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CONVERGENCE ANALYSIS FOR THE TWO-STEP NEWTON METHOD OF ORDER FOUR

IOANNIS K. ARGYROS* and SANJAY K. KHATTRI†

Abstract. We provide a tighter than before convergence analysis for the two-step Newton method of order four using recurrent functions. Numerical examples are also provided in this study.

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1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is Fréchet-differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Many problems in computational mathematics can be brought in the form (1.1). The solutions of these equations are rarely found in closed form. Therefore most solution methods for these equations are iterative. Newton's method

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad (n \ge 0), \quad (x_0 \in \mathcal{D}) \tag{1.2}$$

is undoubtedly the most popular method for generating a sequence $\{x_n\}$ converging quadratically to $x^\star$ [5, 13, 15]. Two-step Newton method (TSNM)

$$\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad (n \ge 0), \quad (x_0 \in \mathcal{D}), \\ x_{n+1} &= y_n - \mathcal{F}'(y_n)^{-1}\mathcal{F}(y_n), \end{aligned} \tag{1.3}$$

generates a converging sequence $\{x_n\}$ to $x^\star$ with order four [5, 9]. The following conditions have been used to show the semilocal convergence for the Newton's

*Department of Mathematical Sciences, Cameron University, Lawton, Oklahoma 73505-6377, USA, e-mail: iargyros@cameron.edu.
†Department of Engineering, Stord Haugesund University College, Norway, e-mail: sanjay.khattri@hsh.no.

method (1.2) and consequently the semilocal convergence of (TSNM) [5, 13, 15, 17] ($\mathbf{C}_{\mathrm{K}}$):

$$\mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}) \quad \text{for some } x_0 \in \mathcal{D};$$

$$\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\| \leq \nu$$

$$\|\mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x) - \mathcal{F}'(y)]\| \leq L\,\|x - y\| \quad \text{for all } x, y \in \mathcal{D};$$

$$h_K = L\eta \leq \tfrac{1}{2} \tag{1.4}$$

and

$$\overline{U}(x_0, \lambda) = \{x \in \mathcal{X} \mid \|x - x_0\| \leq \lambda\} \subseteq \mathcal{D},$$

for specified $\lambda \geq 0$.

Note that (1.4) is the, famous for its simplicity and clarity, Kantorovich sufficient convergence hypothesis for the Newton's method (1.2). A current survey on Newton-type methods can be found in [5] and the references therein (see also [1–4] and [6–17]). We have shown [5] the quadratic convergence of the Newton's method (1.2) using the set of conditions ($\mathbf{C}_{\mathrm{AH}}$)

$$\mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}) \qquad \text{for some } x_0 \in \mathcal{D};$$

$$\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\| \leq \eta$$

$$\|\mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x) - \mathcal{F}'(x_0)]\| \leq L_0\,\|x - x_0\| \quad \text{for all } x \in \mathcal{D};$$

$$\|\mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x) - \mathcal{F}'(y)]\| \leq L\,\|x - y\| \qquad \text{for all } x, y \in \mathcal{D};$$

$$h_{AH} = \overline{L}\eta \leq \tfrac{1}{2} \tag{1.5}$$

and

$$\overline{U}(x_0, \lambda_0) \subseteq \mathcal{D},$$

for some specified $\lambda_0 \geq 0$, where

$$\overline{L} = \tfrac{1}{8}\left(L + 4L_0 + \sqrt{L^2 + 8L_0 L}\right). \tag{1.6}$$

Note that

$$L_0 \leq L \tag{1.7}$$

holds in general, and L/L_0 can be arbitrarily large [4, 5]. Moreover, the L_0 Center-Lipschitz is not an additional condition, since L_0 is a special case of L. Furthermore, we have by (1.4)-(1.7)

$$h_K \leq \tfrac{1}{2} \quad \Longrightarrow \quad h_{AH} \leq \tfrac{1}{2} \tag{1.8}$$

but not necessarily vice versa unless if $L_0 = L$. The error analysis under (1.5) is also tighter than (1.4). Hence, the applicability of Newton's method (1.2) has been extended.

In this study, we provide the sufficient convergence conditions for (TSNM) corresponding to (1.4). The paper is organized as follows: §2 contains the semilocal convergence analysis for (TSNM), whereas the numerical examples are given in §3.

2. SEMILOCAL CONVERGENCE ANALYSIS FOR (TSNM)

We need the following result on majorizing sequence for (TSNM).

