
\]

is strictly decreasing on \((0, \log(1 + \sqrt{2}))\), where \( \sinh(t) = (e^t - e^{-t})/2 \) and \( \cosh(t) = (e^t + e^{-t})/2 \) are the hyperbolic sine and cosine functions, respectively.

**Proof.** Let

\[
(2.2) \\
h_1(t) = 90t + 52t \cosh(2t) - 66 \sinh(2t) + 2t \cosh(4t) - 3 \sinh(4t),
\]

\[
(2.3) \\
h_2(t) = 15t - 20t \cosh(2t) + 5t \cosh(4t).
\]

Then making use of power series formulas we have

\[
(2.4) \\
h_1(t) = 90t + 52t \sum_{n=0}^\infty \frac{(2t)^{2n}}{(2n)!} - 66 \sum_{n=0}^\infty \frac{(2t)^{2n+1}}{(2n+1)!} + 2t \sum_{n=2}^\infty \frac{(4t)^{2n}}{(2n)!} - 3 \sum_{n=0}^\infty \frac{(4t)^{2n+1}}{(2n+1)!}
\]

\[
= 52t \sum_{n=2}^\infty \frac{(2t)^{2n}}{(2n)!} - 66 \sum_{n=2}^\infty \frac{(2t)^{2n+1}}{(2n+1)!} + 2t \sum_{n=2}^\infty \frac{(4t)^{2n}}{(2n)!}
\]

\[
- 3 \sum_{n=2}^\infty \frac{(4t)^{2n+1}}{(2n+1)!}
\]

\[
= \sum_{n=0}^\infty \frac{[16t + 13n + (2n-1)2^{2n+2} + 2^{2n+7}]t^{2n+5}}{(2n+5)!}
\]

and

\[
(2.5) \\
h_2(t) = 15t - 20t \sum_{n=0}^\infty \frac{(2t)^{2n}}{(2n)!} + 5t \sum_{n=0}^\infty \frac{(4t)^{2n}}{(2n)!}
\]

\[
= - 20t \sum_{n=2}^\infty \frac{(2t)^{2n}}{(2n)!} + 5t \sum_{n=2}^\infty \frac{(4t)^{2n}}{(2n)!} = \sum_{n=0}^\infty \frac{5(2^{2n+2} - 1)2^{2n+6}t^{2n+5}}{(2n+4)!}
\]

It follows from (2.1)-(2.5) that

\[
(2.6) \\
h(t) = \frac{\sum_{n=0}^\infty \frac{a_n t^{2n}}{\sum_{n=0}^\infty b_n t^{2n}}}{\sum_{n=0}^\infty b_n t^{2n}}.
\]
where
\begin{align*}
    a_n &= \frac{16 + 13n + (2n-1)2^{2n+1}2^{2n+7}}{(2n+3)!}, \\
    b_n &= \frac{5(2^{2n+2} - 1)2^{2n+6}}{(2n+4)!}.
\end{align*}

Equation (2.7) leads to
\begin{equation}
    \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{6c_n}{5(2n+5)(2n+7)(2^{2n+2} - 1)(2^{2n+4} - 1)},
\end{equation}
where
\begin{equation}
    c_n = (30n^2 + 135n + 110 - 4^{n+3})4^{n+1} + 11.
\end{equation}

From (2.9) we get
\begin{align*}
    c_0 &= 195, \\
    c_1 &= 315, \\
    c_2 &= -33525
\end{align*}
and
\begin{equation}
    c_n < (30n^2 + 135n + 110 - 64n^3)4^{n+1} + 11
\end{equation}
for \( n \geq 3 \).

Equations (2.8) and (2.10) together with inequality (2.11) lead to the conclusion that the sequence \( \{a_n/b_n\} \) is strictly decreasing for \( 0 \leq n \leq 2 \) and strictly increasing for \( n \geq 3 \). Then from Lemma 2.2(2) and (2.6) we clearly see that there exists \( t_0 \in (0, \infty) \) such that \( h(t) \) is strictly decreasing on \( (0, t_0) \) and strictly increasing on \( (t_0, \infty) \).

