REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 42 (2013) no. 2, pp. 161–169
ictp.acad.ro/jnaat

SOME ESTIMATIONS FOR THE TAYLOR'S REMAINDER[‡]

HUI SUN,[*] BO-YONG LONG[†] and YU-MING CHU[†]

Abstract. In this paper, we establish several integral inequalities for the Taylor's remainder by Grüss and Cheyshev inequalities.

MSC 2000. 26D15.

Keywords. Taylor remainder, Grüss inequality, Cheyshev inequality.

1. INTRODUCTION

For two given integrable functions f and g on $[a,b]$, the Chebychev functional $T(f,g)$ is defined by

$$T(f,g) = \tfrac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x \cdot \tfrac{1}{b-a}\int_a^b g(x)\mathrm{d}x.$$

In 1935, Grüss [1] proved that

$$|T(f,g)| \leq \tfrac{1}{4}(M-m)(L-l) \tag{1.1}$$

if

$$m \leq f(x) \leq M, \qquad l \leq g(x) \leq L$$

for all $x \in [a,b]$, where M, m, L and l are constants. Inequality (1.1) is called Grüss inequality.

The well-known Chebyshev inequality [2] can be stated as follows: if both f and g are increasing or decreasing, then

$$T(f,g) \geq 0.$$

If one of the functions f and g is increasing and the other decreasing, then the above inequality is reversed.

[*] School of Mathematics and Computation Science, Hunan City University, Yiyang 413000, Hunan, China, e-mail: `sunhui4624117@163.com`.

[†]Department of Mathematics, Huzhou Teachers College, Huzhou 313000, Zhejiang, China, e-mail: `lbymxy@163.com,chuyuming@hutc.zj.cn`.

[‡] The work of the first author has been supported by the Natural Science Foundation of the Department of Education of Hunan Province (Grant No. 13C127) and the work of the last author has been supported by the Natural Science Foundation of China (Grant No. 61374086).

In what follows n denotes a non-negative integer. We denote by $R_{n,f}(x_0,x)$ the nth Taylor remainder of the function $f(x)$ with center x_0, i.e.

$$R_{n,f}(x_0,x)=f(x)-\sum_{k=0}^{n}\tfrac{(x-x_0)^k}{k!}f^{(k)}(x_0).$$

The Taylor remainder has been the subject of intensive research [3]–[9]. In particular, many remarkable integral inequalities for the Taylor remainder can be found in the literature [5]–[7]. The following Theorems A and B were proved by Gauchman in [6].

THEOREM A. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x)\in C^{n+1}[a,b]$ and $m\le f^{(n+1)}(x)\le M$ for each $x\in[a,b]$, where m and M are constants. Then*

$$\Big|\int_a^b R_{n,f}(a,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\Big|\le\tfrac{(b-a)^{n+2}}{4(n+1)!}(M-m)$$

and

$$\Big|(-1)^{n+1}\int_a^b R_{n,f}(b,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\Big|\le\tfrac{(b-a)^{n+2}}{4(n+1)!}(M-m).$$

THEOREM B. *Let $f(x)$ be a function defined on $[a,b]$ such that $f(x)\in C^{n+1}[a,b]$. If $f^{(n+1)}(x)$ is increasing on $[a,b]$, then*

$$\begin{aligned}&-\tfrac{f^{(n+1)}(b)-f^{(n+1)}(a)}{4(n+1)!}(b-a)^{n+2}\le\\&\le\int_a^b R_{n,f}(a,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\le 0\end{aligned}$$

and

$$\begin{aligned}0&\le(-1)^{n+1}\int_a^b R_{n,f}(b,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\\&\le\tfrac{f^{(n+1)}(b)-f^{(n+1)}(a)}{4(n+1)!}(b-a)^{n+2}.\end{aligned}$$

