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ON APPROXIMATING THE SOLUTIONS OF EQUATIONS BY THE CHORD METHOD AND A METHOD OF AITKEN-STEFFENSEN TYPE*

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Abstract. In [13] we have studied the existence and the convergence of iterative methods that use generalized abstract divided differences (this notion being defined there). We have indicated a construction model for these differences as well. A special place has been given to the iterative method of the chord for which we have established a convergence theorem which in the same time ensures the existence of the solution of the considered equation. We have obtained the convergence order with the value $\frac{1+\sqrt{5}}{2}$. This value is inferior to 2, this last value representing the convergence order of the method of Newton-Kantorovich. This diminuation of the convergence order is the price to pay for the replacement of the Fréchet differential with the generalized abstract divided difference. In this paper we consider the issue of the improvement of the convergence order with respect to the method of Steffensen and Aitken-Steffensen or their generalizations through the method of the auxiliary sequences. This method will be presented in the paper together with the specification of the convergence order of the main sequence and the auxiliary sequences.

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1. INTRODUCTION

In [13] we have studied certain approximation methods for the solutions of equations in linear normed spaces, methods that use the abstract divided differences or the generalized abstract divided differences. The main result concerned the study of the convergence of the iterative method of chord with the fixing of its convergence order.

Let us consider X, Y two linear normed spaces, denote by $\|\cdot\|_X : X \to \mathbb{R}$ and $\|\cdot\|_Y : Y \to \mathbb{R}$ their norms respectively, and by θ_X and θ_Y their null elements respectively. By $(X,Y)^*$ we denote the set of the linear and continuous mappings defined from X to Y. The set $(X,Y)^*$ is a linear normed space as well, if we define the norm $\|\cdot\| : (X,Y)^* \to [0,+\infty[$, by $\|U\| = \sup_{h\in X,\ \|h\|_X=1} \|U(h)\|$ for any $U \in (X,Y)^*$. For the case of $Y = \mathbb{R}$ we denote by X^* the set $(X,\mathbb{R})^*$,

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this set representing the space of real, linear and continuous functionals defined on the linear normed space X.

Let us consider now a set $D \subseteq X$ and a nonlinear mapping $f : D \to Y$. Using this mapping we have the equation:

$$f(x) = \theta_Y. \tag{1}$$

We will study the approximation of its solutions.

In order to clarify the aforementioned notions we have the following definition:

DEFINITION 1.1. *Considering the nonlinear mapping* $f : D \to Y$ *together with the points* $x, y \in D$, $x \neq y$, *any mapping* $\Gamma_{f;x,y} \in (X,Y)^*$ *that verifies the equality:*

$$\Gamma_{f;x,y}(x-y) = f(x) - f(y) \tag{2}$$

is called generalized abstract divided difference of the function $f : D \to Y$ at the points x, y.

In connection with the previous definition we have the following remark:

REMARK 1.2. *a)* If we consider the theorem according to which in every linear normed space $(X, \|\cdot\|_X)$, for any $a \in X \setminus \{\theta_X\}$ there exists a linear and continuous functional $u \in X^*$ such that $\|u\| = 1$ and $u(a) = \|a\|_X$. Therefore, for any $x, y \in X$ with $x \neq y$ there exists the functional $U_{xy} \in X^*$ such that $\|U_{xy}\| = 1$ and $U_{xy}(x-y) = \|x-y\|_X$. At the same time there exists the functional $U_{yx} \in X^*$ such that $\|U_{yx}\| = 1$ and $U_{yx}(y-x) = \|y-x\|_X$ as well. In the paper [13] there appears the mapping $[x, y; f] \in (X,Y)^*$, defined by the equality:

$$[x, y; f]\, h = \frac{U_{xy}(h) f(x) + U_{yx}(h) f(y)}{\|x-y\|_X} \tag{3}$$

for any $h \in X$.
This mapping verifies the equality (2) and it is called *abstract divided difference* of the nonlinear mapping $f : D \to Y$ at the points $x, y \in D$ with $x \neq y$. This mapping is a special case of generalized abstract divided difference.

b) Let us suppose now that the space X is a space with a scalar product $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{R}$. Defining $\|\cdot\|_X : X \to \mathbb{R}$ by $\|x\|_X = \sqrt{\langle x|x\rangle}$, the space $(X, \|\cdot\|_X)$ is a linear normed space.

For any $x, y \in X$ with $x \neq y$ the functional $U_{xy} \in X^*$ from **a)** will be defined by:

$$U_{xy}(h) = \frac{\langle h|x-y\rangle}{\|x-y\|_X}$$

for any $h \in X$. So, for the same elements $x, y \in X$ with $x \neq y$ we have that the abstract divided difference $[x, y; f] \in (X,Y)^*$ is defined by:

$$[x, y; f]\, h = \frac{\langle x-y|h\rangle (f(x) - f(y))}{\|x-y\|_X^2}$$

for any $h \in X$.

□

The main result of the paper [13] concerns the convergence of the chord method for the approximation of a solution of the equation (1). This method consists in the consideration of an approximant sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ (here $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and this notation remains valid for the rest of this paper) and that for any $n \in \mathbb{N}$ verifies the equality:

$$\Gamma_{f;x_{n-1},x_n}(x_{n+1} - x_n) + f(x_n) = \theta_Y. \tag{4}$$

If we suppose the existence for a certain $n \in \mathbb{N}$ of the mapping $\Gamma^{-1}_{f;x_{n-1},x_n} \in (Y,X)^*$ representing the inverse of the mapping $\Gamma_{f;x_{n-1},x_n} \in (X,Y)^*$, for this number $n \in \mathbb{N}$ the equality (4) is equivalent to:

$$x_{n+1} = x_n - \Gamma^{-1}_{f;x_{n-1},x_n} f(x_n) = x_{n-1} - \Gamma^{-1}_{f;x_{n-1},x_n} f(x_{n-1}). \tag{5}$$

2. THEOREM OF CONVERGENCE OF THE CHORD METHOD

The main result with regard to the possibility that the equality (4) can be written under the form (5), together with the convergence of the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ to a solution of the equation (1) the existence of which is also proved, is the following theorem:

