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LOCALIZATION RESULTS FOR THE LAGRANGE MAX-PRODUCT INTERPOLATION OPERATOR BASED ON EQUIDISTANT KNOTS[‡]

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Abstract. In the class of strictly positive functions strong localization results are obtained in approximation by the Lagrange max-product interpolation operators based on equidistant nodes. The results allow to approximate locally bounded strictly positive functions with very good accuracy. Then, it is observed that the results can be extended to bounded functions of variable sign.

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1. INTRODUCTION

Based on the Open Problem 5.5.4, pp. 324-326 in [14], in a series of recent papers we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), see [1], [3], Meyer-König and Zeller operators, see [4], Baskakov operators, see [6], [7] and Bleimann-Butzer-Hahn operators, see [5].

For example, in the recent paper [2], starting from the linear Bernstein operators $B_n(f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f(k/n)$, where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, written in the equivalent form

$$B_n(f)(x) = \frac{\sum\limits_{k=0}^{n} b_{n,k}(x) f(k/n)}{\sum\limits_{k=0}^{n} b_{n,k}(x)}$$

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and then replacing the sum operator Σ by the maximum operator $\bigvee$, one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x)},$$

where the notation $\bigvee_{k=0}^{n} b_{n,k}(x)$ means $\max\{b_{n,k}(x); k \in \{0, ..., n\}\}$ and similarly for the numerator.

For this max-product operator, nice approximation and shape preserving properties were found in the class of positive valued functions, in e.g. [2], [12].

In other two recent papers [9] and [10], this idea is applied to the Lagrange interpolation based on the Chebyshev nodes of second kind plus the endpoints, and to the Hermite-Fejér interpolation based on the Chebyshev nodes of first kind respectively, obtaining max-product interpolation operators which, in general, (for example, in the class of positive Lipschitz functions) approximates essentially better than the corresponding Lagrange and Hermite-Fejér interpolation polynomials.

Let $I = [a, b]$, $a < b$ and $f : [a, b] \to \mathbb{R}$. The max-product Lagrange interpolation operator on equidistant knots attached to the function f is given by (see [11])

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} l_{n,k}(x)}, \quad x \in I,\ n \in \mathbb{N},$$

where $x_{n,k} = a + (b - a)k/n$ for all $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n\}$ and

$$l_{n,k}(x) = (-1)^{n-k} \left(\prod_{i=0}^{n} (x - x_{n,i}) \right) \cdot \frac{1}{x - x_{n,k}}$$

for all $x \in I$, $n \in \mathbb{N}$ and $k \in \{0, 1, ..., n\}$. Note that $L_n^{(M)}(f)$ is a well defined function. Indeed, using the fundamental Lagrange polynomials,

$$p_{n,k}(x) = \frac{(x-x_{n,0})(x-x_{n,1})...(x-x_{n,k-1})(x-x_{n,k+1})...(x-x_{n,n})}{(x_{n,k}-x_{n,0})(x_{n,k}-x_{n,1})...(x_{n,k}-x_{n,k-1})(x_{n,k}-x_{n,k+1})...(x_{n,k}-x_{n,n})},$$

we observe that we can rewrite $l_{n,k}(x)$, $x \in I$, in the form

$$l_{n,k}(x) = c_{n,k} \cdot p_{n,k}(x),$$

where

$$c_{n,k} = (x_{n,k} - x_{n,0})(x_{n,k} - x_{n,1})...(x_{n,k} - x_{n,k-1})(x_{n,k+1} - x_{n,k})...(x_{n,n} - x_{n,k}).$$

Then, since for any $x \in I$ we have $\sum_{i=0}^{n} p_{n,i}(x) = 1$ it follows the existence of $i(x) \in \{0, 1, ..., n\}$ such that $p_{n,i(x)}(x) > 0$ and noting that $c_{n,i(x)} > 0$ it easily results that $l_{n,i(x)}(x) > 0$ and this implies that $\bigvee_{k=0}^{n} l_{n,k}(x) > 0$ for all $x \in I$, which means that indeed $L_n^{(M)}(f)$ is a well defined function on $[a, b]$.

