

MULTIVARIATE ERROR FUNCTION BASED  
NEURAL NETWORK APPROXIMATIONS

GEORGE A. ANASTASSIOU\*

**Abstract.** Here we present multivariate quantitative approximations of real and complex valued continuous multivariate functions on a box or  $\mathbb{R}^N$ ,  $N \in \mathbb{N}$ , by the multivariate quasi-interpolation, Baskakov type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last three types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the Gaussian error special function. The approximations are point-wise and uniform. The related feed-forward neural network is with one hidden layer.

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## 1. INTRODUCTION

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the  $N$ th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is motivated by the article [12] of Z. Chen and F. Cao, also by [4], [5], [6], [7], [8], [9], [10], [13], [14].

The author here performs multivariate error function based neural network approximations to continuous functions over boxes or over the whole  $\mathbb{R}^N$ ,  $N \in \mathbb{N}$ , then he extends his results to complex valued multivariate functions. Also he does iterated approximation. All convergences here are with rates

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\*Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A., e-mail: [ganastss@memphis.edu](mailto:ganastss@memphis.edu).

expressed via the multivariate modulus of continuity of the involved function or its high order partial derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the "right" precisely defined multivariate quasi-interpolation neural network operators related to boxes or  $\mathbb{R}^N$ , as well as Baskakov type and quadrature type related operators on  $\mathbb{R}^N$ . Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by error function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for  $0 \leq j \leq n$ ,  $b_j \in \mathbb{R}$  are the thresholds,  $a_j \in \mathbb{R}^s$  are the connection weights,  $c_j \in \mathbb{R}$  are the coefficients,  $\langle a_j \cdot x \rangle$  is the inner product of  $a_j$  and  $x$ , and  $\sigma$  is the activation function of the network. In many fundamental network models, the activation function is the error function. About neural networks read [15], [16], [17].

## 2. BASICS

We consider here the (Gauss) error special function ([1], [11])

$$(1) \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R},$$

which is a sigmoidal type function and is a strictly increasing function.

It has the basic properties

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(+\infty) = 1, \quad \operatorname{erf}(-\infty) = -1.$$

We consider the activation function ([10])

$$(2) \quad \chi(x) = \frac{1}{4} (\operatorname{erf}(x+1) - \operatorname{erf}(x-1)) > 0, \quad \forall x \in \mathbb{R},$$

which is an even function.

Next we follow [10] on  $\chi$ . We got there  $\chi(0) \simeq 0.4215$ , and that  $\chi$  is strictly decreasing on  $[0, \infty)$  and strictly increasing on  $(-\infty, 0]$ , and the  $x$ -axis is the horizontal asymptote on  $\chi$ , i.e.  $\chi$  is a bell symmetric function.

**THEOREM 1.** [10] *We have that*

$$(3) \quad \sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \quad \forall x \in \mathbb{R},$$

$$(4) \quad \sum_{i=-\infty}^{\infty} \chi(nx-i) = 1, \quad \forall x \in \mathbb{R}, \quad n \in \mathbb{N},$$

and

$$(5) \quad \int_{-\infty}^{\infty} \chi(x) dx = 1,$$

that is  $\chi(x)$  is a density function on  $\mathbb{R}$ .

We need the important

**THEOREM 2.** [10] *Let  $0 < \alpha < 1$ , and  $n \in \mathbb{N}$  with  $n^{1-\alpha} \geq 3$ . It holds*

$$(6) \quad \sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \chi(nx-k) < \frac{1}{2\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}.$$

Denote by  $\lfloor \cdot \rfloor$  the integral part of the number and by  $\lceil \cdot \rceil$  the ceiling of the number.

**THEOREM 3.** [10] *Let  $x \in [a, b] \subset \mathbb{R}$  and  $n \in \mathbb{N}$  so that  $\lceil na \rceil \leq \lfloor nb \rfloor$ . It holds*

$$(7) \quad \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} < \frac{1}{\chi(1)} \simeq 4.019, \quad \forall x \in [a, b].$$

Also from [10] we get

$$(8) \quad \lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \neq 1,$$

at least for some  $x \in [a, b]$ .

For large enough  $n$  we always obtain  $\lceil na \rceil \leq \lfloor nb \rfloor$ . Also  $a \leq \frac{k}{n} \leq b$ , iff  $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$ . In general it holds by (4) that

$$(9) \quad \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \leq 1.$$

We introduce

$$(10) \quad Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^N \chi(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}.$$

It has the properties:

- (i)  $Z(x) > 0, \quad \forall x \in \mathbb{R}^N,$
- (ii)

$$(11) \quad \sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, \dots, x_N-k_N) = 1,$$

where  $k := (k_1, \dots, k_n) \in \mathbb{Z}^N, \forall x \in \mathbb{R}^N,$   
hence

(iii)

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} Z(nx - k) = \\
(12) \quad & = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(nx_1 - k_1, \dots, nx_N - k_N) = 1, \quad \forall x \in \mathbb{R}^N; n \in \mathbb{N},
\end{aligned}$$

and

(iv)

$$(13) \quad \int_{\mathbb{R}^N} Z(x) dx = 1,$$

that is  $Z$  is a multivariate density function.

Here  $\|x\|_{\infty} := \max\{|x_1|, \dots, |x_N|\}$ ,  $x \in \mathbb{R}^N$ , also set  $\infty := (\infty, \dots, \infty)$ ,  $-\infty := (-\infty, \dots, -\infty)$  upon the multivariate context, and

$$(14) \quad \begin{aligned} \lceil na \rceil & : = (\lceil na_1 \rceil, \dots, \lceil na_N \rceil), \\ \lfloor nb \rfloor & : = (\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor), \end{aligned}$$

where  $a := (a_1, \dots, a_N)$ ,  $b := (b_1, \dots, b_N)$ .

We obviously see that

$$\begin{aligned}
& \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \chi(nx_i - k_i) \right) \\
(15) \quad & = \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left( \prod_{i=1}^N \chi(nx_i - k_i) \right) = \prod_{i=1}^N \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right).
\end{aligned}$$

For  $0 < \beta < 1$  and  $n \in \mathbb{N}$ , a fixed  $x \in \mathbb{R}^N$ , we have that

$$\begin{aligned}
& \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) = \\
(16) \quad & = \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}}^{\lfloor nb \rfloor}} \chi(nx - k) + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}}^{\lfloor nb \rfloor}} \chi(nx - k).
\end{aligned}$$

In the last two sums the counting is over disjoint vector sets of  $k$ 's, because the condition  $\|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}$  implies that there exists at least one  $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^{\beta}}$ , where  $r \in \{1, \dots, N\}$ .