THEOREM 2.1. *We suppose that the following assumptions hold:*

i) $(X, \|\cdot\|_X)$ *is a Banach space;*

ii) *the nonlinear mapping $f : D \to Y$, for any $x, y \in D$ with $x \neq y$ admits a generalized abstract divided difference $\Gamma_{f;x_{n-1},x_n} \in (X,Y)^*$ and there exists a number $L > 0$ such that for any $x, y, z \in D$ with $x \neq y$ and $y \neq z$ we have the following inequality:*

$$\|\Gamma_{f;x,y} - \Gamma_{f;y,z}\| \leq L\|x - z\|_X; \tag{6}$$

iii) *the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ is such that for any $n \in \mathbb{N}$ we have $x_{n-1} \neq x_n$ and the equality (4) is verified;*

iv) *referring to the initial elements $x_0, x_1 \in D$ of the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ we suppose the carrying out of the following conditions:*

iv$_1$) *the mapping $\Gamma_{f;x_0,x_1} \in (X,Y)^*$ is invertible and $\Gamma^{-1}_{f;x_0,x_1} \in (Y,X)^*$;*

iv$_2$) *there exist the numbers $h_0 \in]0,1[$ and $B_0 > 0$ such that we have the following inequality:*

$$\left\|\Gamma^{-1}_{f;x_0,x_1}\right\| \leq \tfrac{B_0}{1-h_0}; \tag{7}$$

iv$_3$) *if we note $R_0 = \|f(x_0)\|_Y$ and $h_1 = \frac{LB_0^2R_0}{(1-h_0)^2}$ there exists a number $q \in \left[\frac{1}{2}, 1\right[$ such that we have the following inequality:*

$$d = \max\left\{\tfrac{h_0}{(1-q)^2}, \left[\tfrac{h_1}{(1-q)^2}\right]^{\frac{1}{\alpha}}\right\} < 1, \tag{8}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$;

iv$_4$) *if* $\delta = \frac{(1-q)^2}{LB_0} \cdot \frac{d^\alpha}{1-d}$ *and* $S(x_0,\delta) = \{x \in X / \|x - x_0\|_X \le \delta\}$ *the relation* $x_1 \in S(x_0,\delta) \subseteq D$ *is true.*

Then the following conclusions are true:

j) *for any* $n \in \mathbb{N}$ *we have that* $x_n \in S(x_0,\delta)$, *there exists the mapping* $\Gamma^{-1}_{f;x_{n-1},x_n} \in (Y,X)^*$, *and the sequence* $(x_n)_{n\in\mathbb{N}^*}$ *verifies the equality (5);*

jj) *the sequence* $(x_n)_{n\in\mathbb{N}^*} \subseteq X$ *is convergent;*

jjj) *if* $\overline{x} = \lim\limits_{n\to\infty} x_n$, *then* $\overline{x} \in S(x_0,\delta)$ *and* $f(\overline{x}) = \theta_Y$;

jv) *for any* $n \in \mathbb{N}^*$ *the following estimate of the error of approximation is true:*

$$\|\overline{x} - x_n\|_X \le \tfrac{(1-q)^2}{LB_0} \cdot \frac{d^{\alpha^{n+1}}}{1-d^{\alpha^n}}. \tag{9}$$

The proof of this theorem is given in [13].

Regarding this theorem we have the following remark:

Remark 2.2. From the conclusions of the theorem 2.1, corroborated with the fact that $d < 1$, we deduce that there exists a number $N \in \mathbb{N}$ such that for any number $n \in \mathbb{N}$, $n \ge N$ we have the following inequality:

$$\|\overline{x} - x_n\|_X \le \frac{d^{\alpha^n}}{LB_0}. \tag{10}$$

This inequality indicates that the convergence order of the iterative method of the chord is $\alpha = \frac{1+\sqrt{5}}{2}$. □

The inequality (10) is obvious from (9).

3. CERTAIN REMARKS IN CONNECTION WITH THE PREVIOUS RESULT

We have the following statements:

Remark 3.1. Under the hypotheses of Theorem 2.1 the sequence of real and positive numbers $\left(\|\Gamma^{-1}_{f;x_{n-1},x_n}\|\right)_{n\in\mathbb{N}}$ has an upper bound, for any $n \in \mathbb{N}$ the following inequality taking place:

$$\|\Gamma^{-1}_{f;x_{n-1},x_n}\| \le B_0 \mathrm{e}^{\frac{(1-q)d}{1-d^{\alpha-1}}}. \tag{11}$$

□

Indeed, as $\|\Gamma^{-1}_{f;x_{n-1},x_n}\| \le B_n$, and using the recurrence relation of the sequence $(B_n)_{n\in\mathbb{N}^*}$, we have $\frac{B_n}{B_{n-1}} \le \frac{1}{1-h_{n-1}}$ and so:

$$B_n \le \tfrac{B_0}{(1-h_0)(1-h_1)\cdot\ldots\cdot(1-h_{n-1})}.$$

Evidently:

$$\tfrac{1}{(1-h_0)(1-h_1)\cdot\ldots\cdot(1-h_{n-1})} \leq \left[\tfrac{1}{n}\sum_{j=0}^{n-1}\tfrac{1}{1-h_j}\right]^n =$$

$$= \left[1+\tfrac{1}{n}\sum_{j=0}^{n-1}\tfrac{h_j}{1-h_j}\right]^n \leq \left[1+\tfrac{1}{n}\cdot\tfrac{1}{1-q}\sum_{j=0}^{n-1}h_j\right]^n.$$

But:

$$\sum_{j=0}^{n-1}h_j \leq (1-q)^2\sum_{j=0}^{n-1}d^{\alpha^j} \leq (1-q)^2\,\tfrac{d}{1-d^{\alpha-1}},$$

therefore:

$$\tfrac{1}{(1-h_0)(1-h_1)\cdot\ldots\cdot(1-h_{n-1})} \leq \left[1+\tfrac{1}{n}\cdot\tfrac{(1-q)d}{1-d^{\alpha-1}}\right]^n < \exp\left(\tfrac{(1-q)d}{1-d^{\alpha-1}}\right),$$

from where the inequality (11) is evident.

Remark 3.2. If in the hypotheses of Theorem 2.1 we choose $q=\frac{1}{2}$, the values of the other constants from this theorem are:

$$d=\max\left\{4h_0,(4h_1)^{\frac{1}{\alpha}}\right\}<1,\ \delta=\tfrac{d^\alpha}{4LB_0(1-d)}, \tag{12}$$

and the inequality that expressed an upper bound of the error by which x_n approximates $\overline{x}$ is:

$$\|\overline{x}-x_n\|_X \leq \frac{d^{\alpha^{n+1}}}{4LB_0\left(1-d^{\alpha^n}\right)}. \tag{13}$$

For the upper bound of the sequence $\left(\left\|\Gamma^{-1}_{f;x_{n-1},x_n}\right\|\right)_{n\in\mathbb{N}}$ for any $n\in\mathbb{N}$, we have the following inequality:

$$\left\|\Gamma^{-1}_{f;x_{n-1},x_n}\right\| \leq B_0\mathrm{e}^{\frac{d}{2(1-d^{\alpha-1})}}. \tag{14}$$

□

The statements are obvious:

Remark 3.3. For any $n\in\mathbb{N}$ there exists the generalized abstract divided difference $\Gamma_{f;x_n,\overline{x}}\in(X,Y)^*$, this mapping is invertible, so the mapping $\Gamma^{-1}_{f;x_n,\overline{x}}\in(Y,X)^*$ exists and the sequence $\left(\left\|\Gamma^{-1}_{f;x_n,\overline{x}}\right\|\right)_{n\in\mathbb{N}}$ has an upper bound, more precisely, there exists a number $p\in\mathbb{N}$ such that for any $n\in\mathbb{N},\ n\geq p$ we have the inequality:

$$\left\|\Gamma^{-1}_{f;x_n,\overline{x}}\right\| \leq 2B_0\mathrm{e}^{\frac{(1-q)d}{1-d^{\alpha-1}}}. \tag{15}$$

□

It is obvious that $\overline{x} \neq x_n$ for any $n \in \mathbb{N}$. Indeed, if there exists a number $n_0 \in \mathbb{N}$ such that $\overline{x} = x_{n_0}$ then:

$$x_{n_0+1} = x_{n_0} - \Gamma^{-1}_{f;x_{n_0-1},x_{n_0}} f\left(x_{n_0}\right) = x_{n_0},$$

which is impossible.