The max-product operator $L_n^{(M)}(f)(x)$ is continuous on $[a, b]$ and has the interpolation properties $L_n^{(M)}(f)(x_{n,j}) = f(x_{n,j})$ for all $j \in \{0, 1, ..., \}$.

Also, according to Corollary 3.2, (i), in [11], for positive valued functions, i.e. for $f : [a, b] \to \mathbb{R}_+$, it satisfies the Jackson-type estimate

$$|L_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1\left(f; \frac{b-a}{n}\right)_{[a,b]}, \text{ for all } x \in [a, b],\, n \in \mathbb{N},$$

where $\omega_1\left(f; \frac{b-a}{n}\right)_{[a,b]}$ denotes the modulus of continuity of f on $[a, b]$. This estimate for the Lagrange max-product operator essentially improves for positive valued functions the order of approximation by the classical Lagrange interpolation polynomials on equidistant nodes, when as it is well-know, we can also have a very pronounced divergence phenomenon in $[a, b]$ (see e.g. Chapter 4 in the book [17], see also [16], [8]).

It is worth noting that saturation and local inverse results for $L_n^{(M)}(f)(x)$ were obtained in [13].

The plan of the paper goes as follows. In Section 2 an interesting strong localization result for the Lagrange max-product operator $L_n^{(M)}$ is obtained. At the end of the section and as consequences of this localization result, a local direct result and an interesting local shape preserving property are proved. Section 3 contains comparisons with some linear interpolation operators of rational type.

It is worth noting in Section 2 the strong localization result expressed by Theorem 2.1, that shows that if the continuous strictly positive functions f and g coincide on a subinterval $[\alpha, \beta] \subset [0, 1]$, then for sufficiently large values of n, $L_n^{(M)}(f)$ and $L_n^{(M)}(g)$ coincide on subintervals sufficiently close to $[\alpha, \beta]$. Clearly, Corollary 2.4 shows that $L_n^{(M)}(f)$ is very suitable to approximate continuous functions which are constant on some subintervals. Namely, if f is a continuous strictly positive function which is constant on some subintervals $[\alpha_i, \beta_i]$, $i = 1, ..., p$, of $[a, b]$, then for sufficiently large n, $L_n^{(M)}(f)$ takes the same constant values on subintervals sufficiently close to each $[\alpha_i, \beta_i]$, $i = 1, ..., p$.

2. LOCALIZATION RESULTS

Let $l_{n,k}$ denote the fundamental Lagrange polynomials attached to the knots $x_{n,k} = k/n$, $k \in \{0, 1, ..., n\}$, $n \in \mathbb{N}$.

The main result of this section is the following localization result.

THEOREM 2.1. *Let $f, g : [0, 1] \to [0, \infty)$ be both bounded on $[0, 1]$ with strictly positive lower bounds and suppose that there exist $a, b \in [0, 1]$, $0 < a < b < 1$ such that $f(x) = g(x)$ for all $x \in [a, b]$. Then for all $c, d \in [a, b]$ satisfying*

$a < c < d < b$ there exists $\widetilde{n} \in \mathbb{N}$ which depends only on f, g, a, b, c, d such that $L_n^{(M)}(f)(x) = L_n^{(M)}(g)(x)$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \widetilde{n}$.

Proof. Let us choose arbitrary $x \in [c, d]$ and for each $n \in \mathbb{N}$ let $j_x \in \{0, 1, ..., n\}$ (j_x depends on n too, but there is no need at all to complicate on the notations) be such that $x \in [j_x/n, (j_x + 1)/n]$. Then we know that