We treat

$$\begin{aligned}
& \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\lfloor nb \rfloor}} Z(nx-k) = \prod_{i=1}^N \left( \sum_{\substack{k_i=\lceil na_i \rceil \\ \|\frac{k_i}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\lfloor nb_i \rfloor}} \chi(nx_i - k_i) \right) \\
(17) \quad & \leq \left( \prod_{\substack{i=1 \\ i \neq r}}^N \left( \sum_{k_i=-\infty}^{\infty} \chi(nx_i - k_i) \right) \right) \cdot \left( \sum_{\substack{k_r=\lceil na_r \rceil \\ |\frac{k_r}{n}-x_r| > \frac{1}{n^\beta}}^{\lfloor nb_r \rfloor}} \chi(nx_r - k_r) \right) \\
& = \left( \sum_{\substack{k_r=\lceil na_r \rceil \\ |\frac{k_r}{n}-x_r| > \frac{1}{n^\beta}}^{\lfloor nb_r \rfloor}} \chi(nx_r - k_r) \right) \leq \sum_{\substack{k_r=-\infty \\ |\frac{k_r}{n}-x_r| > \frac{1}{n^\beta}}^{\infty}} \chi(nx_r - k_r) \\
(18) \quad & = \sum_{\substack{k_r=-\infty \\ |nx_r - k_r| > n^{1-\beta}}}^{\infty} \chi(nx_r - k_r) \stackrel{(6)}{\leq} \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}},
\end{aligned}$$

when  $n^{1-\beta} \geq 3$ .

We have proved that

(v)

$$(19) \quad \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\lfloor nb \rfloor}} Z(nx-k) \leq \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}},$$

$$0 < \beta < 1, n \in \mathbb{N}; n^{1-\beta} \geq 3, x \in \left( \prod_{i=1}^N [a_i, b_i] \right).$$

By Theorem 3 clearly we obtain

$$\begin{aligned}
(20) \quad 0 & < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} = \frac{1}{\prod_{i=1}^N \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right)} \\
& < \frac{1}{(\chi(1))^N} \simeq (4.019)^N.
\end{aligned}$$

That is,

(vi) it holds

$$(21) \quad 0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{(\chi(1))^N} \simeq (4.019)^N, \quad \forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right), n \in \mathbb{N}.$$

It is also clear that

(vii)

$$(22) \quad \sum_{k=-\infty}^{\infty} Z(nx - k) \leq \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}},$$

$$\left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}$$

$$0 < \beta < 1, n \in \mathbb{N} : n^{1-\beta} \geq 3, x \in \left( \prod_{i=1}^N [a_i, b_i] \right).$$

Also we get that

$$(23) \quad \lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \neq 1,$$

for at least some  $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ .Let  $f \in C \left( \prod_{i=1}^N [a_i, b_i] \right)$  and  $n \in \mathbb{N}$  such that  $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$ ,  $i = 1, \dots, N$ .We introduce and define the multivariate positive linear neural network operator ( $x := (x_1, \dots, x_N) \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ )

$$(24) \quad A_n(f, x_1, \dots, x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left( \prod_{i=1}^N \chi(nx_i - k_i) \right)}{\prod_{i=1}^N \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right)}.$$

For large enough  $n$  we always obtain  $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$ ,  $i = 1, \dots, N$ . Also  $a_i \leq \frac{k_i}{n} \leq b_i$ , iff  $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$ ,  $i = 1, \dots, N$ .

For convenience we call

$$(25) \quad A_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)$$

$$:= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left( \prod_{i=1}^N \chi(nx_i - k_i) \right),$$

$$\forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right).$$

That is

$$(26) \quad A_n(f, x) := \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)},$$

$$\forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right), n \in \mathbb{N}.$$

Hence

$$(27) \quad A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x) \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}.$$

Consequently we derive

$$(28) \quad |A_n(f, x) - f(x)| \leq (4.019)^N \left| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right|,$$

$$\forall x \in \left( \prod_{i=1}^N [a_i, b_i] \right).$$

We will estimate the right hand side of (28).

For the last we need, for  $f \in C \left( \prod_{i=1}^N [a_i, b_i] \right)$  the first multivariate modulus of continuity

$$(29) \quad \omega_1(f, h) := \sup_{\substack{x, y \in \prod_{i=1}^N [a_i, b_i] \\ \|x-y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0.$$

It holds that

$$(30) \quad \lim_{h \rightarrow 0} \omega_1(f, h) = 0.$$

Similarly it is defined for  $f \in C_B(\mathbb{R}^N)$  (continuous and bounded functions on  $\mathbb{R}^N$ ) the  $\omega_1(f, h)$ , and it has the property (30), given that  $f \in C_U(\mathbb{R}^N)$  (uniformly continuous functions on  $\mathbb{R}^N$ ).

When  $f \in C_B(\mathbb{R}^N)$  we define,

$$(31) \quad \begin{aligned} B_n(f, x) &:= B_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) := \\ &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \dots, \frac{k_N}{n}\right) \left( \prod_{i=1}^N \chi(nx_i - k_i) \right), \end{aligned}$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$ , the multivariate quasi-interpolation neural network operator.

Also for  $f \in C_B(\mathbb{R}^N)$  we define the multivariate Kantorovich type neural network operator

(32)

$$\begin{aligned} C_n(f, x) &:= C_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \left( n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) Z(nx - k) := \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left( n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \cdot \\ &\quad \cdot \left( \prod_{i=1}^N \chi(nx_i - k_i) \right), \end{aligned}$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N$ .

Again for  $f \in C_B(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$ , we define the multivariate neural network operator of quadrature type  $D_n(f, x)$ ,  $n \in \mathbb{N}$ , as follows. Let  $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{N}^N$ ,  $r = (r_1, \dots, r_N) \in \mathbb{Z}_+^N$ ,  $w_r = w_{r_1, r_2, \dots, r_N} \geq 0$ , such that  $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} = 1$ ;  $k \in \mathbb{Z}^N$  and

$$\begin{aligned} \delta_{nk}(f) &:= \delta_{n, k_1, k_2, \dots, k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) := \\ (33) \quad &:= \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \end{aligned}$$

where  $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right)$ .

We put

$$\begin{aligned} (34) \quad D_n(f, x) &:= D_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) := \\ &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, \dots, k_N}(f) \left( \prod_{i=1}^N \chi(nx_i - k_i) \right), \end{aligned}$$

$\forall x \in \mathbb{R}^N$ .

Let fixed  $j \in \mathbb{N}$ ,  $0 < \beta < 1$ , and  $A, B > 0$ . For large enough  $n \in \mathbb{N} : n^{1-\beta} \geq 3$ , in the linear combination  $\left(\frac{A}{n^{\beta j}} + \frac{B}{(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}\right)$ , the dominant rate of convergence, as  $n \rightarrow \infty$ , is  $n^{-\beta j}$ . The closer  $\beta$  is to 1 we get faster and better rate of convergence to zero.