As $\overline{x} \neq x_n$ for any $n \in \mathbb{N}$ we deduce the existence of the mapping $\Gamma_{f;x_n,\overline{x}} \in (X,Y)^*$ representing the generalized abstract divided difference of the function $f : D \to Y$ on the points x_n and $\overline{x}$.

In order to prove the invertibility of the mapping $\Gamma_{f;x_n,\overline{x}} \in (X,Y)^*$ for any any $n \in \mathbb{N}$ let us consider the following expression:

$$W_n = \Gamma^{-1}_{f;x_{n-1},x_n}\left(\Gamma_{f;x_{n-1},x_n} - \Gamma_{f;x_n,\overline{x}}\right) \in (X,X)^*.$$

We have:

$$\Gamma_{f;x_n,\overline{x}} = \Gamma_{f;x_{n-1},x_n}\left(\mathbf{I}_X - W_n\right).$$

It is clear that:

$$\|W_n\| \leq B_n L \left\|\overline{x} - x_{n-1}\right\|_X \leq B_0 L \mathrm{e}^{\frac{(1-q)d}{1-d^{\alpha-1}}} \left\|\overline{x} - x_{n-1}\right\|_X.$$

As $\lim\limits_{n\to\infty} \left\|\overline{x} - x_{n-1}\right\|_X = 0$, we deduce that there exists a number $p \in \mathbb{N}$ such that for any $n \in \mathbb{N},\ n \geq p$, we have the inequality:

$$B_0 L \mathrm{e}^{\frac{(1-q)d}{1-d^{\alpha-1}}} \left\|\overline{x} - x_{n-1}\right\|_X \leq \tfrac{1}{2} < 1,$$

therefore $\|W_n\| \leq \frac{1}{2} < 1$ and so there exists the mapping $\left(\mathbf{I}_X - W_n\right)^{-1} \in (X,X)^*$ and:

$$\left\|\left(\mathbf{I}_X - W_n\right)^{-1}\right\| \leq \tfrac{1}{1-\|W_n\|} \leq 2.$$

From the existence of the mapping $\Gamma^{-1}_{f;x_{n-1},x_n} \in (Y,X)^*$ and using the inequality $\left\|\Gamma^{-1}_{f;x_{n-1},x_n}\right\| \leq B_n$ we deduce the existence of the mapping $\Gamma^{-1}_{f;x_n,\overline{x}} \in (Y,X)^*$ by the equality:

$$\Gamma^{-1}_{f;x_n,\overline{x}} = \left(\mathbf{I}_X - W_n\right)^{-1} \Gamma^{-1}_{f;x_{n-1},x_n},$$

again:

$$\left\|\Gamma^{-1}_{f;x_n,\overline{x}}\right\| \leq \left\|\left(\mathbf{I}_X - W_n\right)^{-1}\right\| \cdot \left\|\Gamma^{-1}_{f;x_{n-1},x_n}\right\| \leq 2B_n \leq 2B_0 \mathrm{e}^{\frac{(1-q)d}{1-d^{\alpha-1}}}.$$

Remark 3.4. The sequence $\left(\|\Gamma_{f;x_n,\overline{x}}\|\right)_{n\in\mathbb{N}}$ also has an upper bound and for any $n \in \mathbb{N}$ we have the inequality:

$$\|\Gamma_{f;x_n,\overline{x}}\| \leq \|\Gamma_{f;x_0,\overline{x}}\| + \tfrac{(1-q)^2}{B_0} \cdot \tfrac{d^\alpha}{1-d}. \tag{16}$$

□

Indeed, for any $n \in \mathbb{N}$, we have:

$$\begin{aligned}\|\Gamma_{f;x_n,\overline{x}}\| &= \|\Gamma_{f;x_0,\overline{x}}\| + \sum_{k=1}^{n} \left(\|\Gamma_{f;x_k,\overline{x}}\| - \left\|\Gamma_{f;x_{k-1},\overline{x}}\right\|\right) \\ &\leq \|\Gamma_{f;x_0,\overline{x}}\| + \sum_{k=1}^{n} \left\|\Gamma_{f;x_k,\overline{x}} - \Gamma_{f;x_{k-1},\overline{x}}\right\| \\ &\leq \|\Gamma_{f;x_0,\overline{x}}\| + L\sum_{k=1}^{n} \|x_k - x_{k-1}\|_X \\ &\leq \|\Gamma_{f;x_0,\overline{x}}\| + \tfrac{(1-q)^2}{B_0}\sum_{k=1}^{n} d^{\alpha^k} \\ &\leq \|\Gamma_{f;x_0,\overline{x}}\| + \tfrac{(1-q)^2}{B_0} \cdot \tfrac{d^\alpha}{1-d}.\end{aligned}$$

REMARK 3.5. There exists a number $N \in \mathbb{N}$ such that for the same values $n \in \mathbb{N}$ with $n \geq N$ we have that:

$$\|\overline{x} - x_n\|_X \leq \frac{d^{\alpha^n}}{LB_0}, \tag{17}$$

and this inequality proves that the convergence order of this method is at least $\alpha = \frac{1+\sqrt{5}}{2}$. □

Indeed, as $\alpha^{n+1} = \alpha^n + \alpha^{n-1}$ it is clear that $\frac{d^{\alpha^{n+1}}}{1-d^{\alpha^n}} = d^{\alpha^n} \cdot \frac{d^{\alpha^{n-1}}}{1-d^{\alpha^n}}$ and from the fact that $\lim\limits_{n\to\infty} \frac{d^{\alpha^{n-1}}}{1-d^{\alpha^n}} = 0$ the conclusion is obvious.

4. THE ACCELERATION OF THE CONVERGENCE. ITERATIVE METHODS OF THE AITKEN-STEFFENSEN TYPE

The main conclusion of the introduction is the fact that the convergence order of the chord method is $\alpha = \frac{1+\sqrt{5}}{2}$.

At the same time it is well known that the convergence order of the Newton-Kantorovich method is 2, therefore greater. But the Newton-Kantorovich method uses the Fréchet differential instead of the divided difference.

One naturally thinks of improving the convergence order of the chord method, without renouncing at the divided difference in the favor of the Fréchet differential.