Let \( t^* = \log(1 + \sqrt{2}) \). Then simple computations lead to
\begin{equation}
\begin{align*}
    \sinh(2t^*) &= 2\sqrt{2}, \\
    \cosh(2t^*) &= 3, \\
    \sinh(4t^*) &= 12\sqrt{2}, \\
    \cosh(4t^*) &= 17.
\end{align*}
\end{equation}

Differentiating (2.1) yields
\begin{equation}
    h'(t) = \frac{90 - 80 \cosh(2t) + 104t \sinh(2t) - 10 \cosh(4t) + 8t \sinh(4t)}{h_2(t)}
\end{equation}
\begin{equation}
\begin{align*}
    &- \frac{15 - 20 \cosh(2t) - 40 \sinh(2t) + 5 \cosh(4t) + 20t \sinh(4t)}{h_2(t)^2} h_1(t).
\end{align*}
\end{equation}

From (2.2) and (2.3) together with (2.12) and (2.13) we get
\begin{equation}
    h'(t^*) = \frac{-102\sqrt{2}t^* + 93t^* + 21\sqrt{2}}{5t^*} = -0.10035 \cdots < 0.
\end{equation}

From the piecewise monotonicity of \( h(t) \) and inequality (2.14) we clearly see that \( t_0 > t^* = \log(1 + \sqrt{2}) \), and the proof of Lemma 2.3 is completed.

3. PROOFS OF THEOREMS 1.1 AND 1.2

3.1. Proof of Theorem 1.1 Since \( M(a, b), C(a, b) \) and \( A(a, b) \) are symmetric and homogeneous of degree 1. Without loss of generality, we assume that \( a > b \). Let \( x = (a - b)/(a + b) \) and \( t = \text{arcsinh}(x) \). Then \( x \in (0, 1), t \in (0, \log(1 + \sqrt{2})) \) and
\begin{equation}
\begin{align*}
    \log M(a, b) - \log A(a, b) &= \log \frac{x}{\text{arcsinh}(x)} \\
    \log C(a, b) - \log A(a, b) &= \log \frac{\sinh(t)/t}{\cosh(t)}.
\end{align*}
\end{equation}
Let \( f_1(t) = \log[\sinh(t)/t] \), \( f_2(t) = \log[\cosh(t)] \) and
\[
(3.2) \quad f(t) = \frac{\log[\sinh(t)/t]}{\log[\cosh(t)]}.
\]
Then \( f_1(0^+) = f_2(0) = 0 \), \( f(t) = f_1(t)/f_2(t) \) and
\[
(3.3) \quad \frac{f_1'(t)}{f_2'(t)} = \frac{t \cosh^2(t) - \sinh(t) \cosh(t)}{t \sinh^2(t)} = \frac{t(\cosh(2t) + 1) - \sinh(2t)}{t(\cosh(2t) - 1)} = \frac{t}{2} \left( \frac{\sum_{n=0}^{\infty} 2^{2n} e^{2n}/(2n)!}{t \sum_{n=1}^{\infty} 2^{2n} e^{2n}/(2n)!} \right) = \frac{A_n e^{2n}}{B_n e^{2n}},
\]
where \( A_n = 2^{2n+2}(2n+1)/(2n+3) \) and \( B_n = 2^{2n+2}/(2n+2) \).

