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SHARP INEQUALITIES FOR THE NEUMAN-SÁNDOR MEAN IN TERMS OF ARITHMETIC AND CONTRA-HARMONIC MEANS[‡]

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Abstract. In this paper, we find the greatest values α and λ, and the least values β and μ such that the double inequalities

$$C^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < C^{\beta}(a,b)A^{1-\beta}(a,b)$$

and

$$[C(a,b)/6+5A(a,b)/6]^{\lambda}\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]^{1-\lambda} < M(a,b) <$$
$$< [C(a,b)/6+5A(a,b)/6]^{\mu}\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]^{1-\mu}$$

hold for all $a,b>0$ with $a\neq b$, where $M(a,b)$, $A(a,b)$ and $C(a,b)$ denote the Neuman-Sándor, arithmetic, and contra-harmonic means of a and b, respectively.

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1. INTRODUCTION

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\operatorname{arcsinh}[(a-b)/(a+b)]},$$

where $\operatorname{arcsinh}(x)=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1]–[4].

Let $A(a,b)=(a+b)/2$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the arithmetic and contra-harmonic means of a and b, respectively. Then from [1], [2] we clearly see that the double inequality

$$A(a,b) < M(a,b) < C(a,b)$$

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holds for all $a, b > 0$ with $a \neq b$.

In [4], Neuman proved that the double inequality

$$\alpha C(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta C(a,b) + (1-\beta)A(a,b) \tag{1.1}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \left(1 - \log(\sqrt{2}+1)\right)/\log(\sqrt{2}+1) = 0.1345\cdots$ and $\beta \geq 1/6$, and the inequality

$$C^{\lambda}(a,b)A^{1-\lambda}(a,b) < M(a,b) < C^{\mu}(a,b)A^{1-\mu}(a,b) \tag{1.2}$$

holds true for all $a, b > 0$ with $a \neq b$ if $\mu \geq \log\left((\sqrt{2}+2)/3\right)/\log 2 = 0.1865\cdots$ and $\lambda \leq 1/6$.

The main purpose of this paper is to give some refinements and improvements for inequalities (1.1) and (1.2). Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. *The double inequality*

$$C^{\alpha}(a,b)A^{1-\alpha}(a,b) < M(a,b) < C^{\beta}(a,b)A^{1-\beta}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 1/6$ and $\beta \geq -\log(\log(1+\sqrt{2}))/\log 2 = 0.1821\cdots$.

Theorem 1.2. *The double inequality*

$$\begin{aligned}[C(a,b)/6 + 5A(a,b)/6]^{\lambda}\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]^{1-\lambda} &< M(a,b)\\ < [C(a,b)/6 + 5A(a,b)/6]^{\mu}\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]^{1-\mu}&\end{aligned}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq -[6\log(\log(1+\sqrt{2})) + \log 2]/[6\log(7/6) - \log 2] = 0.27828\cdots$ and $\mu \geq 8/25$.

2. LEMMAS

In order to prove our main results we need three lemmas, which we present in this section.

Lemma 2.1. *(See* [5, Theorem 1.25]*). For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *(See* [6, Lemma 1.1]*). Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $b_n > 0$ for all $n \in \{0, 1, 2, \cdots\}$. Let $h(x) = f(x)/g(x)$, then the following statements are true:*

(1) If the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing), then* $h(x)$ *is also (strictly) increasing (decreasing) on* $(0,r)$*;*

(2) If the sequence $\{a_n/b_n\}$ *is (strictly) increasing (decreasing) for* $0 < n \le n_0$ *and (strictly) decreasing (increasing) for* $n > n_0$*, then there exists* $x_0 \in (0,r)$ *such that* $h(x)$ *is (strictly) increasing (decreasing) on* $(0,x_0)$ *and (strictly) decreasing (increasing) on* (x_0,r)*.*

LEMMA 2.3. *The function*

$$h(t) = \frac{90t+52t\cosh(2t)-66\sinh(2t)+2t\cosh(4t)-3\sinh(4t)}{15t-20t\cosh(2t)+5t\cosh(4t)} \tag{2.1}$$

is strictly decreasing on $(0,\log(1+\sqrt{2}))$*, where* $\sinh(t) = (\mathrm{e}^t - \mathrm{e}^{-t})/2$ *and* $\cosh(t) = (\mathrm{e}^t+\mathrm{e}^{-t})/2$ *are the hyperbolic sine and cosine functions, respectively.*