One way of doing this is using Steffensen's method, that uses a mapping $Q : X \to X$ that verifies the inclusion $Q(D) \subseteq D$ for a set $D \subseteq X$, generates the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$, starting from an arbitrary $x_0 \in D$, by the verification for any $n \in \mathbb{N}^*$ of the equality:

$$\Gamma_{f;x_n,Q(x_n)}(x_{n+1} - x_n) + f(x_n) = \theta_Y. \tag{18}$$

If for any $n \in \mathbb{N}^*$ there exists the mapping $\Gamma^{-1}_{f;x_n,Q(x_n)} \in (Y,X)^*$ the last equality is equivalent to:

$$x_{n+1} = x_n - \Gamma^{-1}_{f;x_n,Q(x_n)} f(x_n). \tag{19}$$

A more general case is the one which use two mappings $Q_1, Q_2 : X \to X$ which verify for $i \in \{1,2\}$ the relations $Q_i(D) \subseteq D$. Starting from an arbitrary $x_0 \in D$ one build the sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ by the verification for any $n \in \mathbb{N}^*$ of the equality:

$$\Gamma_{f;Q_1(x_n),Q_2(x_n)} (x_{n+1} - Q_1(x_n)) + f(Q_1(x_n)) = \theta_Y, \tag{20}$$

equality which in the hypothesis of the existence of $\Gamma^{-1}_{f;Q_1(x_n),Q_2(x_n)} \in (Y,X)^*$ is equivalent to:

$$x_{n+1} = Q_1(x_n) - \Gamma^{-1}_{f;Q_1(x_n),Q_2(x_n)} f(Q_1(x_n)). \tag{21}$$

A study of the convergence of this method, known as the iterative method of Aitken-Steffensen was made by Păvăloiu in [21]. The established result requests very strong conditions imposed on the mappings $f : D \to Y$ together with $Q_1, Q_2 : X \to X$ and one requests the verification of these conditions on every point of a set $D \subseteq X$.

We propose a more general frame and we will build an iterative process after as follows.

Let us consider a initial element $x_0 \in D$. Besides the main sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ we also use two auxiliary sequences $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D$.

For these auxiliary sequences we request the existence of the numbers K_1, $K_2, p, q > 0$ such that for any $n \in \mathbb{N}^*$ the following inequalities are verified:

$$\|f(y_n)\|_Y \le K_1 \|f(x_n)\|_Y^p, \quad \|f(z_n)\|_Y \le K_1 \|f(x_n)\|_Y^q. \tag{22}$$

Then, if for a number $n \in \mathbb{N}^*$ we have available the elements $y_n, z_n \in D$ starting from $x_n \in D$, we will generate the new iterate $x_{n+1} \in D$ by the following relation:

$$\Gamma_{f;y_n,z_n} (x_{n+1} - y_n) + f(y_n) = \theta_Y. \tag{23}$$

On account of the property of definition of the mapping $\Gamma_{f;y_n,z_n} \in (X,Y)^*$ the equality (23) is equivalent with:

$$\Gamma_{f;y_n,z_n} (x_{n+1} - z_n) + f(z_n) = \theta_Y. \tag{24}$$

If for any $n \in \mathbb{N}^*$ there exists the mapping $\Gamma^{-1}_{f;y_n,z_n} \in (Y,X)^*$ we have:

$$x_{n+1} = y_n - \Gamma^{-1}_{f;y_n,z_n} f(y_n) = z_n - \Gamma^{-1}_{f;y_n,z_n} f(z_n). \tag{25}$$

In connection to the main sequence $(x_n)_{n\in\mathbb{N}^*}$ and the auxiliary sequences $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D$ we have the following remarks. Here for a number $k \in \mathbb{N}$ we denote by $\left(X^{(k)}, Y\right)^*$ the set of mappings defined from X^k to Y that are k-linear and continuous. This set is a linear normed space as well with the norm $\|\cdot\| : \left(X^{(k)}, Y\right)^* \to Y$ is defined for $U \in \left(X^{(k)}, Y\right)^*$ by $\|U\| = \sup\limits_{h_i \in X;\ \|h_i\|=1;\ i=\overline{1,k}} \|U(h_1, ..., h_k)\|_Y$.

Remark 4.1. If the mapping $f : D \to Y$ admits Fréchet differentials up to the order $s-1$, where $s = \max\{p,q\}$ and $p, q \in \mathbb{N}$, the mapping $f^{(s-1)} : D \to \left(X^{(s-1)}, Y\right)^*$ is a Lipschitz mapping, namely there exist the constants $M^* > 0$ such that for any $x, y \in D$ the inequality:

$$\left\|f^{(s-1)}(x) - f^{(s-1)}(y)\right\| \le M^* \|x-y\|_X, \tag{26}$$

is verified and for any $n \in \mathbb{N}^*$ the inequalities:

$$\begin{aligned} \left\|\sum_{j=0}^{p-1} \tfrac{1}{j!} f^{(j)}(x_n)(y_n - x_n)^j\right\|_Y &\le \alpha \|f(x_n)\|_Y^p, \\ \left\|\sum_{j=0}^{q-1} \tfrac{1}{j!} f^{(j)}(x_n)(z_n - x_n)^j\right\|_Y &\le \beta \|f(x_n)\|_Y^q, \\ \|y_n - x_n\|_X &\le a \|f(x_n)\|_Y, \\ \|z_n - x_n\|_X &\le b \|f(x_n)\|_Y, \end{aligned} \tag{27}$$

then:

j) the mappings $f^{(p-1)} : D \to \left(X^{(p-1)}, Y\right)^*$ and $f^{(q-1)} : D \to \left(X^{(q-1)}, Y\right)^*$ are Lipschitz mappings, namely there exist the constants $M_1, M_2 > 0$ such that for any $x, y \in D$ the following inequalities are true:

$$\begin{aligned} \|f^{(p-1)}(x) - f^{(p-1)}(y)\| &\le M_1 \|x-y\|_X, \\ \|f^{(q-1)}(x) - f^{(q-1)}(y)\| &\le M_2 \|x-y\|_X; \end{aligned} \tag{28}$$

jj) the sequences $(x_n)_{n\in\mathbb{N}^*}$, $(y_n)_{n\in\mathbb{N}^*}$, $(z_n)_{n\in\mathbb{N}^*} \subseteq D$ verify the equalities (22) with:

$$K_1 = \alpha + \tfrac{M_1 a^p}{p!},\ K_2 = \beta + \tfrac{M_2 b^q}{q!}.$$

□

Indeed, from the hypothesis (26) we deduce that for any $k \in \mathbb{N},\ k \le s-1$ the mapping $f^{(k)} : D \to \left(X^{(k)}, Y\right)^*$ is a Lipschitz mapping as well (we can use the well known theorem of Lagrange), particularly, the mappings $f^{(p-1)} : D \to \left(X^{(p-1)}, Y\right)^*$ and $f^{(q-1)} : D \to \left(X^{(q-1)}, Y\right)^*$ are Lipschitz mappings and the rest of the conclusion j) is obvious especially the inequalities (28).

From these inequalities we deduce easily that for any $x, y \in D$ the following inequalities are also verified:

$$\left\| f(y) - \sum_{j=0}^{p-1} \tfrac{1}{j!} f^{(j)}(x)(y-x)^j \right\|_Y \leq \tfrac{M_1}{p!} \|y-x\|_X^p, \tag{29}$$
$$\left\| f(y) - \sum_{j=0}^{q-1} \tfrac{1}{j!} f^{(j)}(x)(y-x)^j \right\|_Y \leq \tfrac{M_2}{q!} \|y-x\|_X^q$$

(we can use Taylors' formula for mappings).