If $f^{(n+1)}(x)$ is decreasing on $[a,b]$, then

$$\begin{aligned}0&\le\int_a^b R_{n,f}(a,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\le\\&\le\tfrac{f^{(n+1)}(a)-f^{(n+1)}(b)}{4(n+1)!}(b-a)^{n+2}\end{aligned}$$

and

$$\begin{aligned}&-\tfrac{f^{(n+1)}(a)-f^{(n+1)}(b)}{4(n+1)!}(b-a)^{n+2}\le\\&\le(-1)^{n+1}\int_a^b R_{n,f}(b,x)\mathrm{d}x-\tfrac{f^{(n)}(b)-f^{(n)}(a)}{(n+2)!}(b-a)^{n+1}\le 0.\end{aligned}$$

It is the aim of this paper to establish several new inequalities for the Taylor remainder by Grüss and Cheyshev inequalities.

2. MAIN RESULTS

LEMMA 2.1. *Let $f(x)$ be a function defined on $[a,b]$ and $x_0 \in (a,b)$. If $f(x) \in C^{n+1}[a,b]$, then*

$$\int_{x_0}^{b} R_{n,f}(x_0,x)\mathrm{d}x = \int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x \tag{2.1}$$

and

$$\int_{a}^{x_0} R_{n,f}(x_0,x)\mathrm{d}x = \int_{a}^{x_0} \tfrac{(a-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x. \tag{2.2}$$

Proof. We only give the proof of (2.1) in detail, the similar argument leads to (2.2). It follows from the formula of integration by parts that

$$\begin{aligned}
&\int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x = \\
&= \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n)}(x)|_{x_0}^{b} + \int_{x_0}^{b} \tfrac{(b-x)^{n}}{n!} f^{(n)}(x)\mathrm{d}x \\
&= -\tfrac{(b-x_0)^{n+1}}{(n+1)!} f^{(n)}(x_0) + \tfrac{(b-x)^{n}}{n!} f^{(n-1)}(x)|_{x_0}^{b} + \int_{x_0}^{b} \tfrac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\mathrm{d}x \\
&= -\tfrac{(b-x_0)^{n+1}}{(n+1)!} f^{(n)}(x_0) - \tfrac{(b-x_0)^{n}}{n!} f^{(n-1)}(x_0) - \cdots - (b-x_0)f(x_0) + \int_{x_0}^{b} f(x)\mathrm{d}x \\
&= \int_{x_0}^{b} [f(x) - \sum_{k=0}^{n} \tfrac{f^{(k)}(x_0)}{k!}(x-x_0)^k]\mathrm{d}x \\
&= \int_{x_0}^{b} R_{n,f}(x_0,x)\mathrm{d}x.
\end{aligned}$$

□

THEOREM 2.2. *Let $f(x) \in C^{n+1}[a,b]$, such that $m_1 \le f^{(n+1)}(x) \le M_1$ for $x \in [a,x_0]$ and $m_2 \le f^{(n+1)}(x) \le M_2$ for $x \in [x_0,b]$, where m_1, m_2, M_1 and M_2 are constants. Then*

$$\begin{aligned}
&\Big| \int_{a}^{b} R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \\
&\quad - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \Big| \\
&\le \tfrac{(x_0-a)^{n+2}}{4(n+1)!}(M_1-m_1) + \tfrac{(b-x_0)^{n+2}}{4(n+1)!}(M_2-m_2).
\end{aligned} \tag{2.3}$$

Proof. Let

$$F(x) = f^{(n+1)}(x), \qquad G(x) = \tfrac{(b-x)^{n+1}}{(n+1)!}.$$

Then for any $x \in [x_0, b]$, we clearly see that

$$m_2 \le F(x) \le M_2, \qquad 0 \le G(x) \le \tfrac{(b-x_0)^{n+1}}{(n+1)!}.$$

Making use of Grüss inequality (1.1) one has

$$\begin{aligned} &\Big|\int_{x_0}^{b} F(x)G(x)\mathrm{d}x - \tfrac{1}{b-x_0}\int_{x_0}^{b} F(x)\mathrm{d}x \int_{x_0}^{b} G(x)\mathrm{d}x\Big| = \\ &= \Big|\int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x - \tfrac{1}{b-x_0}\int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!}\mathrm{d}x \int_{x_0}^{b} f^{(n+1)}(x)\mathrm{d}x\Big| \\ &\le \tfrac{(b-x_0)^{n+2}}{4(n+1)!}(M_2 - m_2). \end{aligned} \tag{2.4}$$