LEMMA 1. *Let L_0, L, η be constants. Assume: there exist parameters α and ϕ such that*

$$\frac{L\eta}{2(1-L_0\eta)} \leq \alpha, \tag{2.1}$$

$$\frac{L_1\eta}{2(1-L_2\eta)} \leq \phi \leq \phi_0 \tag{2.2}$$

and

$$\eta \leq \eta_0 \tag{2.3}$$

where,

$$L_1 = \alpha^2 L, \qquad L_2 = (1+\alpha)L_0, \tag{2.4}$$

$$\phi_1 = \frac{4L_0\alpha}{2(L_0+L_2)\alpha - L + \sqrt{[2(L_0+L)\alpha-L]^2+8L_0L\alpha}}, \tag{2.5}$$

$$\phi_2 = \frac{2L_1}{L_1+\sqrt{L_1^2+8L_1L_2}}, \qquad \phi_3 = \frac{2\alpha[1-(L_0+L_2)\eta]}{L\eta}, \tag{2.6}$$

$$\phi_0 = \min\left\{\phi_1, \phi_2, \phi_3\right\}, \tag{2.7}$$

$$\eta_1 = \frac{2}{L_1+2L_2(1+\phi)}, \qquad \eta_2 = \frac{1}{L_0+L_2}, \tag{2.8}$$

$$\eta_0 = \min\left\{\eta_1, \eta_2\right\}. \tag{2.9}$$

Then, sequences $\{s_n\}$, $\{t_n\}$ generated by

$$\begin{aligned} &t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{L(s_n-t_n)^2}{2(1-L_0 s_n)}, \\ &s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-s_n)^2}{2(1-L_0 t_{n+1})}, \end{aligned} \tag{2.10}$$

are non-decreasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\alpha}{1-\phi}\right)\eta, \tag{2.11}$$

and converge to their common least upper bound $t^\star \in [0, t^{\star\star}]$. Moreover, the following estimates holds

$$0 \leq t_{n+1} - s_n \quad \leq \alpha(s_n - t_n), \tag{2.12}$$

and

$$0 \leq s_{n+1} - t_{n+1} \leq \phi(s_n - t_n). \tag{2.13}$$

Proof. We shall show using induction on k:

$$0 \leq \frac{L(s_k-t_k)}{2(1-L_0 s_k)} \quad \leq \alpha, \tag{2.14}$$

and

$$0 \leq \frac{L_1(s_k - t_k)}{2(1 - L_0 t_{k+1})} \leq \phi. \tag{2.15}$$

Note that estimates (2.12) and (2.13) will then follow from (2.14) and (2.15), respectively. Estimates (2.14) and (2.15) hold by the left hand side hypotheses in (2.1) and (2.2), respectively. It follows from (2.10), (2.14) and (2.15) that estimates (2.12) and (2.13) hold for $n = 0$. Let us assume estimates (2.14) and (2.15) hold for all $k \leq n$. It then follows that estimates (2.12) and (2.13) hold for $n = k$. We then have:

$$0 \leq s_k - t_k \leq \phi(s_{k-1} - t_{k-1}) \leq \phi \cdot \phi(s_{k-2} - t_{k-2}) \leq \cdots \leq \phi^k \eta, \tag{2.16}$$

$$0 \leq t_{k+1} - s_k \leq \alpha(s_k - t_k) \leq \alpha\phi^k \eta, \tag{2.17}$$

and

$$\begin{aligned} t_{k+1} \leq s_k + \alpha\phi^k\eta \leq t_k + \alpha\phi^k\eta + \phi^k\eta & \\ \leq s_{k-1} + \alpha\phi^{k-1}\eta + \alpha\phi^k\eta + \phi^k\eta & \\ \leq t_{k-1} + \phi^{k-1}\eta + \alpha\phi^{k-1}\eta + \alpha\phi^k\eta + \phi^k\eta & \\ = t_{k-1} + (\phi^{k-1} + \phi^k)\eta + \alpha(\phi^{k-1} + \phi^k)\eta \leq \cdots & \\ \leq s_0 + \alpha(\eta + \phi\eta + \cdots + \phi^k\eta) + \alpha(\phi\eta + \cdots + \phi^k\eta) & \\ = (1 + \alpha)(1 + \phi + \cdots + \phi^k\eta) \leq t^{\star\star}. & \qquad (2.18) \end{aligned}$$

In view of (2.16) and (2.18), estimate (2.14) certainly holds if

$$0 \leq \frac{L\phi^k\eta}{2\left[1 - L_2(1 + \phi + \cdots + \phi^{k-1})\eta - L_0 t^{k-1}\eta\right]} \leq \alpha, \tag{2.19}$$

or

$$L\phi^k\eta + 2\alpha L_2(1 + \phi + \cdots + \phi^{k-1})\eta - 2\alpha + 2L_0\alpha t^{k-1}\eta \leq 0. \tag{2.20}$$