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x)f(\frac{k}{n})}{\bigvee_{k=0}^{n} l_{n,k}(x)} = \frac{\bigvee_{k\in J_n(x)} l_{n,k}(x)f(\frac{k}{n})}{\bigvee_{k\in J_n(x)} l_{n,k}(x)}$$

where $J_n(x) = \{k \in \{0, 1, ..., n\} : l_{n,k}(x) > 0\}$ and $l_{n,k}$, $k \in \{0, 1, ..., n\}$ are the Lagrange fundamental polynomials attached to the knots $x_{n,k} = k/n$, $k \in \{0, 1, ..., n\}$. Since $x \in [c, d] \cap [j_x/n, (j_x + 1)/n]$ and since $a < c < d < b$ it is immediate that for $n \geq n_0$ where n_0 is chosen such that $1/n_0 < \min\{c - a, d - b\}$, we obtain $a < j_x/n < b$ which gives $na < j_x < nb$ for all $n \geq n_0$ (indeed, if we would suppose that there exists $n > n_0$ which does not satisfy the previous double inequalities, then we would easily get a contradiction).

It is important to notice here that n_0 does not depend on x. From the inequalities $na < j_x < b$ it follows that if $n \geq n_0$ then for any $x \in [c, d]$ there exists $\alpha_x \in [a, b]$ such that $j_x = n\alpha_x$.

In what follows, it will serve to our purpose to use the sequence $(a_n)_{n\geq 1}$, $a_n = \sqrt{n}$. For this sequence there exists $n_1 \in \mathbb{N}$ such that $na - a_n > 0$ for all $n \geq n_1$.

Our intention is to prove as an intermediate result, that there exists an absolute constant $N_0 \in \mathbb{N}$ which does not depend of $x \in [c, d]$ such that for any $n \geq N_0$ and $x \in [c, d]$ we have $\bigvee_{k=0}^{n} l_{n,k}(x)f(\frac{k}{n}) = \bigvee_{k\in I_{n,x}} l_{n,k}(x)f(\frac{k}{n})$ where $I_{n,x} = \{k \in J_n(x) : j_x - a_n \leq k \leq j_x + a_n\}$. In order to obtain this conclusion, for $n \geq \max\{n_0, n_1\}$ let us choose $k \in J_n(x) \setminus I_{n,x}$. We have two cases: i) $k + a_n < j_x$ and ii) $j_x + a_n < k$.

Case i) Firstly, note that $j_x \in J_n(x)$, because $sign(l_{n,j}(x)) = (-1)^{n-j} \cdot (-1)^{n-j} = 1$. Noting that $k/n < (j_x - a_n)/n$ and $nx \geq j_x$, we get

$$\begin{aligned}
\frac{l_{n,j_x}(x)f(\frac{j_x}{n})}{l_{n,k}(x)f(\frac{k}{n})} &= \frac{|x - k/n|}{|x - j_x/n|} \cdot \frac{f(\frac{j_x}{n})}{f(\frac{k}{n})} = \frac{x - k/n}{x - j_x/n} \cdot \frac{f(\frac{j_x}{n})}{f(\frac{k}{n})} \\
&\geq \frac{x - (j_x - a_n)/n}{1/n} \cdot \frac{f(\frac{j_x}{n})}{f(\frac{k}{n})} = (nx - j_x + a_n) \cdot \frac{f(\frac{j_x}{n})}{f(\frac{k}{n})} \\
&\geq \sqrt{n} \cdot \frac{f(\frac{j_x}{n})}{f(\frac{k}{n})}.
\end{aligned}$$

Then, denoting the infimum and the supremum of f on $[a, b]$ with m_f and M_f respectively (according to the hypotheses these values are strictly positive), we get that

$$\frac{l_{n,j_x}(x)f(\frac{j_x}{n})}{l_{n,k}(x)f(\frac{k}{n})} \geq \sqrt{n}\cdot\frac{m_f}{M_f}$$

Since $\lim_{n\to\infty}\sqrt{n}\cdot\frac{m_f}{M_f}=\infty$, it follows that there exists $n_2\in\mathbb{N}$, $n_2\geq\max\{n_0,n_1\}$ such that $\frac{l_{n,j_x}(x)f(\frac{j_x}{n})}{l_{n,k}(x)f(\frac{k}{n})}>1$ for all $x\in[c,d]$, $n\geq n_2$ and $k\in\{0,1,...,n\}$, $k<j_x-a_n$ (as $k\notin I_{n,x}$). In addition, it is important to notice that n_2 does not depend on $x\in[c,d]$, but of course it depends on f.