Let  $f \in C^m \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $m, N \in \mathbb{N}$ . Here  $f_\alpha$  denotes a partial derivative of  $f$ ,  $\alpha := (\alpha_1, \dots, \alpha_N)$ ,  $\alpha_i \in \mathbb{Z}_+$ ,  $i = 1, \dots, N$ , and  $|\alpha| := \sum_{i=1}^N \alpha_i = l$ , where  $l = 0, 1, \dots, m$ . We write also  $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$  and we say it is of order  $l$ .

We denote

$$(35) \quad \omega_{1,m}^{\max}(f_\alpha, h) := \max_{|\alpha|=m} \omega_1(f_\alpha, h).$$

Call also

$$(36) \quad \|f_\alpha\|_{\infty, m}^{\max} := \max_{|\alpha|=m} \{\|f_\alpha\|_\infty\},$$

$\|\cdot\|_\infty$  is the supremum norm.

In this article we study the basic approximation properties of  $A_n, B_n, C_n, D_n$  neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator  $I$ . We study also the complex functions related approximation.

### 3. MULTIDIMENSIONAL REAL NEURAL NETWORK APPROXIMATIONS

Here we present a series of neural network approximations to a function given with rates.

We give

**THEOREM 4.** *Let  $f \in C \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $0 < \beta < 1$ ,  $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} \geq 3$ . Then*

$$(37) \quad 1) \quad |A_n(f, x) - f(x)| \leq (4.019)^N \left[ \omega_1 \left( f, \frac{1}{n^\beta} \right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] =: \lambda_1,$$

and

$$(38) \quad 2) \quad \|A_n(f) - f\|_\infty \leq \lambda_1.$$

We notice that  $\lim_{n \rightarrow \infty} A_n(f) = f$ , pointwise and uniformly.

*Proof.* We observe that

$$\Delta(x) := A_n^*(f, x) - f(x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) =$$

$$\begin{aligned}
&= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x) Z(nx - k) \\
(39) \quad &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k).
\end{aligned}$$

Thus

$$\begin{aligned}
|\Delta(x)| &\leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| Z(nx - k) = \\
&= \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| Z(nx - k) \\
&\quad + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left| f\left(\frac{k}{n}\right) - f(x) \right| Z(nx - k) \\
&\stackrel{\text{(by (12))}}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + 2\|f\|_\infty \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) \\
(40) \quad &\stackrel{\text{(by (19))}}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}.
\end{aligned}$$

So that

$$|\Delta| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}.$$

Now using (28) we finish proof.  $\square$

We continue with

**THEOREM 5.** *Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} \geq 3$ . Then*

1)

$$(41) \quad |B_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} =: \lambda_2,$$

2)

$$(42) \quad \|B_n(f) - f\|_\infty \leq \lambda_2.$$

Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} B_n(f) = f$ , uniformly.

*Proof.* We have that

$$(43) \quad \begin{aligned} B_n(f, x) - f(x) &\stackrel{(12)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) \\ &= \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx - k). \end{aligned}$$

Hence

$$(44) \quad \begin{aligned} |B_n(f, x) - f(x)| &\leq \sum_{k=-\infty}^{\infty} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx - k) \\ &= \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}} }^{\infty} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx - k) \\ &\quad + \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}} }^{\infty} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx - k) \\ &\stackrel{(12)}{\leq} \omega_1\left(f, \frac{1}{n^{\beta}}\right) + 2\|f\|_{\infty} \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}} }^{\infty} Z(nx - k) \\ &\stackrel{(19)}{\leq} \omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}, \end{aligned}$$

proving the claim.  $\square$

We give

**THEOREM 6.** *Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} \geq 3$ . Then*

1)

$$(45) \quad |C_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} =: \lambda_3,$$

2)

$$(46) \quad \|C_n(f) - f\|_{\infty} \leq \lambda_3.$$

*Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} C_n(f) = f$ , uniformly.*

*Proof.* We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N) dt_1 dt_2 \dots dt_N =$$

$$(47) \quad = \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt.$$

Thus it holds

$$(48) \quad C_n(f, x) = \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k).$$

We observe that

$$(49) \quad \begin{aligned} & |C_n(f, x) - f(x)| = \\ & = \left| \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right| \\ & = \left| \sum_{k=-\infty}^{\infty} \left( \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z(nx - k) \right| \\ & = \left| \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left( f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z(nx - k) \right| \\ & \leq \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} |f\left(t + \frac{k}{n}\right) - f(x)| dt \right) Z(nx - k) \\ & = \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}} }^{\infty} \left( n^N \int_0^{\frac{1}{n}} |f\left(t + \frac{k}{n}\right) - f(x)| dt \right) Z(nx - k) \\ & \quad + \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}} }^{\infty} \left( n^N \int_0^{\frac{1}{n}} |f\left(t + \frac{k}{n}\right) - f(x)| dt \right) Z(nx - k) \\ & \leq \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}} }^{\infty} \left( n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_{\infty} + \left\| \frac{k}{n} - x \right\|_{\infty}\right) dt \right) Z(nx - k) \\ & \quad + 2 \|f\|_{\infty} \left( \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}} }^{\infty} Z(|nx - k|) \right) \\ (50) \quad & \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}, \end{aligned}$$

proving the claim.  $\square$

We also present

THEOREM 7. Let  $f \in C_B(\mathbb{R}^N)$ ,  $0 < \beta < 1$ ,  $x \in \mathbb{R}^N$ ,  $N, n \in \mathbb{N}$  with  $n^{1-\beta} \geq 3$ . Then

1)

$$(51) \quad |D_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} = \lambda_3,$$

2)

$$(52) \quad \|D_n(f) - f\|_\infty \leq \lambda_3.$$

Given that  $f \in (C_U(\mathbb{R}^N) \cap C_B(\mathbb{R}^N))$ , we obtain  $\lim_{n \rightarrow \infty} D_n(f) = f$ , uniformly.

*Proof.* We have that

$$(53) \quad \begin{aligned} & |D_n(f, x) - f(x)| \\ &= \left| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right| \\ &= \left| \sum_{k=-\infty}^{\infty} (\delta_{nk}(f) - f(x)) Z(nx - k) \right| \\ &= \left| \sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left( f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right) \right) Z(nx - k) \right| \\ &\leq \sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - k) \\ &= \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - k) \\ &\quad + \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) Z(nx - k) \\ &\leq \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \omega_1\left(f, \left\| \frac{k}{n} - x \right\|_\infty + \left\| \frac{r}{n\theta} \right\|_\infty\right) \right) Z(nx - k) + \end{aligned}$$

$$\begin{aligned}
& + 2 \|f\|_\infty \left( \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\infty}} Z(nx-k) \right) \\
(54) \quad & \leq \omega_1 \left( f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}},
\end{aligned}$$

proving the claim.  $\square$

In the next we discuss high order of approximation by using the smoothness of  $f$ .