Estimate (2.20) motivates us to introduce functions f_k on $[0, 1)$ by

$$f_k(t) = L\eta t^k + 2\alpha L_2(1 + t + \cdots + t^{k-1})\eta + 2L_0\alpha t^{k-1}\eta - 2\alpha. \tag{2.21}$$

We need a relationship between two consecutive functions f_k:

$$\begin{aligned} f_{k+1}(t) &= Lt^{k+1}\eta + +2\alpha L_0 t^k\eta + 2\alpha L_2(1 + t + \cdots + t^k)\eta - 2\alpha - Lt^k\eta \\ &\quad - 2\alpha L_2(1 + t + \cdots + t^{k-1})\eta - 2L_0\alpha t^{k-1}\eta + 2\alpha + f_k(t) \\ &= f_k(t) + Lt^{k+1}\eta - Lt^k\eta + 2\alpha L_2 t^k\eta + 2L_0\alpha t^k\eta - 2L_0\alpha t^{k-1}\eta \\ &= f_k(t) + g(t)t^{k-1}\eta, \qquad (2.22) \end{aligned}$$

where

$$g(t) = Lt^2 + \left[2\alpha(L_2 + L_0) - L\right]t - 2L_0\alpha. \tag{2.23}$$

Using (2.21), we see that (2.20) holds

$$\text{if} \quad f_k(\phi) \le 0 \tag{2.24}$$

$$\text{or} \quad f_1(\phi) \le 0, \tag{2.25}$$

$$\text{since,} \quad g(\phi) \le 0 \quad \text{and} \quad f_{k+1}(\phi) = f_k(\phi) + g(\phi)\phi^k\eta \le f_k(\phi) \tag{2.26}$$

where ϕ is chosen as in the right hand side inequality of (2.1). But (2.23) also holds by (2.1). Moreover, define function f_∞ on $[0, 1)$ by

$$f_\infty(t) = \lim_{k\to\infty} f_k(t). \tag{2.27}$$

Then, we have by (2.24)

$$f_\infty(\phi) \le 0.$$

Hence, (2.12) and (2.14) hold for all k. Similarly, (2.15) holds if

$$L_1\phi^k\eta \le 2\phi\left[1 - L_2(1 + \phi + \cdots + \phi^k)\eta\right] \tag{2.28}$$

or

$$L_1\phi^k\eta + 2\phi L_2(1 + \phi + \cdots + \phi^k)\eta - 2\phi \le 0. \tag{2.29}$$

As in (2.21) we define functions h_k on $[0, 1)$ by

$$h_k(t) = L_1t^k\eta + 2tL_2(1 + t + \cdots + t^k)\eta - 2\phi. \tag{2.30}$$

We need a relationship between two consecutive functions h_k:

$$\begin{aligned} h_{k+1}(t) &= L_1t^{k+1}\eta + 2tL_2(1 + t + \cdots + t^{k+1})\eta - 2\phi - L_1t^k\eta - \\ &\quad - 2tL_2(1 + t + \cdots + t^k)\eta + 2\phi + h_k(t) \\ &= h_k(t) + L_1t^{k+1}\eta - L_1t^k\eta + 2L_2t^{k+2}\eta \\ &= h_k(t) + g_1(t)t^k\eta \end{aligned} \tag{2.31}$$

where

$$g_1(t) = 2L_2t^2 + L_1t - L_1. \tag{2.32}$$

In view of (2.30), estimate (2.29) holds if

$$\text{if} \quad h_k(\phi) \le 0 \quad \text{or} \quad h_1(\phi) \le 0 \tag{2.33}$$

$$\text{since,} \quad g_1(\phi) \le 0 \quad \text{and} \quad h_{k+1}(\phi) = h_k(\phi) + g_1(\phi)\phi^k\eta \le h_k(\phi) \tag{2.34}$$

where ϕ is chosen as in the right hand side of (2.2). Note now that (2.33) holds by (2.3). Furthermore, define functions h_∞ on $[0, 1)$ by

$$h_\infty(t) = \lim_{k\to\infty} h_k(t). \tag{2.35}$$

We then have

$$h_\infty(\phi) \le 0. \tag{2.36}$$

That completes the induction for (2.13) and (2.15). Finally, in view of (2.12), (2.13) and (2.18), sequences $\{t_n\}$, $\{s_n\}$ converge to $t^\star$. That completes the proof of the Lemma. □

We need an Ostrowski-type relationship between iterates $\{x_n\}$ and $\{y_n\}$ [5, 14].