On account of the fact that x_n, y_n and $z_n \in D$, using the inequalities (27) and (29) we deduce that for any $n \in \mathbb{N}^*$ the following inequalities are verified:

$$\begin{aligned} \|f(y_n)\|_Y &\leq \left\| f(y_n) - \sum_{j=0}^{p-1} \tfrac{1}{j!} f^{(j)}(x_n)(y_n-x_n)^j \right\|_Y \\ &\quad + \left\| \sum_{j=0}^{p-1} \tfrac{1}{j!} f^{(j)}(x_n)(y_n-x_n)^j \right\|_Y \\ &\leq \tfrac{M_1}{p!} \|y_n - x_n\|_X^p + \alpha \|f(x_n)\|_Y^p \\ &\leq \left(\alpha + \tfrac{M_1 a^p}{p!}\right) \|f(x_n)\|_Y^p, \end{aligned} \tag{30}$$

and similarly:

$$\begin{aligned} \|f(z_n)\|_Y &\leq \tfrac{M_2}{q!} \|z_n - x_n\|_X^q + \beta \|f(x_n)\|_Y^q \\ &\leq \left(\beta + \tfrac{M_2 b^q}{q!}\right) \|f(x_n)\|_Y^q, \end{aligned} \tag{31}$$

and these letter inequalities justify the statement of the present remark.

Remark 4.2. It is clear that if the first of the inequalities (22) is verified for any $n \in \mathbb{N}$ with a certain $K_1 > 0$, this inequality is verified with any number $K \geq \max\{1, K_1\}$. The situation is identical regarding the second inequality from (22). In conclusion we can suppose that in these relations we have $K_1 = K_2 = K \geq 1$.

Identically, we can suppose that in the inequalities:

$$\|y_n - x_n\|_X \leq a \|f(x_n)\|_Y, \; \|z_n - x_n\|_X \leq b \|f(x_n)\|_Y,$$

that are true for any $n \in \mathbb{N}$, we can have $b = a \geq 1$.

In conclusion, for the main sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ together with the auxiliary sequences $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D$ we can suppose that for any $n \in \mathbb{N}^*$ we have $y_n \neq z_n$ and there exist the numbers $K, a \geq 1$ such that for any $n \in \mathbb{N}^*$

the following inequalities are verified:

$$\begin{aligned} &\|f(y_n)\|_Y \le K\|f(x_n)\|_Y^p, \\ &\|f(z_n)\|_Y \le K\|f(x_n)\|_Y^q, \\ &\|y_n - x_n\|_X \le a\|f(x_n)\|_Y, \\ &\|z_n - x_n\|_X \le a\|f(x_n)\|_Y \end{aligned} \tag{32}$$

□

The statements are obvious.

5. THE CONVERGENCE OF SOME AUXILIARY REAL NUMBER SEQUENCES

In connection with the enounced problem we consider, for the real numbers $p, q \ge 1$, the following equation in x on the interval $[0, +\infty[$:

$$x^{p+q-1} + 2x^2 + 2x - 1 = 0. \tag{33}$$

We have the following remarks:

REMARKS 5.1. **a)** The equation (33) has an unique positive root and this root is $\alpha \in]0,1[$.
b) If $\alpha \in]0,1[$ is the root of the equation (33) one verifies the following inequalities as well:

$$\alpha^2 + \alpha - 1 < 0,\ \alpha^2 + 2\alpha - 1 < 0,\ 2\alpha^2 + 2\alpha - 1 < 0 \tag{34}$$

and these inequalities are equivalent to the following inequalities respectively:

$$0 < \tfrac{\alpha^2}{1-\alpha} < 1,\ 0 < \tfrac{\alpha}{1-\alpha-\alpha^2} < 1,\ 0 < \tfrac{\alpha^2}{1-2\alpha-\alpha^2} < 1. \tag{35}$$

□

Indeed, let us consider the function $\varphi : [0, +\infty[\to \mathbb{R}$ defined by $\varphi(x) = x^{p+q-1} + 2x^2 + 2x - 1$. It is obvious that for any $x \in [0, +\infty[$ there exists the derivative $\varphi'(x)$ at the point x and:

$$\varphi'(x) = (p + q - 1)\, x^{p+q-2} + 4x + 2.$$

As it is obvious that for any $x \in [0, +\infty[$ we have that $\varphi'(x) > 0$, therefore the function $\varphi : [0, +\infty[\to \mathbb{R}$ is a strictly increasing function, therefore an injective function, thus the equation $\varphi(x) = 0$ has at most one root.

As $\varphi(0) = -1$, $\varphi(1) = 4$, this root exists indeed and it belongs to the interval $]0, 1[$.

If $\alpha \in]0,1[$ is the root of the equation (33) it is clear that $2\alpha^2 + 2\alpha - 1 = -\alpha^{p+q-1} < 0$, whence $\alpha^2 + 2\alpha - 1 = -\alpha^{p+q-1} - \alpha^2 < 0$ and $\alpha^2 + \alpha - 1 = -\alpha^{p+q-1} - \alpha^2 - \alpha < 0$, therefore the relations (34) are true.

The inequality $\alpha^2 + \alpha - 1 < 0$ is equivalent to $0 < \alpha^2 < 1 - \alpha$ therefore to $0 < \frac{\alpha^2}{1-\alpha} < 1$, the inequality $\alpha^2 + 2\alpha - 1 < 0$ is equivalent to $0 < \alpha < 1 - \alpha - \alpha^2$, therefore to $0 < \frac{\alpha}{1-\alpha-\alpha^2} < 1$, again the inequality $2\alpha^2 + 2\alpha - 1 < 0$ is equivalent to $0 < \alpha^2 < 1 - 2\alpha - \alpha^2$ therefore to $0 < \frac{\alpha^2}{1-2\alpha-\alpha^2} < 1$.

Let us consider now the numbers $a, K, L, B_0, R_0 > 0$ and the numbers $p, q \geq 1$ and using these numbers we build the real number sequences $(u_n)_{n\in\mathbb{N}^*}$, $(s_n)_{n\in\mathbb{N}^*}$, $(v_n)_{n\in\mathbb{N}^*}$, $(w_n)_{n\in\mathbb{N}^*}$, $(t_n)_{n\in\mathbb{N}^*}$, $(B_n)_{n\in\mathbb{N}^*}$ and $(R_n)_{n\in\mathbb{N}^*}$ using the following recurrence relations:

$$
\begin{aligned}
u_n &= LKB_n^2R_n^p,\\
s_n &= LKB_n^2R_n^q,\\
v_n &= aL^2K^2 \cdot \tfrac{B_n^3R_n^{p+q}}{1-u_n},\\
w_n &= \tfrac{LKB_n^2R_n^q}{(1-u_n)(1-v_n)},\\
t_n &= \tfrac{aL^2K^2B_n^3R_n^{p+q}}{(1-u_n)(1-v_n)(1-w_n)},\\
B_{n+1} &= \tfrac{B_n}{(1-u_n)(1-v_n)(1-w_n)(1-t_n)},\\
R_{n+1} &= LK^2B_n^2R_n^{p+q}.
\end{aligned}
\tag{36}
$$

It is obvious that this construction has a meaning if for any $n \in \mathbb{N}^*$ we have that $u_n,\ v_n,\ w_n,\ t_n \in \mathbb{R}\setminus\{1\}$ and $B_n, R_n > 0$.