Equation (2.1) and inequality (2.4) lead to the conclusion that

$$\begin{aligned} &\Big|\int_{x_0}^{b} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1}\Big| \le \\ &\le \tfrac{(b-x_0)^{n+2}}{4(n+1)!}(M_2 - m_2). \end{aligned} \tag{2.5}$$

Similarly, if $x \in [a, x_0]$, then Grüss inequality (1.1) leads to

$$\begin{aligned} &\Big|\int_{a}^{x_0} \tfrac{(a-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x \\ &\quad - \tfrac{1}{x_0-a}\int_{a}^{x_0} \tfrac{(a-x)^{n+1}}{(n+1)!}\mathrm{d}x \int_{a}^{x_0} f^{(n+1)}(x)\mathrm{d}x\Big| \le \\ &\le \tfrac{(x_0-a)^{n+2}}{4(n+1)!}(M_1 - m_1). \end{aligned} \tag{2.6}$$

Equation (2.2) and inequality (2.6) imply that

$$\begin{aligned} &\Big|\int_{a}^{x_0} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1}\Big| \le \\ &\le \tfrac{(x_0-a)^{n+2}}{4(n+1)!}(M_1 - m_1). \end{aligned} \tag{2.7}$$

Therefore, inequality (2.3) follows form inequalities (2.5) and (2.7). □

If take $n = 1$ in Theorem 2.2, then we have

Corollary 2.3. *Let $f(x) \in C^2[a, b]$ and $m_1 \le f^{(2)}(x) \le M_1$ for any $x \in [a, x_0]$, $m_2 \le f^{(2)}(x) \le M_2$ for any $x \in [x_0, b]$, where m_1, m_2, M_1 and M_2 are constants. Then*

$$\begin{aligned} &\Big|\int_{a}^{b} f(x)\mathrm{d}x - (b-a)f(x_0) - \tfrac{f'(b)}{6}(b-x_0)^2 + \tfrac{f'(a)}{6}(a-x_0)^2 \\ &\quad - \tfrac{f'(x_0)}{3}(a+b-2x_0)(b-a)\Big| \le \\ &\le \tfrac{(x_0-a)^3}{8}(M_1 - m_1) + \tfrac{(b-x_0)^3}{8}(M_2 - m_2). \end{aligned}$$

In particular, if $x_0 = \frac{a+b}{2}$, then Corollary 2.3 becomes

Corollary 2.4. *Let $f(x) \in C^2[a,b]$ and $m_1 \le f^{(2)}(x) \le M_1$ for any $x \in [a, \frac{a+b}{2}]$, $m_2 \le f^{(2)}(x) \le M_2$ for any $x \in [\frac{a+b}{2}, b]$, where m_1, m_2, M_1 and M_2 are constants. Then*

$$\Big|\tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f(\tfrac{a+b}{2}) - \tfrac{1}{24}(b-a)(f'(b)-f'(a))\Big| \le$$
$$\le \tfrac{(b-a)^2}{64}(M_1 - m_1 + M_2 - m_2).$$

If take $n = 0$ in Theorem 2.2, then we have

Corollary 2.5. *Let $f(x) \in C^1[a,b]$ and $m_1 \le f'(x) \le M_1$ for any $x \in [a, x_0]$, $m_2 \le f'(x) \le M_2$ for any $x \in [x_0, b]$, where m_1, m_2, M_1 and M_2 are constants. Then*

$$\Big|\int_a^b f(x)\mathrm{d}x - (b-a)f(x_0) - \tfrac{f(b)-f(x_0)}{2}(b-x_0) - \tfrac{f(x_0)-f(a)}{2}(a-x_0)\Big| \le$$
$$\le \tfrac{(x_0-a)^2}{4}(M_1 - m_1) + \tfrac{(b-x_0)^2}{4}(M_2 - m_2).$$