We give

**THEOREM 8.** *Let  $f \in C^m \left( \prod_{i=1}^N [a_i, b_i] \right)$ ,  $0 < \beta < 1$ ,  $n, m, N \in \mathbb{N}$ ,  $n^{1-\beta} \geq 3$ ,  $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ . Then*

$$\begin{aligned}
(55) \quad & i) \\
& \left| A_n(f, x) - f(x) - \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \\
& \leq (4.019)^N \cdot \left\{ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\},
\end{aligned}$$

ii)  
(56)

$$\begin{aligned}
& |A_n(f, x) - f(x)| \leq (4.019)^N \cdot \\
& \cdot \left\{ \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta_j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \cdot \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] \right) \right. \\
& \left. + \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\},
\end{aligned}$$

iii)  
(57)

$$\begin{aligned}
& \|A_n(f) - f\|_\infty (4.019)^N \cdot \\
& \left\{ \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta_j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \cdot \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] \right) \right\} +
\end{aligned}$$

$$+ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \Big\} =: K_n,$$

iv) Assume  $f_\alpha(x_0) = 0$ , for all  $\alpha : |\alpha| = 1, \dots, m; x_0 \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ . Then

(58)

$$|A_n(f, x_0) - f(x_0)| \leq (4.019)^N \left\{ \frac{N^m}{m!n^{m\beta}} \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\},$$

notice in the last the extremely high rate of convergence at  $n^{-\beta(m+1)}$ .

*Proof.* Consider  $g_z(t) := f(x_0 + t(z - x_0))$ ,  $t \geq 0$ ;  $x_0, z \in \prod_{i=1}^N [a_i, b_i]$ .

Then

(59)

$$g_z^{(j)}(t) = \left[ \left( \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \dots, x_{0N} + t(z_N - x_{0N})),$$

for all  $j = 0, 1, \dots, m$ .

We have the multivariate Taylor's formula

$$(60) \quad \begin{aligned} f(z_1, \dots, z_N) &= g_z(1) = \\ &= \sum_{j=0}^m \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} (g_z^{(m)}(\theta) - g_z^{(m)}(0)) d\theta. \end{aligned}$$

Notice  $g_z(0) = f(x_0)$ . Also for  $j = 0, 1, \dots, m$ , we have

$$(61) \quad g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}^+, \\ i=1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j}} \left( \frac{j!}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0).$$

Furthermore

(62)

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \alpha_i \in \mathbb{Z}^+, \\ i=1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = m}} \left( \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)),$$

$0 \leq \theta \leq 1$ .

So we treat  $f \in C^m \left( \prod_{i=1}^N [a_i, b_i] \right)$ .

Thus, we have for  $\frac{k}{n}, x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$  that

$$(63) \quad \begin{aligned} & f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) - f(x) = \\ & = \sum_{j=1}^m \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \\ \alpha_i \in \mathbb{Z}^+, \\ i=1, \dots, N, |\alpha| := \sum_{i=1}^N \alpha_i = j}} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N \left( \frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha(x) + R, \end{aligned}$$

where

$$(64) \quad \begin{aligned} R := m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha := (\alpha_1, \dots, \alpha_N), \\ \alpha_i \in \mathbb{Z}^+, i=1, \dots, N, \\ |\alpha| := \sum_{i=1}^N \alpha_i = m}} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N \left( \frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \\ \cdot \left[ f_\alpha \left( x + \theta \left( \frac{k}{n} - x \right) \right) - f_\alpha(x) \right] d\theta. \end{aligned}$$

We see that

$$(65) \quad \begin{aligned} |R| &\leq m \int_0^1 (1-\theta)^{m-1} \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \\ &\cdot \left| f_\alpha \left( x + \theta \left( \frac{k}{n} - x \right) \right) - f_\alpha(x) \right| d\theta \leq m \int_0^1 (1-\theta)^{m-1} \\ &\cdot \left( \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \omega_1 \left( f_\alpha, \theta \left\| \frac{k}{n} - x \right\|_\infty \right) \right) d\theta \leq (*). \end{aligned}$$

Notice here that

$$(66) \quad \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \Leftrightarrow \left| \frac{k_i}{n} - x_i \right| \leq \frac{1}{n^\beta}, \quad i = 1, \dots, N.$$

We further see that

$$(67) \quad \begin{aligned} (*) &\leq m \cdot \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) \int_0^1 (1-\theta)^{m-1} \left( \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N \left( \frac{1}{n^\beta} \right)^{\alpha_i} \right) \right) d\theta \\ &= \left( \frac{\omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right)}{(m!) n^{m\beta}} \right) \left( \sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \left( \frac{\omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right)}{(m!) n^{m\beta}} \right) N^m. \end{aligned}$$

Conclusion: When  $\left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$ , we proved that

$$(68) \quad |R| \leq \left( \frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right).$$



In general we notice that

$$\begin{aligned}
|R| &\leq m \int_0^1 (1-\theta)^{m-1} \left( \sum_{|\alpha|=m} \left( \frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) 2 \|f_\alpha\|_\infty \right) d\theta \\
&= 2 \sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^N \alpha_i!} \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \|f_\alpha\|_\infty \\
(69) \quad &\leq \left( \frac{2 \|b-a\|_\infty^m \|f_\alpha\|_{\infty, m}^{\max}}{m!} \right) \left( \sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) = \frac{2 \|b-a\|_\infty^m \|f_\alpha\|_{\infty, m}^{\max} N^m}{m!}.
\end{aligned}$$

We proved in general that

$$(70) \quad |R| \leq \frac{2 \|b-a\|_\infty^m \|f_\alpha\|_{\infty, m}^{\max} N^m}{m!} := \rho.$$

Next we see that

$$\begin{aligned}
U_n &:= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) R \\
&= \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}}}^{\lfloor nb \rfloor} Z(nx - k) R + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) R.
\end{aligned}$$

Consequently

$$\begin{aligned}
|U_n| &\leq \left( \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) \right) \frac{N^m}{m! n^{m\beta}} \omega_{1, m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) \\
&\quad + \rho \frac{1}{2\sqrt{\pi} (n^{1-\beta} - 2) e^{(n^{1-\beta} - 2)^2}} \\
(71) \quad &\leq \frac{N^m}{m! n^{m\beta}} \omega_{1, m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \rho \frac{1}{2\sqrt{\pi} (n^{1-\beta} - 2) e^{(n^{1-\beta} - 2)^2}}.
\end{aligned}$$

We have established that

$$(72) \quad |U_n| \leq \frac{N^m}{m! n^{m\beta}} \omega_{1, m}^{\max} \left( f_\alpha, \frac{1}{n^\beta} \right) + \left( \frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty, m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi} (n^{1-\beta} - 2) e^{(n^{1-\beta} - 2)^2}}.$$

We observe that

$$\begin{aligned}
& \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) = \\
& = \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \left( \prod_{i=1}^N \left( \frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \right) \right) \\
(73) \quad & + \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) R.
\end{aligned}$$

The last says

$$\begin{aligned}
& A_n^*(f, x) - f(x) \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \right) - \\
(74) \quad & - \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n^* \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) = U_n.
\end{aligned}$$

Clearly  $A_n^*$  is a positive linear operator.