Proof. Let

$$h_1(t) = 90t + 52t\cosh(2t) - 66\sinh(2t) + 2t\cosh(4t) - 3\sinh(4t), \tag{2.2}$$

$$h_2(t) = 15t - 20t\cosh(2t) + 5t\cosh(4t). \tag{2.3}$$

Then making use of power series formulas we have

$$\begin{aligned} h_1(t) =& 90t + 52t\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!} - 66\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!} \\ &+ 2t\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!} - 3\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!} \\ =& 52t\sum_{n=2}^{\infty}\frac{(2t)^{2n}}{(2n)!} - 66\sum_{n=2}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!} + 2t\sum_{n=2}^{\infty}\frac{(4t)^{2n}}{(2n)!} \\ &- 3\sum_{n=2}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!} \\ =& \sum_{n=0}^{\infty}\frac{[16+13n+(2n-1)2^{2n+2}]2^{2n+7}}{(2n+5)!}t^{2n+5} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} h_2(t) =& 15t - 20t\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!} + 5t\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n)!} \\ =& -20t\sum_{n=2}^{\infty}\frac{(2t)^{2n}}{(2n)!} + 5t\sum_{n=2}^{\infty}\frac{(4t)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{5(2^{2n+2}-1)2^{2n+6}}{(2n+4)!}t^{2n+5}. \end{aligned} \tag{2.5}$$

It follows from (2.1)-(2.5) that

$$h(t) = \frac{\sum_{n=0}^{\infty} a_n t^{2n}}{\sum_{n=0}^{\infty} b_n t^{2n}}, \tag{2.6}$$

where

$$a_n = \frac{[16+13n+(2n-1)2^{2n+2}]2^{2n+7}}{(2n+5)!}, \quad b_n = \frac{5(2^{2n+2}-1)2^{2n+6}}{(2n+4)!}. \tag{2.7}$$

Equation (2.7) leads to

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{6c_n}{5(2n+5)(2n+7)(2^{2n+2}-1)(2^{2n+4}-1)}, \tag{2.8}$$

where

$$c_n = (30n^2 + 135n + 110 - 4^{n+3})4^{n+1} + 11. \tag{2.9}$$

From (2.9) we get

$$c_0 = 195, \quad c_1 = 315, \quad c_2 = -33525 \tag{2.10}$$

and

$$\begin{aligned} c_n &< (30n^2 + 135n + 110 - 64n^3)4^{n+1} + 11 \\ &= \left[10n^2(3-n) + 15n(9-n^2) + 5(22-n^3) - 34n^3\right] 4^{n+1} + 11 \\ &< -34n^3 \cdot 4^{n+1} + 11 < 0 \end{aligned} \tag{2.11}$$

for $n \geq 3$.

Equations (2.8) and (2.10) together with inequality (2.11) lead to the conclusion that the sequence $\{a_n/b_n\}$ is strictly decreasing for $0 \leq n \leq 2$ and strictly increasing for $n \geq 3$. Then from Lemma 2.2(2) and (2.6) we clearly see that there exists $t_0 \in (0, \infty)$ such that $h(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing on (t_0, ∞).

Let $t^* = \log(1+\sqrt{2})$. Then simple computations lead to

$$\sinh(2t^*) = 2\sqrt{2},\ \cosh(2t^*) = 3,\ \sinh(4t^*) = 12\sqrt{2},\ \cosh(4t^*) = 17. \tag{2.12}$$

Differentiating (2.1) yields

$$\begin{aligned} h'(t) =& \frac{90-80\cosh(2t)+104t\sinh(2t)-10\cosh(4t)+8t\sinh(4t)}{h_2(t)} \\ &- \frac{15-20\cosh(2t)-40t\sin(2t)+5\cosh(4t)+20t\sinh(4t)}{h_2(t)^2} h_1(t). \end{aligned} \tag{2.13}$$

From (2.2) and (2.3) together with (2.12) and (2.13) we get

$$h'(t^*) = \frac{-102\sqrt{2}t^{*2}+93t^*+21\sqrt{2}}{5t^{*2}} = -0.10035\cdots < 0. \tag{2.14}$$

From the piecewise monotonicity of $h(t)$ and inequality (2.14) we clearly see that $t_0 > t^* = \log(1+\sqrt{2})$, and the proof of Lemma 2.3 is completed. □

3. PROOFS OF THEOREMS 1.1 AND 1.2

3.1. Proof of Theorem 1.1 Since $M(a,b)$, $C(a,b)$ and $A(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$ and

$$\frac{\log[M(a,b)] - \log[A(a,b)]}{\log[C(a,b)] - \log[A(a,b)]} = \frac{\log[x/\operatorname{arcsinh}(x)]}{\log(1+x^2)} = \frac{\log[\sinh(t)/t]}{2\log[\cosh(t)]}. \tag{3.1}$$

Let $f_1(t) = \log[\sinh(t)/t]$, $f_2(t) = \log[\cosh(t)]$ and

$$f(t) = \frac{\log[\sinh(t)/t]}{\log[\cosh(t)]}. \tag{3.2}$$

Then $f_1(0^+) = f_2(0) = 0$, $f(t) = f_1(t)/f_2(t)$ and

$$\begin{aligned}\frac{f_1{}'(t)}{f_2{}'(t)} &= \frac{t\cosh^2(t) - \sinh(t)\cosh(t)}{t\sinh^2(t)} = \frac{t[\cosh(2t)+1] - \sinh(2t)}{t[\cosh(2t)-1]} \\ &= \frac{t\left(\sum_{n=0}^{\infty} 2^{2n}t^{2n}/(2n)! + 1\right) - \sum_{n=0}^{\infty} 2^{2n+1}t^{2n+1}/(2n+1)!}{t\sum_{n=1}^{\infty} 2^{2n}t^{2n}/(2n)!} \\ &= \frac{\sum_{n=1}^{\infty} 2^{2n}t^{2n+1}/(2n)! - \sum_{n=1}^{\infty} 2^{2n+1}t^{2n+1}/(2n+1)!}{t\sum_{n=1}^{\infty} 2^{2n}t^{2n}/(2n)!} = \frac{\sum_{n=0}^{\infty} A_n t^{2n}}{\sum_{n=0}^{\infty} B_n t^{2n}},\end{aligned} \tag{3.3}$$