LEMMA 2. *Let us assume iterates $\{x_n\}$ and $\{y_n\}$ in (TSNM) are well defined for all $n \geq 0$. Then, the following identities hold:*

$$\mathcal{F}(x_{n+1}) = \int_0^1 \left[\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)\right](x_{n+1} - y_n)\mathrm{d}\theta, \tag{2.37}$$

and

$$\mathcal{F}(y_n) = \int_0^1 \left[\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)\right](y_n - x_n)\mathrm{d}\theta. \tag{2.38}$$

Proof. Identity (2.37) follows from the Taylor's theorem and the first iteration in (TSNM), whereas (2.38) follows from Taylor's theorem and the second iteration in (TSNM). That completes the proof of the Lemma. □

We can show the following semilocal convergence result for (TSNM).

LEMMA 3. *Let $\mathcal{F} : \mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ be Fréchet-differentiable operator. Assume: there exist $x_0 \in \mathcal{D}$, $L_0 > 0$, $L > 0$ and $\eta \geq 0$ such that for all $x, y \in \mathcal{D}$:*

$$\mathcal{F}'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X}), \tag{2.39}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta, \tag{2.40}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x) - \mathcal{F}'(x_0)\right)\right\| \leq L_0 \left\|x - x_0\right\|, \tag{2.41}$$

$$\left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x) - \mathcal{F}'(y)\right)\right\| \leq L \left\|x - y\right\|, \tag{2.42}$$

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D}; \tag{2.43}$$

hypotheses of Lemma 2.1 hold, where $t^\star$ is given in Lemma 2.1. Then, sequences $\{x_n\}$ and $\{y_n\}$ generated by (TSNM) are well defined, remain in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converge to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold

$$\|y_n - x_n\| \leq s_n - t_n, \tag{2.44}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{2.45}$$

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \tag{2.46}$$

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n, \tag{2.47}$$

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{2.48}$$

$$\|y_n - x^\star\| \leq t^\star - s_n. \tag{2.49}$$

Furthermore, if there exists $R \geq t^\star$ such that

$$\overline{U}(x_0, R) \subseteq \mathcal{D} \tag{2.50}$$

and

$$L_0(t^\star + R) < 2, \tag{2.51}$$

then, $x^\star$ is the only solution of $\mathcal{F}(x) = 0$ in $\overline{U}(x_0, R)$

Proof. We shall show using induction on k that (TSNM) is well defined, the iterates remain in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and estimates (2.44) and (2.45) hold for all $n \geq 0$. Iterate y_0 is well defined by the first equation in (TSNM) for $n = 0$ and (2.39). We also have by (2.6) and (2.40)

$$\|y_0 - x_0\| = \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta = s_0 = s_0 - t_0 \leq t^\star.$$

That is (2.44) holds for $n = 0$ and $y_0 \in \overline{U}(x_0, t^\star)$. Using (TSNM) for $n = 0$, we see that x_1 is well defined. Let $w \in \overline{U}(x_0, t^\star)$. Then, we have by Lemma 2.1 and (2.41):

$$\left\|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(w) - \mathcal{F}'(x_0)\right]\right\| \leq L_0 \|w - x_0\| \leq L_0 t^\star < 1. \tag{2.52}$$

It follows from (2.52) and the Banach lemma on invertible operators [5, 13, 15] that $\mathcal{F}'(w)^{-1}$ exists and

$$\left\|\mathcal{F}'(w)^{-1}\mathcal{F}'(x_0)\right\| \leq \tfrac{1}{1 - L_0\|w - x_0\|}. \tag{2.53}$$

In particular, for $x_1 \in \overline{U}(x_0, t^\star)$, we have

$$\left\|\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\right\| \leq \tfrac{1}{1 - L_0\|x_1 - x_0\|} \leq \tfrac{1}{1 - L_0(t_1 - t_0)} = \tfrac{1}{1 - L_0 t_1}. \tag{2.54}$$