Thus (here  $\alpha_i \in \mathbb{Z}^+ : |\alpha| = \sum_{i=1}^N \alpha_i = j$ )

$$\begin{aligned}
& \left| A_n^* \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq A_n^* \left( \prod_{i=1}^N |\cdot - x_i|^{\alpha_i}, x \right) \\
& = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx - k) \\
& = \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx - k) \\
& \quad + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx - k) \\
& \leq \frac{1}{n^{\beta j}} + \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \left( \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) \right) \leq
\end{aligned}$$

$$(75) \quad \leq \frac{1}{n^{\beta_j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}.$$

So we have proved that

$$(76) \quad \left| A_n^* \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq \frac{1}{n^{\beta_j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}},$$

for all  $j = 1, \dots, m$ .

At last we observe

$$(77) \quad \left| A_n(f, x) - f(x) - \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq$$

$$\leq (4.019)^N \cdot \left| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) - \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n^* \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right|.$$

Putting all of the above together we prove theorem.  $\square$

We make

DEFINITION 9. Let  $f \in C_B(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$ . We define the general neural network operator

$$(78) \quad F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) =$$

$$= \begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases}$$

Clearly  $l_{nk}(f)$  is a positive linear functional such that  $|l_{nk}(f)| \leq \|f\|_\infty$ .

Hence  $F_n(f)$  is a positive linear operator with  $\|F_n(f)\|_\infty \leq \|f\|_\infty$ , a continuous bounded linear operator.

We need

THEOREM 10. Let  $f \in C_B(\mathbb{R}^N)$ ,  $N \geq 1$ . Then  $F_n(f) \in C_B(\mathbb{R}^N)$ .

*Proof.* Clearly  $F_n(f)$  is a bounded function.

Next we prove the continuity of  $F_n(f)$ . Notice for  $N = 1$ ,  $Z = \chi$  by (10).

We will use the Weierstrass  $M$  test: If a sequence of positive constants  $M_1, M_2, M_3, \dots$ , can be found such that in some interval

(a)  $|u_n(x)| \leq M_n$ ,  $n = 1, 2, 3, \dots$

(b)  $\sum M_n$  converges,

then  $\sum u_n(x)$  is uniformly and absolutely convergent in the interval.

Also we will use:

If  $\{u_n(x)\}$ ,  $n = 1, 2, 3, \dots$  are continuous in  $[a, b]$  and if  $\sum u_n(x)$  converges uniformly to the sum  $S(x)$  in  $[a, b]$ , then  $S(x)$  is continuous in  $[a, b]$ . I.e. a uniformly convergent series of continuous functions is a continuous function.

First we prove claim for  $N = 1$ .

We will prove that  $\sum_{k=-\infty}^{\infty} l_{nk}(f) \chi(nx - k)$  is continuous in  $x \in \mathbb{R}$ .

There always exists  $\lambda \in \mathbb{N}$  such that  $nx \in [-\lambda, \lambda]$ .

Since  $nx \leq \lambda$ , then  $-nx \geq -\lambda$  and  $k - nx \geq k - \lambda \geq 0$ , when  $k \geq \lambda$ .

Therefore

$$(79) \quad \sum_{k=\lambda}^{\infty} \chi(nx - k) = \sum_{k=\lambda}^{\infty} \chi(k - nx) \leq \sum_{k=\lambda}^{\infty} \chi(k - \lambda) = \sum_{k'=0}^{\infty} \chi(k') \leq 1.$$

So for  $k \geq \lambda$  we get

$$|l_{nk}(f)| \chi(nx - k) \leq \|f\|_{\infty} \chi(k - \lambda),$$

and

$$\|f\|_{\infty} \sum_{k=\lambda}^{\infty} \chi(k - \lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass  $M$  test we obtain that  $\sum_{k=\lambda}^{\infty} l_{nk}(f) \chi(nx - k)$  is uniformly and absolutely convergent on  $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$ .

Since  $l_{nk}(f) \chi(nx - k)$  is continuous in  $x$ , then  $\sum_{k=\lambda}^{\infty} l_{nk}(f) \chi(nx - k)$  is continuous on  $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$ .

Because  $nx \geq -\lambda$ , then  $-nx \leq \lambda$ , and  $k - nx \leq k + \lambda \leq 0$ , when  $k \leq -\lambda$ .

Therefore

$$\sum_{k=-\infty}^{-\lambda} \chi(nx - k) = \sum_{k=-\infty}^{-\lambda} \chi(k - nx) \leq \sum_{k=-\infty}^{-\lambda} \chi(k + \lambda) = \sum_{k'=-\infty}^0 \chi(k') \leq 1.$$

So for  $k \leq -\lambda$  we get

$$(80) \quad |l_{nk}(f)| \chi(nx - k) \leq \|f\|_{\infty} \chi(k + \lambda),$$

and

$$\|f\|_{\infty} \sum_{k=-\infty}^{-\lambda} \chi(k + \lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass  $M$  test we obtain that  $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \chi(nx - k)$  is uniformly and absolutely convergent on  $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$ .

Since  $l_{nk}(f)\chi(nx-k)$  is continuous in  $x$ , then  $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k)$  is continuous on  $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$ .

So we proved that  $\sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k)$  and  $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k)$  are continuous on  $\mathbb{R}$ . Since  $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\chi(nx-k)$  is a finite sum of continuous functions on  $\mathbb{R}$ , it is also a continuous function on  $\mathbb{R}$ .

Writing

$$(81) \quad \begin{aligned} \sum_{k=-\infty}^{\infty} l_{nk}(f)\chi(nx-k) &= \sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k) + \\ &+ \sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\chi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k) \end{aligned}$$

we have it as a continuous function on  $\mathbb{R}$ . Therefore  $F_n(f)$ , when  $N = 1$ , is a continuous function on  $\mathbb{R}$ .