It is clear that for any $n \in \mathbb{N}^*$ we have:

$$
\begin{aligned}
v_n &= \tfrac{a}{B_n} \cdot \tfrac{u_ns_n}{1-u_n},\\
w_n &= \tfrac{s_n}{(1-u_n)(1-v_n)},\\
t_n &= \tfrac{v_n}{(1-v_n)(1-w_n)},\\
R_{n+1} &= \tfrac{u_ns_n}{LB_n^2},
\end{aligned}
\tag{37}
$$

as well.

Referring to the sequences that are defined by the relations (36) we have the following proposition:

Proposition 5.2. *If the following inequalities are verified:*

$$
a \leq B_0 \leq \tfrac{1}{\sqrt{L}} \cdot \min\left\{K^{\frac{p-q+1}{2(q-1)}}, K^{\frac{q-p+1}{2(p-1)}}\right\}
\tag{38}
$$

(with the specification that for $q = 1$ the expression that has $q - 1$ in its denominator is $+\infty$, and the same for the expression that has $p - 1$ in its denominator) and:

$$
d = \frac{LKB_0^2}{\alpha^2} \cdot \max{}^{\frac{1}{p+q-1}}\left\{\frac{R_0^{p(p+q-1)}K^{p-q+1}}{\left(LB_0^2\right)^{q-1}}, \frac{R_0^{q(p+q-1)}K^{q-p+1}}{\left(LB_0^2\right)^{p-1}}\right\} < 1
\tag{39}
$$

where $\alpha \in\ ]0,1[$ is the unique root of the equation (33), then for any $n \in \mathbb{N}^*$ we have the following inequalities:

$$
u_n \leq \alpha d^{(p+q)^n} < \alpha < 1,
\tag{40}
$$

$$
\begin{aligned}
s_n &\leq \alpha d^{(p+q)^n} < \alpha < 1,\\
v_n &\leq \tfrac{\alpha^2}{1-\alpha} \cdot d^{2(p+q)^n} < \tfrac{\alpha^2}{1-\alpha} < 1,\\
w_n &\leq \tfrac{\alpha}{1-\alpha-\alpha^2} \cdot d^{(p+q)^n} < \tfrac{\alpha}{1-\alpha-\alpha^2} < 1,\\
t_n &\leq \tfrac{\alpha^2}{1-2\alpha-\alpha^2} d^{2(p+q)^n} < \tfrac{\alpha^2}{1-2\alpha-\alpha^2} < 1,\\
B_{n+1} &\leq \tfrac{B_n}{1-2\alpha-2\alpha^2}, \qquad (41)\\
R_{n+1} &\leq \tfrac{\alpha^2}{LB_0^2} d^{2(p+q)^n}.
\end{aligned}
$$

Proof. Let us consider first that $q > 1$.

From the definition of d it is clear that:

$$\frac{LKB_0^2R_0^2}{\alpha^2}\left[\frac{K^{p-q+1}}{\left(LB_0^2\right)^{q-1}}\right]^{\frac{1}{p+q-1}} \leq d,$$

therefore:

$$u_0 \leq \alpha \cdot \alpha\left(\frac{LB_0^2}{K^{\frac{p-q+1}{q-1}}}\right)^{\frac{q-1}{p+q-1}} \cdot d.$$

From $B_0 \leq \frac{1}{\sqrt{L}} K^{\frac{p-q+1}{2(q-1)}}$ we deduce that $\frac{K^{\frac{p-q+1}{q-1}}}{LB_0^2} \geq 1$, therefore as $\frac{q-1}{p+q-1} \geq 0$ it is clear that $\left(\frac{K^{\frac{p-q+1}{q-1}}}{LB_0^2}\right)^{\frac{q-1}{p+q-1}} \geq 1 > \alpha$, therefore:

$$\alpha\left(\frac{LB_0^2}{K^{\frac{p-q+1}{q-1}}}\right)^{\frac{q-1}{p+q-1}} < 1, \tag{42}$$

therefore:

$$u_0 \leq \alpha d = \alpha d^{(p+q)^0} < \alpha < 1.$$

For $q = 1$ from the same definition of d we have:

$$\frac{LKB_0^2}{\alpha^2} R_0^p K \leq d,$$

namely $u_0 \leq \alpha \cdot \frac{\alpha}{K} \cdot d$.

But $\alpha < 1 \leq K$, so $\frac{\alpha}{K} < 1$, therefore $u_0 \leq \alpha d$.

Identically if $p > 1$ we have that:

$$\frac{LKB_0^2R_0^q}{\alpha^2}\left[\frac{K^{q-p+1}}{\left(LB_0^2\right)^{p-1}}\right]^{\frac{1}{p+q-1}} \leq d,$$

whence, in the same manner as in the case of u_0, by inverting the roles of the numbers p and q, we deduce that $s_0 \leq \alpha d = \alpha d^{(p+q)^0} < \alpha < 1$.

For $p = 1$ one can show the inequality $s_0 \leq \alpha d$ in the same manner as the inequality $u_0 \leq \alpha d$ for the case of $q = 1$.

As $v_0 = \frac{a}{B_0} \cdot \frac{u_0 s_0}{1-u_0}$ and using $a \leq B_0$ and the inequalities concerning to u_0 and s_0, we have that:

$$v_0 \leq \frac{\alpha^2}{1-\alpha} \cdot d^2 = \frac{\alpha^2}{1-\alpha} \cdot d^{2(p+q)^0} < \frac{\alpha^2}{1-\alpha} < 1,$$

the last inequality being the first from (35).

Also:

$$w_0 = \frac{s_0}{(1-u_0)(1-v_0)} \le \frac{\alpha d}{(1-\alpha)\left(1-\frac{\alpha^2}{1-\alpha}\right)} =$$
$$= \frac{\alpha}{1-\alpha-\alpha^2} \cdot d^{(p+q)^0} < \frac{\alpha}{1-\alpha-\alpha^2} < 1,$$

the last inequality being the second from (35), while:

$$t_0 = \frac{v_0}{(1-v_0)(1-w_0)} \le \frac{\frac{\alpha^2}{1-\alpha}\cdot d}{\left(1-\frac{\alpha^2}{1-\alpha}\right)\left(1-\frac{\alpha}{1-\alpha-\alpha^2}\right)} =$$
$$= \frac{\alpha^2}{1-2\alpha-\alpha^2} \cdot d^{(p+q)^0} < \frac{\alpha^2}{1-2\alpha-\alpha^2} < 1,$$

the last inequality being the third from (35).

As $u_0, v_0 \in \,]0,1[$ it is clear that $v_0, w_0, t_0 \in \,]0,1[$ as well.

Then we have:

$$\begin{aligned}\frac{B_1}{B_0} &= \frac{1}{(1-u_0)(1-v_0)(1-w_0)(1-t_0)} \\ &\le \frac{1}{(1-\alpha)\left(1-\frac{\alpha^2}{1-\alpha}\right)\left(1-\frac{\alpha}{1-\alpha-\alpha^2}\right)\left(1-\frac{\alpha^2}{1-2\alpha-\alpha^2}\right)} \\ &= \frac{1}{1-2\alpha-2\alpha^2},\end{aligned}$$

therefore:

$$B_1 \le \frac{B_0}{1-2\alpha-2\alpha^2}.$$

As:

$$R_1 = \frac{1}{LB_0^2} \cdot u_0 s_0 = \frac{\alpha^2}{LB_0^2} \cdot d^2 = \frac{\alpha^2}{LB_0^2} \cdot d^{2(p+q)^0}.$$

From the afore established relations we deduce that the properties (40)-(41) are true in the case of $n = 0$.