Moreover, in view of (2.38) for $n = 0$, (TSNM), (2.6) and (2.40)-(2.42), we get

(2.55)

$$\begin{aligned}
&\|x_1 - y_0\| = \\
&= \left\|\int_0^1 \left[\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0)\right]\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(x_0 + \theta(y_0 - x_0)) - \mathcal{F}'(x_0)\right]\mathrm{d}\theta(y_0 - x_0)\right\| \\
&\leq \tfrac{L_0}{1 - L_0\|y_0 - x_0\|}\int_0^1 \theta\|y_0 - x_0\|^2\mathrm{d}\theta \\
&= \tfrac{L_0}{2(1 - L_0\|y_0 - x_0\|)}\|y_0 - x_0\|^2 \\
&\leq \tfrac{L_0}{2(1 - L_0 s_0)}(s_0 - t_0)^2 = t_1 - s_0,
\end{aligned}$$

which shows (2.45) for $n = 0$. We also have

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 - t_0 \leq t^\star,$$

which implies (2.46) holds for $n = 0$ and $x_1 \in \overline{U}(x_0, t^\star)$.

Using (TSNM), (2.6), (2.37) (for $n = 0$) and (2.54), we get

$$\begin{aligned}\|y_1 - x_1\| &= \left\|\left[\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\right]\left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_1)\right]\right\|\\ &\leq \left\|\mathcal{F}'(x_1)^{-1}\mathcal{F}'(x_0)\right\|\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_1)\right\|\\ &\leq \tfrac{1}{1-L_0t_1}\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(y_0+\theta(x_1-y_0)) - \mathcal{F}'(y_0)\right]\mathrm{d}\theta(x_1-y_0)\right\|\\ &\leq \tfrac{L_0}{1-L_0t_1}\int_0^1 \theta\|x_1-y_0\|\mathrm{d}\theta\|x_1-y_0\|\\ &\leq \tfrac{L}{1-L_0t_1}\tfrac{1}{2}(t_1-s_0)(t_1-s_0) = s_1 - t_1,\end{aligned}$$

which implies (2.44) for $n = 1$. We then have:

$$\begin{aligned}\|y_1 - y_0\| &\leq \|y_1 - x_1\| + \|x_1 - y_0\| \leq s_1 - t_1 + t_1 - s_0 = s_1 - s_0,\\ \|y_1 - x_0\| &\leq \|y_1 - y_0\| + \|y_0 - x_0\| \leq s_1 - s_0 + s_0 - t_0 = s_1 \leq t^\star,\end{aligned}$$

which imply (2.47) for $n = 0$ and $y_1 \in \overline{U}(x_0, t^\star)$. Let us now assume (2.44)-(2.47), $y_n, x_k \in \overline{U}(x_0, t^\star)$ for all $n \leq k$. Using (TSNM), (2.6), (2.37), (2.38), (2.42), (2.53) and the induction hypotheses, we have in turn:

$$\begin{aligned}\|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|\\ &\leq t_{k+1} - t_k + t_k - t_{k-1} + \cdots + t_1 - t_0 = t_{k+1} \leq t^\star, \qquad (2.56)\end{aligned}$$

$$\begin{aligned}\|y_k - x_0\| &\leq \|y_k - x_k\| + \|x_k - x_0\| \qquad (2.57)\\ &\leq s_k - t_k + t_k - t_0\\ &= s_k \leq t^\star\end{aligned}$$

(2.58)

$$\begin{aligned}&\|y_{k+1} - x_{k+1}\| =\\ &= \left\|\left[\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right]\left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right]\right\|\\ &\leq \left\|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right\|\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\|\\ &\leq \tfrac{1}{1-L_0\|x_{k+1}-x_0\|}\int_0^1 \|\mathcal{F}'(x_0)^{-1}\left[\mathcal{F}'(y_k+\theta(x_{k+1}-y_k)) - \mathcal{F}'(y_k)\right]\mathrm{d}\theta(x_{k+1}-y_k)\|\\ &\leq \tfrac{L}{1-L_0t_{k+1}}\int_0^1 \theta\|x_{k+1}-y_k\|^2\mathrm{d}\theta\\ &\leq \tfrac{L}{1-L_0t_{k+1}}\tfrac{1}{2}(t_{k+1}-s_k)^2\\ &= s_{k+1} - t_{k+1},\end{aligned}$$

$$\|x_{k+2} - y_{k+1}\| = \left\|\left[\mathcal{F}'(y_{k+1})^{-1}\mathcal{F}'(x_0)\right]\left[\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_{k+1})\right]\right\| \leq \qquad (2.59)$$

$$\leq \tfrac{1}{1-L_0 s_{k+1}}\int_0^1 \|\mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_{k+1}+\theta(y_{k+1}-x_{k+1}))-\mathcal{F}'(x_{k+1})]\mathrm{d}\theta(y_{k+1}-x_{k+1})\|$$
$$\leq \tfrac{L}{1-L_0 s_{k+1}}\int_0^1 \theta\|y_{k+1}-x_{k+1}\|^2\mathrm{d}\theta$$
$$\leq \tfrac{L}{2(1-L_0 s_{k+1})}(s_{k+1}-t_{k+1})^2 = t_{k+2}-s_{k+1},$$