When  $N = 2$  we have

$$\begin{aligned} F_n(f, x_1, x_2) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk}(f)\chi(nx_1-k_1)\chi(nx_2-k_2) = \\ &= \sum_{k_1=-\infty}^{\infty} \chi(nx_1-k_1) \left( \sum_{k_2=-\infty}^{\infty} l_{nk}(f)\chi(nx_2-k_2) \right) \end{aligned}$$

(there always exist  $\lambda_1, \lambda_2 \in \mathbb{N}$  such that  $nx_1 \in [-\lambda_1, \lambda_1]$  and  $nx_2 \in [-\lambda_2, \lambda_2]$ )

$$\begin{aligned} &= \sum_{k_1=-\infty}^{\infty} \chi(nx_1-k_1) \left[ \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\chi(nx_2-k_2) + \right. \\ &\quad \left. + \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\chi(nx_2-k_2) + \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\chi(nx_2-k_2) \right] \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\chi(nx_1-k_1)\chi(nx_2-k_2) + \\ &\quad + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\chi(nx_1-k_1)\chi(nx_2-k_2) + \\ &\quad + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\chi(nx_1-k_1)\chi(nx_2-k_2) =: (*). \end{aligned}$$

(For convenience call

$$F(k_1, k_2, x_1, x_2) := l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2).)$$

Thus

$$\begin{aligned}
(*) &= \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \\
&+ \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) + \\
&+ \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) + \\
(82) \quad &+ \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) + \\
&+ \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2).
\end{aligned}$$

Notice that the finite sum of continuous functions  $F(k_1, k_2, x_1, x_2)$ ,

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2)$$

is a continuous function.

The rest of the summands of  $F_n(f, x_1, x_2)$  are treated all the same way and similarly to the case of  $N = 1$ . The method is demonstrated as follows.

We will prove that

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2)$$

is continuous in  $(x_1, x_2) \in \mathbb{R}^2$ .

The continuous function

$$|l_{nk}(f)| \chi(nx_1 - k_1) \chi(nx_2 - k_2) \leq \|f\|_{\infty} \chi(k_1 - \lambda_1) \chi(k_2 + \lambda_2),$$

and

$$\begin{aligned} \|f\|_\infty & \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \chi(k_1 - \lambda_1) \chi(k_2 + \lambda_2) = \\ & = \|f\|_\infty \left( \sum_{k_1=\lambda_1}^{\infty} \chi(k_1 - \lambda_1) \right) \left( \sum_{k_2=-\infty}^{-\lambda_2} \chi(k_2 + \lambda_2) \right) \\ & \leq \|f\|_\infty \left( \sum_{k'_1=0}^{\infty} \chi(k'_1) \right) \left( \sum_{k'_2=-\infty}^0 \chi(k'_2) \right) \leq \|f\|_\infty. \end{aligned}$$

So by the Weierstrass  $M$  test we get that

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2)$$

is uniformly and absolutely convergent. Therefore it is continuous on  $\mathbb{R}^2$ .

Next we prove continuity on  $\mathbb{R}^2$  of

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2).$$

Notice here that

$$\begin{aligned} |l_{nk}(f)| \chi(nx_1 - k_1) \chi(nx_2 - k_2) & \leq \|f\|_\infty \chi(nx_1 - k_1) \chi(k_2 + \lambda_2) \\ & \leq \|f\|_\infty \chi(0) \chi(k_2 + \lambda_2) \\ & = 0.4215 \cdot \|f\|_\infty \chi(k_2 + \lambda_2), \end{aligned}$$

and

$$\begin{aligned} & 0.4215 \cdot \|f\|_\infty \left( \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} 1 \right) \left( \sum_{k_2=-\infty}^{-\lambda_2} \chi(k_2 + \lambda_2) \right) = \\ (83) \quad & = 0.4215 \cdot \|f\|_\infty (2\lambda_1 - 1) \left( \sum_{k'_2=-\infty}^0 \chi(k'_2) \right) \leq 0.4215 \cdot (2\lambda_1 - 1) \|f\|_\infty. \end{aligned}$$

So the double series under consideration is uniformly convergent and continuous. Clearly  $F_n(f, x_1, x_2)$  is proved to be continuous on  $\mathbb{R}^2$ .

Similarly reasoning one can prove easily now, but with more tedious work, that  $F_n(f, x_1, \dots, x_N)$  is continuous on  $\mathbb{R}^N$ , for any  $N \geq 1$ . We choose to omit this similar extra work.  $\square$

REMARK 1. By (24) it is obvious that  $\|A_n(f)\|_\infty \leq \|f\|_\infty < \infty$ , and  $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ , given that  $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ .

Call  $L_n$  any of the operators  $A_n, B_n, C_n, D_n$ .

Clearly then

$$(84) \quad \left\| L_n^2(f) \right\|_\infty = \left\| L_n(L_n(f)) \right\|_\infty \leq \left\| L_n(f) \right\|_\infty \leq \|f\|_\infty,$$

etc.

Therefore we get

$$(85) \quad \left\| L_n^k(f) \right\|_\infty \leq \|f\|_\infty, \quad \forall k \in \mathbb{N},$$

the contraction property.

Also we see that

$$(86) \quad \left\| L_n^k(f) \right\|_\infty \leq \left\| L_n^{k-1}(f) \right\|_\infty \leq \dots \leq \left\| L_n(f) \right\|_\infty \leq \|f\|_\infty.$$

Also  $L_n(1) = 1$ ,  $L_n^k(1) = 1$ ,  $\forall k \in \mathbb{N}$ .

Here  $L_n^k$  are positive linear operators. □

NOTATION 11. Here  $N \in \mathbb{N}$ ,  $0 < \beta < 1$ . Denote by

$$(87) \quad c_N := \begin{cases} (4.019)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases}$$

$$(88) \quad \varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases}$$

$$(89) \quad \Omega := \begin{cases} C \left( \prod_{i=1}^N [a_i, b_i] \right), & \text{if } L_n = A_n, \\ C_B(\mathbb{R}^N), & \text{if } L_n = B_n, C_n, D_n, \end{cases}$$

and

$$(90) \quad Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases}$$

We give the condensed

THEOREM 12. Let  $f \in \Omega$ ,  $0 < \beta < 1$ ,  $x \in Y$ ;  $n, N \in \mathbb{N}$  with  $n^{1-\beta} \geq 3$ .

Then

(i)

$$(91) \quad |L_n(f, x) - f(x)| \leq c_N \left[ \omega_1(f, \varphi(n)) + \frac{\|f\|_\infty}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] =: \tau,$$

(ii)

$$(92) \quad \|L_n(f) - f\|_\infty \leq \tau.$$

For  $f$  uniformly continuous and in  $\Omega$  we obtain

$$\lim_{n \rightarrow \infty} L_n(f) = f,$$

pointwise and uniformly.