We suppose that these properties are true for any $n \in \mathbb{N}$ with $n \le k$ and we prove that they are also true for $n = k+1$.

For any $i \in \mathbb{N}$ with $i \le k$ we have that:

$$\begin{aligned}u_{i+1} &= LKB_{i+1}^2 R_{i+1}^p = LK \cdot \frac{B_i^2}{(1-2\alpha-2\alpha^2)^2} \cdot L^p K^{2p} B_i^{2p} R_i^{p(p+q)} \\ &= \frac{L^{p+1}K^{2p+1}B_i^{2p+2}R_i^{p(p+q)}}{(1-2\alpha-2\alpha^2)^2} = \frac{L^{p+1}K^{2p+1}}{(1-2\alpha-2\alpha^2)^2} \cdot \frac{\left(B_i^2 R_i^p\right)^{p+q}}{B_i^{2q-2}} \\ &= \frac{L^{p+1}K^{2p+1}}{(1-2\alpha-2\alpha^2)^2} \cdot \frac{1}{B_i^{2q-2}} \cdot \left(\frac{u_i}{LK}\right)^{p+q}.\end{aligned}$$

From the equality that defined the sequence $(B_n)_{n\in\mathbb{N}^*}$ we deduce that:

$$\frac{B_i}{B_{i-1}} = \frac{1}{1-u_{i-1}} \cdot \frac{1}{1-v_{i-1}} \cdot \frac{1}{1-w_{i-1}} \cdot \frac{1}{1-s_{i-1}} \ge 1,$$

every fraction from those that multiply being greater than the unit.

Therefore $B_i \ge B_{i-1}$.

From this we actually have that $B_i \ge B_0$, from where as $2q-2 \ge 0$ we have $B_i^{2q-2} \ge B_0^{2q-2}$, namely $\frac{1}{B_i^{2q-2}} \le \frac{1}{B_0^{2q-2}}$ and so:

$$u_{i+1} \le \frac{L^{1-q}K^{p-q+1}}{(1-2\alpha-2\alpha^2)^2} \cdot \frac{1}{B_0^{2(q-1)}} \cdot u_i^{p+q} = Cu_i^{p+q}, \tag{43}$$

where:

$$C = \frac{K^{p-q+1}}{\left(LB_0^2\right)^{q-1}} \cdot \frac{1}{\left(1-2\alpha-2\alpha^2\right)^2}.$$

If we multiply by $C^{\frac{1}{p+q-1}}$ the two members of the inequality (43) we obtain that:

$$C^{\frac{1}{p+q-1}} u_{i+1} \leq \left(C^{\frac{1}{p+q-1}} u_i\right)^{p+q},$$

namely the inequality $h_{i+1} \leq h_i^{p+q}$ if $h_i = C^{\frac{1}{p+q-1}} u_i$.

As $i \in \{0, 1, ..., k\}$ we immediately deduce that:

$$h_1 \leq h_0^{p+q},\ h_2 \leq h_1^{p+q} \leq h_0^{(p+q)^2},\ \ldots, h_{k+1} \leq h_k^{p+q} = h_0^{(p+q)^{k+1}}.$$

Therefore we have that:

$$u_{k+1} \leq \left(\tfrac{1}{C}\right)^{\frac{1}{p+q-1}} \left(C^{\frac{1}{p+q-1}} u_0\right)^{(p+q)^{k+1}}. \tag{44}$$

But $\alpha \in]0, 1[$ is the root of the equation (33), therefore:

$$\left(1 - 2\alpha - 2\alpha^2\right)^2 = \alpha^{2(p+q-1)},$$

therefore:

$$\begin{aligned} C^{\frac{1}{p+q-1}} u_0 &= \frac{1}{\alpha^2}\left[\frac{K^{p-q+1}}{(LB_0)^{q-1}}\right]^{\frac{1}{p+q-1}} LKB_0^2 R_0^p \\ &= \frac{LKB_0^2}{\alpha}\left[\frac{R_0^{p(p+q-1)}K^{p-q+1}}{(LB_0)^{q-1}}\right]^{\frac{1}{p+q-1}} \leq d. \end{aligned}$$

At the same time, if $q > 1$ we have that:

$$\begin{aligned} \left(\tfrac{1}{C}\right)^{\frac{1}{p+q-1}} &= \left(1 - 2\alpha - 2\alpha^2\right)^{\frac{2}{p+q-1}} \left[\frac{\left(LB_0^2\right)^{q-1}}{K^{p-q+1}}\right]^{\frac{1}{p+q-1}} \\ &= \alpha^2 \left(\frac{LB_0^2}{K^{\frac{p-q+1}{q-1}}}\right)^{\frac{q-1}{p+q-1}} < \alpha; \end{aligned}$$

for the last inequality we have take into account (42).

For $q = 1$ we have:

$$\left(\tfrac{1}{C}\right)^{\frac{1}{p+q-1}} = \left(\tfrac{1}{C}\right)^{\frac{1}{p}} = \frac{\alpha^2}{K} < \alpha.$$

So, from the inequality (44) we obtain that $u_{k+1} \leq \alpha d^{(p+q)^{k+1}}$ and as $d < 1$, we obtain $u_{k+1} \leq \alpha d^{(p+q)^{k+1}} < \alpha < 1$.

One can obtain the inequality $s_{k+1} \leq \alpha d^{(p+q)^{k+1}} < \alpha < 1$ in a similar manner with the one concerning u_{k+1}. It is only necessary to invert the roles of the numbers p and q.

We have:

$$v_{k+1} = \frac{a}{B_{k+1}} \cdot \frac{u_{k+1}s_{k+1}}{1-u_{k+1}}.$$

Identically, as $\frac{B_i}{B_{i-1}} \geq 1$ for any $i \leq k$ we deduce that $\frac{B_{k+1}}{B_k} \geq 1$, therefore $B_{k+1} \geq B_k$ and in fact $B_{k+1} \geq B_0$, namely:

$$v_{k+1} \leq \frac{a}{B_0} \cdot \frac{\alpha^2 d^{2(p+q)^{k+1}}}{1-\alpha}.$$

As $a \leq B_0$ and $d < 1$ we have:

$$v_{k+1} \leq \frac{\alpha^2}{1-\alpha} \cdot d^{2(p+q)^{k+1}} < \frac{\alpha^2}{1-\alpha} < 1.$$