Case ii) The proof is identical with the proof of the above Case i) and therefore we conclude that there exists an absolute constant $n_3\in\mathbb{N}$ which depends only on a,b,c,d,f such that

$$\frac{l_{n,j_x}(x)f(j_x/n)}{l_{n,k}(x)f(k/n)}>1$$

for all $x\in[c,d]$, $n\geq n_3$ and $k\in\{0,1,...,n\}$, $k>j_x+a_n$.

Analyzing the results obtained in cases i)-ii), it results that for all $x\in[c,d]$, $n\geq N_0$, $N_0=\max\{n_2,n_3\}$ and $k\in\{0,1,...,n\}$, with $k<j_x-a_n$ or $k>j_x+a_n$, we have

$$\frac{l_{n,j_x}(x)f(j_x/n)}{l_{n,k}(x)f(k/n)}>1.$$

Since from the Case i) we know that $j_x\in J_n(x)$ and since this easily implies that actually $j_x\in I_{n,x}$, we obtain our preliminary result, that is

$$\bigvee_{k=0}^{n} l_{n,k}(x)f\left(\tfrac{k}{n}\right)=\bigvee_{k\in I_{n,x}} l_{n,k}(x)f\left(\tfrac{k}{n}\right),$$

where $I_{n,x}=\{k\in J_n(x):j_x-a_n\leq k\leq j_x+a_n\}$.

Next, let us choose arbitrary $x\in[c,d]$ and $n\in\mathbb{N}$ so that $n\geq N_0$. If there exists $k\in I_{n,x}$ such that $k/n\notin[c,d]$ then we distinguish two cases. Either $k/n<c$ or $k/n>d$. In the first case we observe that

$$0<c-\tfrac{k}{n}\leq x-\tfrac{k}{n}\leq\tfrac{(j_x+1)}{n+1}-\tfrac{k}{n}\leq\tfrac{(j_x+1)}{n}-\tfrac{k}{n}\leq\tfrac{a_n+1}{n}.$$

Since $\lim_{n\to\infty}\frac{a_n+1}{n}=0$, it results that for sufficiently large n we necessarily have $\frac{a_n+1}{n}<c-a$ which clearly implies that $k/n\in[a,c]$. In the same manner, when $k/n>d$, for sufficiently large n we necessarily have $k/n\in[d,b]$.

Summarizing, there exists $\widetilde{N}_1\in\mathbb{N}$ independent of any $x\in[c,d]$, such that

$$\bigvee_{k=0}^{n} l_{n,k}(x)f\left(\tfrac{k}{n}\right)=\bigvee_{k\in I_{n,x}} l_{n,k}(x)f\left(\tfrac{k}{n}\right),\ n\geq\widetilde{N}_1$$

and in addition for any $x\in[c,d]$, $n\geq\widetilde{N}_1$ and $k\in I_{n,x}$, we have $k/n\in[a,b]$. Also, it is easy to check that $\widetilde{N}_1$ depends only on a,b,c,d,f. We thus obtain that

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k\in I_{n,x}} l_{n,k}(x) f(\frac{k}{n})}{\bigvee_{k=0}^{n} l_{n,k}(x)}, \quad n \geq \widetilde{N}_1, x \in [c,d]$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_1$ and $k \in I_{n,x}$, we have $k/n \in [a,b]$.

Reasoning for the function g exactly as in the case of the function f, it follows that there exists $\widetilde{N}_2 \in \mathbb{N}$ which depends only on a, b, c, d, g such that

$$L_n^{(M)}(g)(x) = \frac{\bigvee_{k\in I_{n,x}} l_{n,k}(x) g(\frac{k}{n})}{\bigvee_{k=0}^{n} l_{n,k}(x)}, \quad n \geq \widetilde{N}_2, x \in [c,d]$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_2$ and $k \in I_{n,x}$, we have $k/n \in [a,b]$. Taking $\widetilde{n} = \max\{\widetilde{N}_1, \widetilde{N}_2\}$, we easily obtain the desired conclusion. □

We can easily extend the above result to arbitrary intervals, as follows.