where $A_n = 2^{2n+2}(2n+1)/(2n+3)!$ and $B_n = 2^{2n+2}/(2n+2)!$.

Note the $A_n/B_n = 1 - 2/(2n+3)$ is strictly increasing for all $n \geq 0$. Then from Lemma 2.2(1) and (3.3) we know that $f_1{}'(t)/f_2{}'(t)$ is strictly increasing on $(0, \infty)$. Hence, $f(t)$ is strictly increasing on $(0, \log(1+\sqrt{2}))$ follows from Lemma 2.1 and the monotonicity of $f_1{}'(t)/f_2{}'(t)$ together with $f(0^+) = f_2(0) = 0$. Moreover,

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{f_1{}'(t)}{f_2{}'(t)} = \frac{A_0}{B_0} = \frac{1}{3}, \tag{3.4}$$

$$\lim_{t\to\log(1+\sqrt{2})} f(t) = -\frac{2\log(\log(1+\sqrt{2}))}{\log 2}. \tag{3.5}$$

Therefore, Theorem 1.1 follows easily from (3.1), (3.2), (3.4) and (3.5) together with the monotonicity of $f(t)$. □

3.2. Proof of Theorem 1.2 Since $M(a,b)$, $C(a,b)$ and $A(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \operatorname{arcsinh}(x)$. Then $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$ and

$$\begin{aligned}&\frac{\log M(a,b) - \log\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]}{\log\left[C(a,b)/6 + 5A(a,b)/6\right] - \log\left[C^{1/6}(a,b)A^{5/6}(a,b)\right]} = \\ &= \frac{\log[x/\operatorname{arcsinh}(x)] - \log(1+x^2)^{1/6}}{\log(1+x^2/6) - \log(1+x^2)^{1/6}} \\ &= \frac{\log[\sinh(t)/t] - [\log\cosh(t)]/3}{\log[1+\sinh^2(t)/6] - [\log\cosh(t)]/3}.\end{aligned} \tag{3.6}$$

Let $g_1(t) = \log[\sinh(t)/t] - [\log\cosh(t)]/3$, $g_2(t) = \log[1+\sinh^2(t)/6] - [\log\cosh(t)]/3$ and

$$g(t) = \frac{\log[\sinh(t)/t] - [\log\cosh(t)]/3}{\log[1+\sinh^2(t)/6] - [\log\cosh(t)]/3}. \tag{3.7}$$

Then $g_1(0^+) = g_2(0) = 0$, $g(t) = g_1(t)/g_2(t)$ and

$$\frac{g_1{}'(t)}{g_2{}'(t)} = \frac{[6+\sinh^2(t)][3t\cosh^2(t) - 3\cosh(t)\sinh(t) - t\sinh^2(t)]}{t\sinh(t)[6\sinh(t)\cosh^2(t) - \sinh(t)(6+\sinh^2(t))]}.$$

Elementary computations lead to

$$[6+\sinh^2(t)][3t\cosh^2(t)-3\cosh(t)\sinh(t)-t\sinh^2(t)]=$$
$$=\tfrac{45}{4}t+\tfrac{13}{2}t\cosh(2t)-\tfrac{33}{4}\sinh(2t)+\tfrac{t}{4}\cosh(4t)-\tfrac{3}{8}\sinh(4t),$$

$$t\sinh(t)[6\sinh(t)\cosh^2(t)-\sinh(t)(6+\sinh^2(t))]=$$
$$=\tfrac{15}{8}t-\tfrac{5}{2}t\cosh(2t)+\tfrac{5}{8}t\cosh(4t)$$

and

$$\frac{g_1{}'(t)}{g_2{}'(t)}=h(t), \tag{3.8}$$

where $h(t)$ is defined as in Lemma 2.3.

It follows from Lemmas 2.1 and 2.3 and (3.8) together with $g_1(0^+)=g_2(0)=0$ that $g(t)$ is strictly decreasing on $(0,\log(1+\sqrt{2}))$. Moreover,

$$\lim_{t\to 0} g(t)=\tfrac{8}{25}, \tag{3.9}$$

$$\lim_{t\to\log(1+\sqrt{2})} g(t)=-\tfrac{6\log(\log(1+\sqrt{2}))+\log 2}{6\log(7/6)-\log 2}. \tag{3.10}$$

Therefore, Theorem 1.2 follows easily from (3.6), (3.7), (3.9) and (3.10) together with the monotonicity of $g(t)$. □

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