$$\|y_{k+2}-y_{k+1}\| \leq \|y_{k+2}-x_{k+2}\| + \|x_{k+2}-y_{k+1}\|$$
$$\leq s_{k+2}-t_{k+2}+t_{k+2}-s_{k+1} = s_{k+2}-s_{k+1}, \tag{2.60}$$
$$\|x_{k+2}-x_{k+1}\| \leq \|x_{k+2}-y_{k+1}\| + \|y_{k+1}-x_{k+1}\|$$
$$\leq t_{k+2}-s_{k+1}+s_{k+1}-t_{k+1} = t_{k+2}-t_{k+1} \tag{2.61}$$

which show (2.44)-(2.47) hold for all $n \geq 0$. Estimates (2.48) and (2.49) follow from (2.46) and (2.47), respectively by using standard majorization technique [5, 13, 15]. It follows from Lemma 2.1 and (2.44)-(2.48) that (TSNM) is Cauchy in a Banach space $\mathcal{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). Moreover, we have by (2.58)

$$\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\| \leq \tfrac{L}{2}\|x_{k+1}-y_k\|\|x_{k+1}-y_k\| \to 0, \quad \text{as} \quad k \to \infty. \tag{2.62}$$

That is $\mathcal{F}(x^\star) = 0$. Finally to show uniqueness, let $y^\star \in \overline{U}(x_0, R)$ be a solution of equation $\mathcal{F}(x) = 0$. Let us define linear operator M by

$$M = \int_0^1 \mathcal{F}'(y^\star + \theta(x^\star - y^\star))\mathrm{d}\theta. \tag{2.63}$$

Then using (2.41), (2.50) and (2.51), we get in turn

$$\|\mathcal{F}'(x_0)\,[M - \mathcal{F}'(x_0)]\| \leq L_0 \int_0^1 \|y^\star + \theta(x^\star - y^\star) - x_0\|\,\mathrm{d}\theta$$
$$\leq L_0 \int_0^1 [(1-\theta)\|y^\star - x_0\| + \theta\|x^\star - x_0\|]\,\mathrm{d}\theta$$
$$\leq \tfrac{L_0}{2}(R + t^\star) < 1. \tag{2.64}$$

It follows from (2.60) and the Banach Lemma on invertible operators that M^{-1} exists. Then, in view of the identity

$$0 = \mathcal{F}(x^\star) - \mathcal{F}(y^\star) = M(x^\star - y^\star), \tag{2.65}$$

we conclude that $x^\star = y^\star$. That completes the proof of the Theorem. □

Remark 4. 1) Limit point $t^\star$ can be replaced by $t^{\star\star}$, given in closed form by (2.7), in hypotheses (2.40) and (2.48).

2) The verification of conditions (2.1)-(2.3) require simple algebra (see also Example 3.1).

3) If $L_0 = L$, then scalar sequences $\{s_n\}$, $\{t_n\}$ given by (2.6) reduce essentially to the ones used in [9]. In particular, we have in this case

$$\begin{aligned} &\bar{t}_0 = 0, \quad \bar{s}_0 = \eta, \quad \bar{t}_{n+1} = \bar{s}_n + \tfrac{L(\bar{s}_n - \bar{t}_n)^2}{2(1-L\bar{s}_n)}, \\ &\bar{s}_{n+1} = \bar{t}_{n+1} + \tfrac{L(\bar{t}_{n+1} - \bar{s}_n)^2}{2(1-L\bar{t}_{n+1})} \end{aligned} \tag{2.66}$$

If $L_0 < L$ iteration (2.6) is tighter than (2.62). Moreover, in view of the proof of the Theorem 2.3, we note that sequence

$$\begin{aligned} &\bar{\bar{t}}_0 = 0, \quad \bar{\bar{s}}_0 = \eta, \quad \bar{\bar{t}}_{n+1} = \bar{\bar{s}}_n + \tfrac{L^\star(\bar{\bar{s}}_n - \bar{\bar{t}}_n)^2}{2(1-L_0\bar{\bar{s}}_n)}, \\ &\bar{\bar{s}}_{n+1} = \bar{\bar{t}}_{n+1} + \tfrac{L^\star(\bar{\bar{t}}_{n+1} - \bar{\bar{s}}_n)^2}{2(1-L_0\bar{\bar{t}}_{n+1})}, \end{aligned} \tag{2.67}$$

is also majorizing for (TSNM), where

$$L^\star = \begin{cases} L_0, & \text{if} \quad n = 0 \\ L, & \text{if} \quad n > 0. \end{cases}$$

In case $L_0 < L$, (2.26) is even a tighter majorizing sequence than (2.62). Furthermore, L, L_1 can be replaced by $L_0, L_1^\star = \alpha^2 L_0$ at the left hand sides of (2.1) and (2.2), respectively.