Theorem 2.2. *Let $f, g : [a,b] \to [0,\infty)$ $(a < b)$ be both bounded on $[a,b]$ with strictly positive lower bounds and suppose that there exist $a', b' \in [a,b]$, $a < a' < b' < b$ such that $f(x) = g(x)$ for all $x \in [a',b']$. Then for all $c, d \in [a',b']$ satisfying $a' < c < d < b'$, there exists $\widetilde{n} \in \mathbb{N}$ which depends only on $f, g, a, b, a', b'c, d$, such that $L_n^{(M)}(f)(x) = L_n^{(M)}(g)(x)$ for all $x \in [c,d]$ and $n \in \mathbb{N}$, $n \geq \widetilde{n}$.*

Proof. We obtain the desired conclusion as a direct consequence of the previous theorem. Indeed, firstly to make a distinction, we denote with $\overline{L}_n^{(M)}$ the Lagrange max-product operator attached to functions defined on the interval $[0,1]$. In addition, in what follows, for all all $n \in \mathbb{N}$ and $k \in \{0,1,...,n\}$ we denote with $l_{n,k}^1$ the fundamental Lagrange polynomials defined on the interval $[0,1]$.

Suppose now that for the two functions $f, g \in C([a,b])$ we have $f(x) = g(x)$, for all $x \in [a',b']$. Let us define the function $h : [0,1] \to [a,b]$, $h(y) = a + (b-a)y$. It is immediate that for any $x \in [a,b]$ there exists an unique $y(x) = h^{-1}(x) \in [0,1]$ such that $f(x) = (f \circ h)(y(x))$ and $g(x) = (g \circ h)(y(x))$.

Then we observe that for any $x \in [a,b]$ we have

$$l_{n,k}(x) = (b-a)^n \cdot l_{n,k}^1(y(x)),\ n \in \mathbb{N},\ k \in \{0,1,...,n\}.$$

The above equalities imply

$$\begin{aligned} L_n^{(M)}(f)(x) &= \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} l_{n,k}(x)} = \frac{(b-a)^n \cdot \bigvee_{k=0}^{n} l^1_{n,k}(y(x))(f\circ h)\left(\frac{k}{n}\right)}{(b-a)^n \cdot \bigvee_{k=0}^{n} l^1_{n,k}(y(x))} \\ &= \overline{L}_n^{(M)}(f\circ h)(y(x)) \end{aligned}$$

and analogously $L_n^{(M)}(g)(x) = \overline{L}_n^{(M)}(g\circ h)(y(x))$, for all $x \in [a,b]$.

Then, our result is immediate by applying Theorem 2.1 to $\overline{L}_n^{(M)}(g\circ h)(y(x))$ and $\overline{L}_n^{(M)}(g\circ h)(y(x))$, where recall that $f\circ h, g\circ h : [0,1] \to [0,+\infty)$, $y(x) = h^{-1}(x)$ and $h : [0,1] \to [a,b]$, $h(x) = a + (b-a)x$. □

The next direct approximation result is now an immediate consequence of the localization result in Theorem 2.2, as follows.

Corollary 2.3. *Let $f : [a,b] \to [0,\infty)$ $(a < b)$ be bounded on $[a,b]$ with strictly positive lower bound and suppose that there exist $a', b' \in [a,b]$, $a < a' < b' < b$ and the constant C_0 which depends only on a,b,a',b', such that*

$$|f(x) - f(y)| \le C_0 |x-y|, \quad \text{for all } x \in [a',b'], \tag{2.1}$$

that is $f|_{[a',b']} \in Lip[a',b']$. Then, for any $c,d \in [a',b']$ satisfying $a' < c < d < b'$, we have

$$\left| L_n^{(M)}(f)(x) - f(x) \right| \le \frac{C}{n}, \quad \text{for all } n \in \mathbb{N} \text{ and } x \in [c,d],$$

where the constant C depends only on f and the values a,b,a',b',c,d.