Note the \( A_n/B_n = 1 - 2/(2n+3) \) is strictly increasing for all \( n \geq 0 \).
Then from Lemma 2.2(1) and (3.3) we know that \( f_1'(t)/f_2'(t) \) is strictly increasing on \( (0, \infty) \). Hence, \( f(t) \) is strictly increasing on \( (0, \log(1 + \sqrt{2})) \) follows from Lemma 2.1 and the monotonicity of \( f_1'(t)/f_2'(t) \) together with \( f(0^+) = f_2(0) = 0 \). Moreover,
\[
(3.4) \quad \lim_{t \to 0^+} f(t) = \frac{A_0}{B_0} = \frac{1}{3},
\]
\[
(3.5) \quad \lim_{t \to \log(1 + \sqrt{2})} f(t) = -\frac{2 \log(1 + \sqrt{2})}{\log x}.
\]

Therefore, Theorem 1.1 follows easily from (3.1), (3.2), (3.4) and (3.5) together with the monotonicity of \( f(t) \). \( \square \)

### 3.2. Proof of Theorem 1.2

Since \( M(a, b) \), \( C(a, b) \) and \( A(a, b) \) are symmetric and homogeneous of degree 1. Without loss of generality, we assume that \( a > b \). Let \( x = (a - b)/(a + b) \) and \( t = \arcsinh(x) \). Then \( x \in (0, 1), t \in (0, \log(1 + \sqrt{2})) \) and
\[
(3.6) \quad \frac{\log M(a, b) - \log [C^{1/6}(a, b) A^{5/6}(a, b)]}{\log [C(a, b) + 5A(a, b)/6] - \log [C^{1/6}(a, b) A^{5/6}(a, b)]} = \frac{\log(x/\arcsinh(x)) - \log(1 + x^2)^{1/6}}{\log(1 + x^2)^{1/6} - \log(1 + x^2)^{1/6}} = \frac{\log(\sinh(t)/t) - \log \cosh(t) + 3/6}{\log[1 + \sinh^2(t)/6] - \log \cosh(t)/3}.
\]

Let \( g_1(t) = \log[\sinh(t)/t] - \log \cosh(t) + 3/6 \), \( g_2(t) = \log[1 + \sinh^2(t)/6] - \log \cosh(t)/3 \) and
\[
(3.7) \quad g(t) = \frac{\log[\sinh(t)/t] - \log \cosh(t) + 3/6}{\log[1 + \sinh^2(t)/6] - \log \cosh(t)/3}.
\]

Then \( g_1(0^+) = g_2(0) = 0 \), \( g(t) = g_1(t)/g_2(t) \) and
\[
\frac{g_1'(t)}{g_2'(t)} = \frac{[6 + \sinh^2(t)][3t \cosh^2(t) - 3 \cosh(t) \sinh(t) - t \sinh^2(t)]}{t \sinh(t) [6 \sinh(t) \cosh(t) - \sinh(t) [6 + \sinh^2(t)]]}.
\]
Elementary computations lead to
\[
[6 + \sinh^2(t)] [3t \cosh^2(t) - 3 \cosh(t) \sinh(t) - t \sinh^2(t)] = \\
= \frac{45}{4} t + \frac{13}{2} t \cosh(2t) - \frac{33}{4} \sinh(2t) + \frac{t}{2} \cosh(4t) - \frac{5}{8} \sinh(4t),
\]
and
\[
t \sinh(t) [6 \sinh(t) \cosh^2(t) - \sinh(t)(6 + \sinh^2(t))] = \\
= \frac{15}{8} t - \frac{5}{2} t \cosh(2t) + \frac{5}{8} t \cosh(4t)
\]

and
\[
g_1'(t) g_2(t) = h(t),
\]
where \( h(t) \) is defined as in Lemma 2.3.

It follows from Lemmas 2.1 and 2.3 and (3.8) together with \( g_1(0^+) = g_2(0) = 0 \) that \( g(t) \) is strictly decreasing on \((0, \log(1 + \sqrt{2}))\). Moreover,
\[
\lim_{t \to 0} g(t) = \frac{8}{25},
\]
\[
\lim_{t \to \log(1 + \sqrt{2})} g(t) = -\frac{6 \log(\log(1 + \sqrt{2})) + \log 2}{6 \log(7/6) - \log 2}.
\]

Therefore, Theorem 1.2 follows easily from (3.6), (3.7), (3.9) and (3.10) together with the monotonicity of \( g(t) \).

REFERENCES


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