*Proof.* By Theorems 4-7.  $\square$

Next we do iterated neural network approximation (see also [9]).  
We make

REMARK 2. Let  $r \in \mathbb{N}$  and  $L_n$  as above. We observe that

$$\begin{aligned} L_n^r f - f &= (L_n^r f - L_n^{r-1} f) + (L_n^{r-1} f - L_n^{r-2} f) + \\ &\quad + (L_n^{r-2} f - L_n^{r-3} f) + \dots + (L_n^2 f - L_n f) + (L_n f - f). \end{aligned}$$

Then

$$\begin{aligned} \|L_n^r f - f\|_\infty &\leq \|L_n^r f - L_n^{r-1} f\|_\infty + \|L_n^{r-1} f - L_n^{r-2} f\|_\infty + \\ &\quad + \|L_n^{r-2} f - L_n^{r-3} f\|_\infty + \dots + \|L_n^2 f - L_n f\|_\infty + \|L_n f - f\|_\infty = \\ &= \|L_n^{r-1} (L_n f - f)\|_\infty + \|L_n^{r-2} (L_n f - f)\|_\infty + \|L_n^{r-3} (L_n f - f)\|_\infty \\ (93) \quad &\quad + \dots + \|L_n (L_n f - f)\|_\infty + \|L_n f - f\|_\infty \leq r \|L_n f - f\|_\infty. \end{aligned}$$

That is

$$(94) \quad \|L_n^r f - f\|_\infty \leq r \|L_n f - f\|_\infty. \quad \square$$

We give

THEOREM 13. All here as in Theorem 12 and  $r \in \mathbb{N}$ ,  $\tau$  as in (91). Then

$$(95) \quad \|L_n^r f - f\|_\infty \leq r\tau.$$

So that the speed of convergence to the unit operator of  $L_n^r$  is not worse than of  $L_n$ .

*Proof.* By (94) and (92).  $\square$

We make

REMARK 3. Let  $m_1, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r$ ,  $0 < \beta < 1$ ,  $f \in \Omega$ . Then  $\varphi(m_1) \geq \varphi(m_2) \geq \dots \geq \varphi(m_r)$ ,  $\varphi$  as in (88).

Therefore

$$(96) \quad \omega_1(f, \varphi(m_1)) \geq \omega_1(f, \varphi(m_2)) \geq \dots \geq \omega_1(f, \varphi(m_r)).$$

Assume further that  $m_i^{1-\beta} \geq 3$ ,  $i = 1, \dots, r$ . Then

$$\begin{aligned} \frac{1}{(m_1^{1-\beta}-2)e^{(m_1^{1-\beta}-2)^2}} &\geq \frac{1}{(m_2^{1-\beta}-2)e^{(m_2^{1-\beta}-2)^2}} \\ (97) \quad &\geq \dots \geq \frac{1}{(m_r^{1-\beta}-2)e^{(m_r^{1-\beta}-2)^2}}. \end{aligned}$$

Let  $L_{m_i}$  as above,  $i = 1, \dots, r$ , all of the same kind.

We write

$$\begin{aligned}
& L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) - f = \\
& = L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) - L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} f)) + \\
& \quad + L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} f)) - L_{m_r} (L_{m_{r-1}} (\dots L_{m_3} f)) + \\
(98) \quad & \quad + L_{m_r} (L_{m_{r-1}} (\dots L_{m_3} f)) - L_{m_r} (L_{m_{r-1}} (\dots L_{m_4} f)) + \dots + \\
& \quad + L_{m_r} (L_{m_{r-1}} f) - L_{m_r} f + L_{m_r} f - f = \\
& = L_{m_r} (L_{m_{r-1}} (\dots L_{m_2})) (L_{m_1} f - f) + L_{m_r} (L_{m_{r-1}} (\dots L_{m_3})) (L_{m_2} f - f) + \\
& \quad + L_{m_r} (L_{m_{r-1}} (\dots L_{m_4})) (L_{m_3} f - f) + \dots + L_{m_r} (L_{m_{r-1}} f - f) + L_{m_r} f - f.
\end{aligned}$$

Hence by the triangle inequality property of  $\|\cdot\|_\infty$  we get

$$\begin{aligned}
& \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) - f\|_\infty \leq \\
& \leq \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_2})) (L_{m_1} f - f)\|_\infty + \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_3})) (L_{m_2} f - f)\|_\infty + \\
& \quad + \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_4})) (L_{m_3} f - f)\|_\infty + \dots + \\
& \quad + \|L_{m_r} (L_{m_{r-1}} f - f)\|_\infty + \|L_{m_r} f - f\|_\infty
\end{aligned}$$

(repeatedly applying (84))

$$\begin{aligned}
& \leq \|L_{m_1} f - f\|_\infty + \|L_{m_2} f - f\|_\infty + \|L_{m_3} f - f\|_\infty + \dots + \\
(99) \quad & \quad + \|L_{m_{r-1}} f - f\|_\infty + \|L_{m_r} f - f\|_\infty = \sum_{i=1}^r \|L_{m_i} f - f\|_\infty.
\end{aligned}$$

That is, we proved

$$(100) \quad \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) - f\|_\infty \leq \sum_{i=1}^r \|L_{m_i} f - f\|_\infty. \quad \square$$

We give

**THEOREM 14.** *Let  $f \in \Omega$ ;  $N, m_1, m_2, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r, 0 < \beta < 1; m_i^{1-\beta} \geq 3, i = 1, \dots, r, x \in Y$ , and let  $(L_{m_1}, \dots, L_{m_r})$  as  $(A_{m_1}, \dots, A_{m_r})$  or  $(B_{m_1}, \dots, B_{m_r})$  or  $(C_{m_1}, \dots, C_{m_r})$  or  $(D_{m_1}, \dots, D_{m_r})$ . Then*

$$\begin{aligned}
& |L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) (x) - f(x)| \leq \\
& \leq \|L_{m_r} (L_{m_{r-1}} (\dots L_{m_2} (L_{m_1} f))) - f\|_\infty \\
& \leq \sum_{i=1}^r \|L_{m_i} f - f\|_\infty \\
& \leq c_N \sum_{i=1}^r \left[ \omega_1(f, \varphi(m_i)) + \frac{\|f\|_\infty}{\sqrt{\pi}(m_i^{1-\beta} - 2)e^{(m_i^{1-\beta} - 2)^2}} \right] \leq
\end{aligned}$$

$$(101) \quad \leq r c_N \left[ \omega_1(f, \varphi(m_1)) + \frac{\|f\|_\infty}{\sqrt{\pi}(m_1^{1-\beta}-2)e^{(m_1^{1-\beta}-2)^2}} \right].$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of  $L_{m_1}$ .