Proof. Let us define the function $F : [a,b] \to \mathbb{R}$,

$$F(x) = \begin{cases} f(x), & \text{if } x \in [a',b'], \\ f(a'), & \text{if } x \in [a,a'], \\ f(b'), & \text{if } x \in [b',b]. \end{cases}$$

The hypothesis imply that F is continuous and strictly positive on $[a,b]$ and according to Corollary 3.2 in [11] it results that

$$\left| L_n^{(M)}(F)(x) - F(x) \right| \le 2\omega_1\left(F, \frac{b-a}{n}\right)_{[a,b]}, \ x \in [a,b], n \in \mathbb{N},$$

Since by the definition of F we have $\omega_1(F, \frac{b-a}{n})_{[a,b]} \le \omega_1(f, \frac{b-a}{n})_{[a,b]}$ and since by the relation (2.1) it easily follows $\omega_1(f, \frac{b-a}{n})_{[a,b]} \le C_0(b-a)/n$, we get

$$\left| L_n^{(M)}(F)(x) - F(x) \right| \le 2C_0(b-a)/n, \ x \in [a,b], n \in \mathbb{N}.$$

Now, let us choose arbitrary $c,d \in [a',b']$ such that $a' < c < d < b'$. Then, by Theorem 2.2 (applicable to f and F) it results the existence of $\widetilde{n} \in \mathbb{N}$ which depends only on $a,b,a',b'c,d,f,F$ such that $L_n^{(M)}(F)(x) = L_n^{(M)}(f)(x)$ for all

$x \in [c, d]$. But since actually the function F depends on the function f, it is clear that in fact $\widetilde{n}$ depends only on $a, b, a', b'c, d$ and f.

Therefore, for arbitrary $x \in [c, d]$ and $n \in \mathbb{N}$ with $n \geq \widetilde{n}$ we obtain

$$\left|L_n^{(M)}(f)(x) - f(x)\right| = \left|L_n^{(M)}(F)(x) - F(x)\right| \leq 2C_0(b-a)/n,$$

where C_0 and $\widetilde{n}$ depend only on $a, b, a', b'c, d$ and f.

Now, denoting

$$C_1 = \max_{1 \leq n < \widetilde{n}} \{n \cdot \|L_n^{(M)}(f) - f\|_{[c,d]}\},$$

we finally obtain

$$|L_n^{(M)}(f)(x) - f(x)| \leq \tfrac{C}{n}, \text{ for all } n \in \mathbb{N}, \, x \in [c, d],$$

with $C = \max\{2C_0(b-a), C_1\}$ depending only on a, b, c, d and f. This proves the corollary. □

At the end of this section, as a consequence of the localization result in Theorem 2.2 we present a locally constant preserving property.

Corollary 2.4. *Let $f : [a, b] \to [0, \infty)$ be bounded on $[a, b]$ with strictly positive lower bound and suppose that there exists $a', b' \in [a, b]$, $a < a' < b' < b$ such that f is constant on $[a', b']$ with the constant value $\alpha > 0$. Then for any $c, d \in [a', b']$ with $a' < c < d < b'$, there exists $\widetilde{n} \in \mathbb{N}$ which depends only on a, b, a', b', c, d and f such that $L_n^{(M)}(f)(x) = \alpha$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \widetilde{n}$.*

Proof. Let $g : [a, b] \to \mathbb{R}_+$ be given by $g(x) = \alpha > 0$ for all $x \in [a, b]$. Since $f(x) = g(x)$ for all $x \in [a', b']$ and since obviously $L_n^{(M)}(g)(x) = \alpha$ for all $x \in [a, b]$, by Theorem 2.2 we easily obtain the desired conclusion. □