In particular, if $x_0 = \frac{a+b}{2}$, then Corollary 2.5 becomes

Corollary 2.6. *Let $f(x) \in C^1[a,b]$ and $m_1 \le f'(x) \le M_1$ for any $x \in [a, \frac{a+b}{2}]$, $m_2 \le f'(x) \le M_2$ for any $x \in [\frac{a+b}{2}, b]$, where m_1, m_2, M_1 and M_2 are constants. Then*

$$\Big|\tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \tfrac{1}{2}f(\tfrac{a+b}{2}) - \tfrac{f(a)+f(b)}{4}\Big| \le$$
$$\le \tfrac{1}{16}(b-a)(M_1 + M_2 - m_1 - m_2).$$

Theorem 2.7. *Let $f(x) \in C^{n+1}[a,b]$ and $x_0 \in [a,b]$, then the following statements are true:*

(1) If n is an odd number and $f^{(n+1)}(x)$ is increasing in $[a,b]$, then

$$\begin{aligned}
&\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0) - f^{(n+1)}(a)) \ge \\
&\ge \int_a^b R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \\
&\quad -\tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \\
&\ge -\tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(b) - f^{(n+1)}(x_0));
\end{aligned} \tag{2.8}$$

(2) If n is an odd number and $f^{(n+1)}(x)$ is decreasing in $[a,b]$, then

$$\begin{aligned}
&-\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(a) - f^{(n+1)}(x_0)) \le \\
&\le \int_a^b R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \\
&\quad -\tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \\
&\le \tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0) - f^{(n+1)}(b));
\end{aligned} \tag{2.9}$$

(3) If n is an even number and $f^{(n+1)}(x)$ is increasing in $[a,b]$, then

$$\begin{aligned}
&-\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0)-f^{(n+1)}(a)) \\
&-\tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(b)-f^{(n+1)}(x_0)) \leq \\
&\leq \int_a^b R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \\
&\quad -\tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \\
&\leq 0;
\end{aligned} \tag{2.10}$$

(4) If n is an even number and $f^{(n+1)}(x)$ is decreasing in $[a,b]$, then

$$\begin{aligned}
&\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(a)-f^{(n+1)}(x_0)) \\
&+\tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0)-f^{(n+1)}(b)) \\
&\geq \int_a^b R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \\
&\quad -\tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \\
&\geq 0.
\end{aligned} \tag{2.11}$$

Proof. We divide the proof into two cases.

Case 1. $x \in [x_0,b]$. Let $F(x) = f^{(n+1)}(x)$ and $G(x) = \frac{(b-x)^{n+1}}{(n+1)!}$, then we clearly see that $G(x)$ is decreasing in $[x_0,b]$. We divide this case into two subcases.

Subcase 1.1. $F(x) = f^{(n+1)}(x)$ is increasing in $[x_0,b]$. It follows from the Chebyshev inequality that

$$\int_{x_0}^b \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x - \tfrac{1}{b-x_0}\int_{x_0}^b \tfrac{(b-x)^{n+1}}{(n+1)!}\mathrm{d}x \int_{x_0}^b f^{(n+1)}(x)\mathrm{d}x \leq 0.$$

Making use of equation (2.1) we get

$$\int_{x_0}^b R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b-x_0)^{n+1} \leq 0. \tag{2.12}$$

From the monotonicity of F and G we have

$$f^{(n+1)}(x_0) \leq F(x) \leq f^{(n+1)}(b)$$

and

$$0 \leq G(x) \leq \tfrac{(b-x_0)^{n+1}}{(n+1)!}$$

for $x \in [x_0, b]$. Therefore, inequalities (2.5) and (2.12) lead to the conclusion that

$$(2.13)\qquad -\tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(b) - f^{(n+1)}(x_0)) \le \le \int_{x_0}^{b} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b - x_0)^{n+1} \le 0.$$