*Proof.* Using (100), (96), (97) and (91), (92).  $\square$

We continue with

**THEOREM 15.** *Let all as in Theorem 8, and  $r \in \mathbb{N}$ . Here  $K_n$  is as in (57). Then*

$$(102) \quad \|A_n^r f - f\|_\infty \leq r \|A_n f - f\|_\infty \leq r K_n.$$

*Proof.* By (94) and (57).  $\square$

#### 4. COMPLEX MULTIVARIATE NEURAL NETWORK APPROXIMATIONS

We make

**REMARK 4.** Let  $Y = \prod_{i=1}^n [a_i, b_i]$  or  $\mathbb{R}^N$ , and  $f : Y \rightarrow \mathbb{C}$  with real and imaginary parts  $f_1, f_2 : f = f_1 + i f_2$ ,  $i = \sqrt{-1}$ . Clearly  $f$  is continuous iff  $f_1$  and  $f_2$  are continuous.

Given that  $f_1, f_2 \in C^m(Y)$ ,  $m \in \mathbb{N}$ , it holds

$$(103) \quad f_\alpha(x) = f_{1,\alpha}(x) + i f_{2,\alpha}(x),$$

where  $\alpha$  indicates a partial derivative of any order and arrangement.

We denote by  $C_B(\mathbb{R}^N, \mathbb{C})$  the space of continuous and bounded functions  $f : \mathbb{R}^N \rightarrow \mathbb{C}$ . Clearly  $f$  is bounded, iff both  $f_1, f_2$  are bounded from  $\mathbb{R}^N$  into  $\mathbb{R}$ , where  $f = f_1 + i f_2$ .

Here  $L_n$  is any of  $A_n, B_n, C_n, D_n$ ,  $n \in \mathbb{N}$ .

We define

$$(104) \quad L_n(f, x) := L_n(f_1, x) + i L_n(f_2, x), \quad \forall x \in Y.$$

We observe that

$$(105) \quad |L_n(f, x) - f(x)| \leq |L_n(f_1, x) - f_1(x)| + |L_n(f_2, x) - f_2(x)|,$$

and

$$(106) \quad \|L_n(f) - f\|_\infty \leq \|L_n(f_1) - f_1\|_\infty + \|L_n(f_2) - f_2\|_\infty. \quad \square$$

We present

**THEOREM 16.** *Let  $f \in C(Y, \mathbb{C})$  which is bounded,  $f = f_1 + i f_2$ ,  $0 < \beta < 1$ ,  $n, N \in \mathbb{N} : n^{1-\beta} \geq 3$ ,  $x \in Y$ . Then*

i)

$$(107) \quad |L_n(f, x) - f(x)| \leq \leq c_N \left[ \omega_1(f_1, \varphi(n)) + \omega_1(f_2, \varphi(n_2)) + \frac{(\|f_1\|_\infty + \|f_2\|_\infty)}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] =: \varepsilon,$$

ii)

$$(108) \quad \|L_n(f) - f\|_\infty \leq \varepsilon.$$

*Proof.* Use of (91).  $\square$

In the next we discuss high order of complex approximation by using the smoothness of  $f$ .

We give

**THEOREM 17.** *Let  $f : \prod_{i=1}^n [a_i, b_i] \rightarrow \mathbb{C}$ , such that  $f = f_1 + if_2$ . Assume  $f_1, f_2 \in C^m \left( \prod_{i=1}^n [a_i, b_i] \right)$ ,  $0 < \beta < 1$ ,  $n, m, N \in \mathbb{N}$ ,  $n^{1-\beta} \geq 3$ ,  $x \in \left( \prod_{i=1}^n [a_i, b_i] \right)$ . Then*

$$(109) \quad \left| A_n(f, x) - f(x) - \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n \left( \prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \leq (4.019)^N \cdot \left\{ \frac{N^m}{m!n^{m\beta}} \left( \omega_{1,m}^{\max} \left( f_{1,\alpha}, \frac{1}{n^\beta} \right) + \omega_{1,m}^{\max} \left( f_{2,\alpha}, \frac{1}{n^\beta} \right) \right) + \left( \frac{\|b-a\|_\infty^m (\|f_{1,\alpha}\|_{\infty,m}^{\max} + \|f_{2,\alpha}\|_{\infty,m}^{\max}) N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\},$$

ii)

(110)

$$|A_n(f, x) - f(x)| \leq (4.019)^N \cdot \left\{ \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{|f_{1,\alpha}(x)| + |f_{2,\alpha}(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] \right) + \frac{N^m}{m!n^{m\beta}} \left( \omega_{1,m}^{\max} \left( f_{1,\alpha}, \frac{1}{n^\beta} \right) + \omega_{1,m}^{\max} \left( f_{2,\alpha}, \frac{1}{n^\beta} \right) \right) + \left( \frac{\|b-a\|_\infty^m (\|f_{1,\alpha}\|_{\infty,m}^{\max} + \|f_{2,\alpha}\|_{\infty,m}^{\max}) N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\},$$

iii)

(111)

$$\begin{aligned} \|A_n(f) - f\|_\infty &\leq (4.019)^N \cdot \\ &\cdot \left\{ \sum_{j=1}^m \left( \sum_{|\alpha|=j} \left( \frac{\|f_{1,\alpha}(x)\|_\infty + \|f_{2,\alpha}(x)\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[ \frac{1}{n^{\beta_j}} + \left( \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right] \right) \right. \\ &+ \frac{N^m}{m!n^{m\beta}} \left( \omega_{1,m}^{\max} \left( f_{1,\alpha}, \frac{1}{n^\beta} \right) + \omega_{1,m}^{\max} \left( f_{2,\alpha}, \frac{1}{n^\beta} \right) \right) \\ &\left. + \left( \frac{\|b-a\|_\infty^m (\|f_{1,\alpha}\|_{\infty,m}^{\max} + \|f_{2,\alpha}\|_{\infty,m}^{\max}) N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\}, \end{aligned}$$

iv) Assume  $f_\alpha(x_0) = 0$ , for all  $\alpha : |\alpha| = 1, \dots, m; x_0 \in \left( \prod_{i=1}^N [a_i, b_i] \right)$ . Then

$$(112) \quad |A_n(f, x_0) - f(x_0)| \leq (4.019)^N \cdot$$

$$\begin{aligned} &\left\{ \frac{N^m}{m!n^{m\beta}} \left( \omega_{1,m}^{\max} \left( f_{1,\alpha}, \frac{1}{n^\beta} \right) + \omega_{1,m}^{\max} \left( f_{2,\alpha}, \frac{1}{n^\beta} \right) \right) + \right. \\ &\left. \left( \frac{\|b-a\|_\infty^m (\|f_{1,\alpha}\|_{\infty,m}^{\max} + \|f_{2,\alpha}\|_{\infty,m}^{\max}) N^m}{m!} \right) \frac{1}{\sqrt{\pi}(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}} \right\}, \end{aligned}$$

notice in the last the extremely high rate of convergence at  $n^{-\beta(m+1)}$ .

*Proof.* By Theorem 8 and Remark 4. □

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