4) If $\alpha = 0$, define $L_1 = L$, then it is simple algebra to show that conditions of Lemma 2.1 reduce to (1.5). Moreover, if $L_0 = L$, these conditions reduce to (1.4). That is we have Newton's method (1.2) and iteration (2.6) reduces to

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \tfrac{L(t_{n+1} - t_n)^2}{2(1-L_0 t_{n+1})}. \tag{2.68}$$

In the case of Newton's method for $L_0 = L$, we have the well-known Kantorovich majorizing sequence.

$$\nu_0 = 0, \quad \nu_1 = \eta, \quad \nu_{n+2} = \nu_{n+1} + \tfrac{L(\nu_{n+1} - \nu_n)^2}{2(1-L_0 \nu_{n+1})}. \tag{2.69}$$

Note that if $L_0 < L$, $\{t_n\}$ is a tighter majorizing sequence than $\{\nu_n\}$ for the Newton's method [5, 13, 15]. □

3. NUMERICAL EXAMPLES

Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ be equipped with the max-norm, $x_0 = (1,1)^T$, $\mathcal{D} = \overline{U}(x_0, 1-p)$, $p \in [0,1)$ and define $\mathcal{F}$ on $\mathcal{D}$ by

$$\mathcal{F}(x) = \left(\xi_1^3 - p, \xi_2^3 - p\right)^T, \quad x = (\xi_1, \xi_2)^T. \tag{3.1}$$

Using (2.35)-(2.37), we get

$$\eta = \tfrac{1-p}{3}, \quad L_0 = 3 - p \quad \text{and} \quad L = 2(2-p) > L_0.$$

Let $p = 0.7$. Then, we get

$$\eta = 0.1, \quad L_0 = 2.3 \qquad \text{and} \quad L = 2.6.$$

The Newton-Kantorovich hypothesis (1.4) is satisfied, since

$$\tfrac{2}{3}(1-p)(2-p) = 0.26 < 1 \quad \text{for all} \quad p \in [0, 1/2).$$

Using Lemma 2.1, for $\alpha = 0.17$ and $\phi = 0.0052$, we get

$$\begin{aligned} L_1 &= 0.07514, & L_2 &= 2.691 & \phi &= 0.756703694, \\ \phi_2 &= 0.111383518, & \phi_3 &= 0.666923077, & \phi_0 &= \phi_2 \\ \eta_1 &= 0.364622409, & \eta_2 &= 0.200360649, & \eta_0 &= \eta_2, \end{aligned}$$

$$L\eta/[2(1-L_0\eta)] = 0.168831169 \quad \text{and} \quad L_1\eta/[2(1-L_2\eta)] = 0.005140238.$$

Hence, the hypotheses of Lemma 2.1 are satisfied. Moreover, we have by (2.11) that

$$t^{\star\star} = 0.11761158 < 1 - p = 0.3.$$

Furthermore, using (2.48) (for $t^\star$ replaced by $t^{\star\star}$), we get

$$t^{\star\star} < R < \tfrac{2}{L_0} - t^{\star\star} = 0.751953637.$$

So, we can choose $R = 0.3$. Hence, hypotheses of Theorem 2.3 hold, and (TSNM) converges to

$$x^\star = \left(\sqrt[3]{0.7}, \sqrt[3]{0.7}\right)^T = (0.887904002, 0.887904002)^T.$$

We compare (2.6) to (2.62).