3. FINAL REMARKS

Let us note that in Hermann-Vértesi [15], starting from a Lagrange interpolatory process (convergent or not)

$$P_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f(x_{n,k}),$$

with

$$p_{n,k}(x) = \frac{(x - x_{n,0})...(x - x_{n,k-1})(x - x_{n,k+1})...(x - x_{n,n})}{(x_{n,k} - x_{n,0})...(x_{n,k} - x_{n,k-1})(x_{n,k} - x_{n,k+1})...(x_{n,k} - x_{n,n})},$$

new linear interpolatory (rational) operators of the form

$$R_n(f)(x) = \frac{\sum_{k=0}^{n} f(x_{n,k})|p_{n,k}(x)|^r}{\sum_{k=0}^{n} f(x_{n,k})|p_{n,k}(x)|^r},$$

are constructed, for which in the case when $r > 2$ and the knots $x_{n,k}$ satisfy some special requirements (e.g. some Jacobi knots), the Jackson-type order of approximation

$$\|R_n(f) - f\| \leq C\omega_1(f; 1/n),$$

is obtained (see e.g. Theorem 3.2 in Hermann-Vértesi [15]).

In other words, for the linear rational construction $R_n(f)(x)$, we get the same order of approximation as for the interpolatory max-product operator (which is piecewise rational)

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} l_{n,k}(x) f(x_{n,k})}{\bigvee_{k=0}^{n} l_{n,k}(x)}.$$

But clearly that with respect to $R_n(f)(x)$, the max-product rational operator $L_n^{(M)}(f)(x)$ presents several advantages, pointed out by the next remarks.

Remark 3.1. For positive continuous functions, it provides an estimate in terms of $\omega_1(f; 1/n)$ for the simplest systems of knots (that is for the equidistant nodes). But, in fact, as it was mentioned in the last Remark in the paper [7], the estimate holds for any kind of interpolatory systems of points with the property that the distance between two consecutive nodes converges to zero as $n \to \infty$. It is worth noting that the operator $R_n(f)(x)$ provides the same Jackson-type estimate, but for systems of interpolatory points satisfying additional requirements (e.g. Theorem 3.2 in [15] for the Jacobi knots). □

Remark 3.2. In our best knowledge, the strong localization results in Theorem 2.1 and Corollary 2.4, have not equivalence for $R_n(f)(x)$. □

Remark 3.3. Although the expression of $L_n^{(M)}(f)(x)$ theoretically looks more complicated than that of $R_n(f)(x)$, however from practical/computational point of view, there not exists any difference between the usage of computer softwares (like Matlab or Mathematica) to trace the graphs of $L_n^{(M)}(f)(x)$ and $R_n(f)(x)$, for any concrete choices of f. In fact, in Computer Science, the sum ($\sum$) operation and the maximum ($\bigvee$) operator have similar levels of computability. □

Remark 3.4. The results in Theorem 2.2 and Corollary 2.4 show the nice property of the max-product interpolation operator $L_n^{(M)}$ to reproduce locally with great accuracy the graph of a strictly positive non-smooth continuous function f. For example, Corollary 2.4 shows that $L_n^{(M)}(f)$ is very suitable to approximate continuous functions which are strictly positive constants on some subintervals. Namely, if f is a continuous strictly positive function which is constant on some subintervals $[\alpha_i, \beta_i]$, $i = 1, ..., p$, of $[a, b]$, then for sufficiently large n, $L_n^{(M)}(f)$ takes the same constant values on subintervals sufficiently close to each $[\alpha_i, \beta_i]$, $i = 1, ..., p$. □

REMARK 3.5. It is easy to see that the results expressed by Theorem 2.2 and Corollaries 2.3-2.4 can be extended to bounded functions of variable sign, for the new max-product operators of the form $\bar{L}_n^{(M)}(f)(x) = L_n^{(M)}(f + \bar{c})(x) - \bar{c}$, where $\bar{c}$ is a constant such that $f(x) + \bar{c} > 0$, for all $x \in [a, b]$. Note that, for example in the case of Theorem 2.2, for f and g bounded and of variable sign, evidently that we may choose a constant $\bar{c}$ such that $f(x) + \bar{c} > 0$ and $g(x) + \bar{c} > 0$, for all $x \in [a, b]$. □

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