Subcase 1.2. $F(x) = f^{(n+1)}(x)$ is decreasing in $[x_0, b]$. The Chebyshev inequality implies that

$$\int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x - \tfrac{1}{b-x_0}\int_{x_0}^{b} \tfrac{(b-x)^{n+1}}{(n+1)!}\mathrm{d}x \int_{x_0}^{b} f^{(n+1)}(x)\mathrm{d}x \ge 0.$$

Then equation (2.1) and inequality (2.5) lead to the conclusion that

$$(2.14)\qquad \tfrac{(b-x_0)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0) - f^{(n+1)}(b)) \ge \ge \int_{x_0}^{b} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(b)-f^{(n)}(x_0)}{(n+2)!}(b - x_0)^{n+1} \ge 0.$$

Case 2. $x \in [a, x_0]$. Let $F(x) = f^{(n+1)}(x)$ and $H(x) = \frac{(a-x)^{n+1}}{(n+1)!}$. We divide the discussion into four subcases.

Subcase 2.1. n is an odd number and $F(x) = f^{(n+1)}(x)$ is increasing in $[a, x_0]$. Then $H(x) = \frac{(a-x)^{n+1}}{(n+1)!}$ is increasing in $[a, x_0]$ and

$$f^{(n+1)}(a) \le F(x) \le f^{(n+1)}(x_0)$$

for all $x \in [a, x_0]$.

Making use of the Chebyshev inequality we get

$$\int_{a}^{x_0} \tfrac{(a-x)^{n+1}}{(n+1)!} f^{(n+1)}(x)\mathrm{d}x - \tfrac{1}{x_0-a}\int_{a}^{x_0} \tfrac{(a-x)^{n+1}}{(n+1)!}\mathrm{d}x \int_{a}^{x_0} f^{(n+1)}(x)\mathrm{d}x \ge 0.$$

Then equation (2.2) and inequality (2.7) imply that

$$(2.15)\qquad \tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0) - f^{(n+1)}(a)) \ge \ge \int_{a}^{x_0} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a - x_0)^{n+1} \ge 0.$$

Subcase 2.2. n is an odd number and $F(x) = f^{(n+1)}(x)$ is decreasing in $[a, x_0]$. Then $H(x)$ is increasing in $[a, x_0]$. It follows from equation (2.2) and inequality (2.7) together with the Chebyshev inequality that

$$(2.16)\qquad -\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(a) - f^{(n+1)}(x_0)) \le \le \int_{a}^{x_0} R_{n,f}(x_0, x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a - x_0)^{n+1} \le 0.$$

Subcase 2.3. n is an even number and $F(x) = f^{(n+1)}(x)$ is increasing in $[a, x_0]$. Then $H(x) = \frac{(a-x)^{n+1}}{(n+1)!}$ is decreasing in $[a, x_0]$. Therefore, equation

(2.2) and inequality (2.7) together with the Chebyshev inequality lead to the conclusion that

$$
\begin{aligned}
(2.17)\qquad & -\tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(x_0)-f^{(n+1)}(a)) \le \\
& \le \int_a^{x_0} R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \le 0.
\end{aligned}
$$

Subcase 2.4. n is an even number and $F(x) = f^{(n+1)}(x)$ is decreasing in $[a, x_0]$. Then $H(x)$ and $F(x)$ have the same monotonicity in $[a, x_0]$. It follows from equation (2.2) and inequality (2.7) together with the Chebyshev inequality that

$$
\begin{aligned}
(2.18)\qquad & \tfrac{(x_0-a)^{n+2}}{4(n+1)!}(f^{(n+1)}(a)-f^{(n+1)}(x_0)) \ge \\
& \ge \int_a^{x_0} R_{n,f}(x_0,x)\mathrm{d}x - \tfrac{f^{(n)}(x_0)-f^{(n)}(a)}{(n+2)!}(a-x_0)^{n+1} \ge 0.
\end{aligned}
$$