In the same manner we have:

$$\begin{aligned} w_{k+1} &= \frac{s_{k+1}}{(1-u_{k+1})(1-v_{k+1})} \leq \frac{\alpha d^{(p+q)^{k+1}}}{(1-\alpha)\left(1-\frac{\alpha^2}{1-\alpha}\right)} \\ &= \frac{\alpha}{1-\alpha-\alpha^2} \cdot d^{(p+q)^{k+1}} < \frac{\alpha}{1-\alpha-\alpha^2} < 1 \end{aligned}$$

and:

$$\begin{aligned} t_{k+1} &= \frac{v_{k+1}}{(1-v_{k+1})(1-w_{k+1})} \leq \frac{\frac{\alpha^2}{1-\alpha} \cdot d^{2(p+q)^{k+1}}}{\left(1-\frac{\alpha^2}{1-\alpha}\right)\left(1-\frac{\alpha}{1-\alpha-\alpha^2}\right)} \\ &= \frac{\alpha^2}{1-2\alpha-\alpha^2} \cdot d^{2(p+q)^{k+1}} < \frac{\alpha^2}{1-2\alpha-\alpha^2} < 1. \end{aligned}$$

For the sequence $(B_n)_{n\in\mathbb{N}^*}$ we have:

$$\begin{aligned} \frac{B_{k+2}}{B_{k+1}} &= \frac{1}{(1-u_{k+1})(1-v_{k+1})(1-w_{k+1})(1-t_{k+1})} \\ &\leq \frac{1}{(1-\alpha)\left(1-\frac{\alpha^2}{1-\alpha}\right)\left(1-\frac{\alpha}{1-\alpha-\alpha^2}\right)\left(1-\frac{\alpha^2}{1-2\alpha-\alpha^2}\right)} \\ &= \frac{1}{1-2\alpha-2\alpha^2}, \end{aligned}$$

therefore:

$$B_{k+2} \leq \frac{B_{k+1}}{1-2\alpha-2\alpha^2}.$$

Finally:

$$R_{k+2} = \frac{u_{k+1}s_{k+1}}{LB_{k+1}^2} \leq \frac{1}{LB_0}\alpha^2 d^{2(p+q)^{k+1}}.$$

Therefore the inequalities (40) are also true for $n = k + 1$.

On the basis of the principle of the mathematical induction these inequalities are true for any $n \in \mathbb{N}^*$. The proposition is proved. □

6. THE MAIN RESULT

We return to the issue of the convergence of the sequences $(x_n)_{n\in\mathbb{N}^*}$, $(y_n)_{n\in\mathbb{N}^*}$, $(x_n)_{n\in\mathbb{N}^*} \subseteq D \subseteq X$ to the solution of the equation $f(x) = \theta_Y$, where $f : D \to Y$.

We have the following fundamental result:

Theorem 6.1. *Suppose that the following assumptions hold:*

i) *The linear normed space $(X, \|\cdot\|_X)$ is a Banach space;*

ii) *The mapping $f : D \to Y$ admits for any $x, y \in D$ with $x \neq y$ a generalized abstract divided difference $\Gamma_{f;x,y} \in (X,Y)^*$ and there exists a number $L > 0$ such that for any $x, y, z \in D$ with $x \neq y,\ y \neq z$ we have the inequality:*

$$\|\Gamma_{f;x,y} - \Gamma_{f;y,z}\| \leq L\, \|x - z\|_X\,;$$

iii) *The main approximant sequence* $(x_n)_{n\in\mathbb{N}^*}$ *together with the secondary sequences* $(y_n)_{n\in\mathbb{N}^*}$ *and* $(z_n)_{n\in\mathbb{N}^*}$ *are such that for any* $n \in \mathbb{N}^*$ *the following equality is fulfilled:*

$$\Gamma_{f;y_n,z_n}(x_{n+1}-y_n)+f(y_n)=\theta_Y \tag{45}$$

and the inequalities (32) with the constants $a, K > 0$. *We also have that* $f(y_n), f(z_n) \in Y \setminus \{\theta_Y\}$, $y_n \neq z_n$ *and we are in one of the following situations:*

iii$_1$) $x_n \neq y_n$ *and* $y_{n+1} \neq z_n$,

or

iii$_2$) $x_n \neq z_n$ *and* $z_{n+1} \neq y_n$.

iv) *The mapping* $\Gamma_{f;y_0,z_0} \in (X,Y)^*$ *is invertible and* $\Gamma^{-1}_{f;y_0,z_0} \in (Y,X)^*$.

v) *Denoting:*

$$B_0 = \max\left\{a, \left\|\Gamma^{-1}_{f;y_0,z_0}\right\|\right\},$$
$$R_0 = \|f(x_0)\|_Y,$$
$$\overline{K} = \max\left\{K, \left(B_0\sqrt{L}\right)^{\frac{2(q-1)}{p-q+1}}, \left(B_0\sqrt{L}\right)^{\frac{2(p-1)}{q-p+1}}\right\},$$
$$d = \frac{L\overline{K}B_0^2}{\alpha^2}\cdot\max\left\{\frac{R_0^{p(p+q-1)}\overline{K}^{p-q+1}}{(LB_0^2)^{q-1}}, \frac{R_0^{q(p+q-1)}\overline{K}^{q-p+1}}{(LB_0^2)^{p-1}}\right\},$$
$$\delta = 2aR_0 + \frac{a\alpha^2}{LB_0^2}\cdot\frac{d^2}{1-d^{2(p+q-1)}} + \frac{2\alpha}{L\overline{K}B_0}\cdot\frac{d}{1-d^{p+q-1}},$$

where $\alpha \in \left]0,1\right[$ *is the unique root of the equation (33), the conditions* $d < 1$ *and* $S(x_0,\delta) = \{x \in X / \|x-x_0\|_X \leq \delta\}$ *are fulfilled.*

Then the following conclusions are true:

j) *for any* $n \in \mathbb{N}^*$ *we have that* $x_n, y_n, z_n \in S(x_0,\delta)$, *there exists the mapping* $\Gamma^{-1}_{f;y_n,z_n} \in (Y,X)^*$ *and:*

$$x_{n+1} = y_n - \Gamma^{-1}_{f;y_n,z_n} f(y_n) = z_n - \Gamma^{-1}_{f;y_n,z_n} f(z_n); \tag{46}$$

jj) *the sequences* $(x_n)_{n\in\mathbb{N}^*}, (y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq X$ *are convergent to the limit* $\overline{x} \in S(x_0,\delta)$ *for that* $f(\overline{x}) = \theta_Y$;

jjj) *for any* $n \in \mathbb{N}^*$ *the following inequalities are fulfilled:*

$$\|x_{n+1}-x_n\|_X \leq \frac{a\alpha^2}{LB_0^2}\cdot d^{2(p+q)^{n-1}} + \frac{\alpha}{L\overline{K}B_0}\cdot d^{(p+q)^n}; \tag{47}$$

$$\|x_n-\overline{x}\|_X \leq \frac{a\alpha^2}{LB_0^2}\cdot\frac{d^{2(p+q)^{n-1}}}{1-d^{2(p+q)^{n-1}(p+q-1)}} + \frac{\alpha}{L\overline{K}B_0}\cdot\frac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}}; \tag{48}$$

$$\max\left\{\|y_n-\overline{x}\|_X, \|z_n-\overline{x}\|_X\right\} \leq \frac{a\alpha^2}{LB_0^2}\cdot d^{2(p+q)^{n-1}}\cdot\frac{2-d^{2(p+q)^{n-1}(p+q-1)}}{1-d^{2(p+q)^{n-1}(p+q-1)}} + \frac{\alpha}{L\overline{K}B_0}\cdot\frac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}}.$$

The proof of this result we will given in the following paper of this paperset.

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