n	$s_n - t_n$	$t_{n+1} - s_n$	$\bar{s}_n - \bar{t}_n$	$\bar{t}_{n+1} - \bar{s}_n$	$\bar{\bar{s}}_n - \bar{\bar{t}}_n$	$\bar{\bar{t}}_{n+1} - \bar{\bar{s}}_n$
0	$1.00 \cdot 10^{-01}$	$1.69 \cdot 10^{-02}$	$1.00 \cdot 10^{-01}$	$1.76 \cdot 10^{-02}$	$1.00 \cdot 10^{-01}$	$1.49 \cdot 10^{-02}$
1	$5.07 \cdot 10^{-04}$	$4.57 \cdot 10^{-07}$	$5.78 \cdot 10^{-04}$	$6.27 \cdot 10^{-07}$	$3.49 \cdot 10^{-04}$	$2.15 \cdot 10^{-07}$
2	$3.73 \cdot 10^{-13}$	$2.47 \cdot 10^{-25}$	$5.37 \cdot 10^{-13}$	$1.02 \cdot 10^{-24}$	$8.19 \cdot 10^{-14}$	$1.19 \cdot 10^{-26}$
3	$1.09 \cdot 10^{-49}$	$2.11 \cdot 10^{-98}$	$1.94 \cdot 10^{-48}$	$1.09 \cdot 10^{-96}$	$2.49 \cdot 10^{-52}$	$1.09 \cdot 10^{-103}$
4	$7.91 \cdot 10^{-196}$	$1.11 \cdot 10^{-390}$	$9.44 \cdot 10^{-191}$	$1.67 \cdot 10^{-380}$	$2.11 \cdot 10^{-206}$	$7.88 \cdot 10^{-412}$
5	$2.21 \cdot 10^{-780}$	$8.70 \cdot 10^{-1560}$	$5.24 \cdot 10^{-760}$	$5.15 \cdot 10^{-1519}$	$1.09 \cdot 10^{-822}$	$2.13 \cdot 10^{-1644}$

Table 1. Comparison among (2.6), (2.66) and (2.67)

As expected from the theoretical results iteration (2.6) is faster than (2.66).

REFERENCES

[1] S. Amat, S. Busquier and J. M. Gutiérrez, *On the local convergence of secant-type methods*, Int. J. Comput. Math., **8**1 (2004), no. 9, pp. 1153–1161.

[2] J. Appell, E. De Pascale, N. A. Evkhuta and P. P. Zabrejko, *On the two-step Newton method for the solution of nonlinear operator equations*, Math. Nachr., **17**2, (1995), pp. 5–14.

[3] I. K. Argyros, *On a multistep Newton method in Banach spaces and the Ptak error estimates*, Adv. Nonlinear Var. Inequal., **6** (2003), no. 2, pp. 121–135.

[4] I. K. Argyros, *A unifying local–semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl., **2**98 (2004), no. 2, pp. 374–397.

[5] I. K. Argyros, J. Y. Cho and S. Hilout, *Numerical Methods for Equations and its Applications*, CRC Press Taylor & Francis Group 2012, New York.

[6] R. P. Brent, *Algorithms for Minimization without Derivatives*, Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[7] E. Cătinaş, *On some iterative methods for solving nonlinear equations*, Rev. Anal. Numér. Théor. Approx., **2**3 (1994), no. 1, pp. 47–53.

[8] J. A. Ezquerro and M. A. Hernández, *Multipoint super-Halley type approximation algorithms in Banach spaces*, Numer. Funct. Anal. Optim., **2**1 (2000), no. 7-8, pp. 845–858.

[9] J. A. Ezquerro, M. A. Hernández and M. A. Salanova, *A Newton-like method for solving some boundary value problems*, Numer. Funct. Anal. Optim., **2**3 (2002), no. 7-8, pp. 791–805.

[10] J. A. Ezquerro, M. A. Hernández and M. A. Salanova, *A discretization scheme for some conservative problems*, J. Comput. Appl. Math., **1**15 (2000), no. 1-2, pp. 181–192.

[11] M. A. Hernández, M. J. Rubio and J.A. Ezquerro, *Secant-like methods for solving nonlinear integral equations of the Hammerstein type*, J. Comput. Appl. Math., **1**15 (2000), no. 1-2, pp. 245–254.

[12] M. A. Hernández and M. J. Rubio, *Semilocal convergence of the secant method under mild convergence conditions of differentiability*, Comput. Math. Appl., **4**4 (2002), no. (3-4), pp. 277-285.

[13] L. V. Kantorovich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[14] A. M. Ostrowski, *Solutions of equations in euclidean and Banach spaces*, A Series of Monographs and Textbooks, Academic Press, New York, 1973.

[15] J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Academic Press, New York 1970.

[16] I. Păvăloiu, *A convergence theorem concerning the method of Chord*, Rev. Anal. Numér. Théor. Approx., **2**1 (1972), no. 1, pp. 59–65.

[17] F. A. Potra and V. Pták, *Nondiscrete induction and iterative processes*, Research Notes in Mathematics, **1**03, Pitman Avanced Publ. Program, Boston, 1984.

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