Therefore, inequality (2.8) follows from inequalities (2.13) and (2.15), inequality (2.9) follows from inequalities (2.14) and (2.16), inequality (2.10) follows from inequalities (2.13) and (2.17), and inequality (2.11) follows from inequalities (2.14) and (2.18). □

If take $n = 0$ and $x_0 = \frac{a+b}{2}$ in Theorem 2.7, then we have

Corollary 2.8. *Let $f(x) \in C^1[a, b]$, then the following statements are true:*
(1) If $f'(x)$ is increasing in $[a, b]$, then

$$
\begin{aligned}
(2.19)\qquad & -\tfrac{1}{16}(b-a)(f'(b)-f'(a)) \le \\
& \le \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \tfrac{1}{2}f(\tfrac{a+b}{2}) - \tfrac{f(a)+f(b)}{4} \le 0.
\end{aligned}
$$

(2) If $f'(x)$ is decreasing in $[a, b]$, then

$$
\begin{aligned}
(2.20)\qquad & \tfrac{1}{16}(b-a)(f'(a)-f'(b)) \ge \\
& \ge \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \tfrac{1}{2}f(\tfrac{a+b}{2}) - \tfrac{f(a)+f(b)}{4} \ge 0.
\end{aligned}
$$

Let us recall the well known Hermite-Hadamard inequality:

$$
(2.21)\qquad f(\tfrac{a+b}{2}) \le (\ge)\tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le (\ge)\tfrac{f(a)+f(b)}{2}
$$

if $f(x)$ is convex (concave) in $[a, b]$.
Inequality (2.19) can be rewritten as

$$
\begin{aligned}
(2.22)\qquad & \tfrac{1}{2}f(\tfrac{a+b}{2}) + \tfrac{f(a)+f(b)}{4} - \tfrac{1}{16}(b-a)(f'(b)-f'(a)) \le \\
& \le \tfrac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \tfrac{1}{2}f(\tfrac{a+b}{2}) + \tfrac{f(a)+f(b)}{4}.
\end{aligned}
$$

We clearly see that $\frac{f(a)+f(b)}{2} \geq \frac{1}{2}f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{4}$ if $f(x)$ is convex in $[a, b]$. Therefore, inequality (2.22) is an improvement of inequality (2.21) if $f'(x)$ is decreasing in $[a, b]$.

Acknowledgement. The authors wish to thank the anonymous referees for their very careful reading of the manuscript and fruitful comments and suggestions.

REFERENCES

[1] G. Grüss, *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\mathrm{d}x \int_a^b g(x)\mathrm{d}x$, Math. Z., **39**(1) (1935), pp. 215-226.

[2] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] S. S. Dragomir, *New estimation of the remainder in Taylor's formula using Grüss' type inequalities and applications*, Math. Inequal. Appl., **2**(2)(1999), pp. 183–193.

[4] G. A. Anastassiou and S. S. Dragomir, *On some estimates of the remainder in Taylor's Formula*, J. Math. Anal. Appl., **263**(1)(2001), pp. 246–263.

[5] H. Gauchman, *Some integral inequalities involving Taylor's remainder I*, JIPAM. J. Inequal. Pure Appl. Math., **3**(2), Article 26, 9 pages, 2002.

[6] H. Gauchman, *Some integral inequalities involving Taylor's remainder II*, JIPAM. J. Inequal. Pure Appl. Math., **4**(1), Article 1, 5 pages, 2003.

[7] L. Bougoffa, *Some estimations for the integral Taylor's remainder*, JIPAM. J. Inequal. Pure Appl. Math., **4**(5), Article 84, 4 pages, 2003.

[8] D.-Y. Hwang, *Improvements of some integral inequalities involving Taylor's remainder*, J. Appl. Math. Comput., **16**(1-2)(2004), pp. 151–163.

[9] E. Talvila, *Estimates of the remainder in Taylor's theorem using the Henstock-Kurzweil integral*, Czechoslovak Math. J., **55**(4) (2005), pp. 933–940.

Received by the editors: April 30, 2013.