REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 43 (2014) no. 1, pp. 45–58
ictp.acad.ro/jnaat

THE EIGENSTRUCTURE OF SOME POSITIVE LINEAR OPERATORS

ANTONIO ATTALIENTI* and IOAN RAŞA**

Abstract. Of concern is the study of the eigenstructure of some classes of positive linear operators satisfying particular conditions. As a consequence, some results concerning the asymptotic behaviour as $t \to +\infty$ of particular strongly continuous semigroups $(T(t))_{t\geq 0}$ expressed in terms of iterates of the operators under consideration are obtained as well. All the analysis carried out herein turns out to be quite general and includes some applications to concrete cases of interest, related to the classical Beta, Stancu and Bernstein operators.

MSC 2000. Primary 41A36; Secondary 47D06.
Keywords. Positive linear operators, eigenvalues and eigenpolynomials, iterates and series of positive linear operators, strongly continuous semigroups, asymptotic behaviour.

1. INTRODUCTION AND NOTATION

The present paper is devoted to the study of the eigenstructure of some classes of positive linear operators L_n acting on the Banach lattice $C([0,1])$ of all real-valued continuous functions on $[0,1]$, endowed with the uniform norm $\|\cdot\|_\infty$ and the usual order.

In order to pursue our main results, we adopt some assumptions over the L_n's. Some of them (see (3) and (4) at the beginning of Section 2) encircle our analysis in a general scheme of investigation initiated by Altomare and continued and developed originally and extensively, in different frameworks, by his school, dealing with the strong interplay between positive linear operators and strongly continuous semigroups: without attempting to be exhaustive in this respect, we confine ourselves to citing [1]-[7], [9]-[11], [14], [15] and all the references quoted therein.

Further conditions over the L_n's, gathered together into three groups, namely Case I, Case II and Case III, are needed to our purposes; as the reader will quickly realize, such additional assumptions, far from being somewhat artificial, turn out to be shared by classical positive linear operators of

*Department of Business and Law-University of Bari, Via Camillo Rosalba 53, 70124, Bari, Italy, e-mail: antonio.attalienti@uniba.it.
**Department of Mathematics-Technical University of Cluj-Napoca str. Memorandumului 28, 400114, Cluj-Napoca, Romania, e-mail: Ioan.Rasa@math.utcluj.ro.

continuous and discrete type, namely the Beta, the Stancu and the Bernstein operators.

Indeed, the whole set of conditions provides a nice eigenstructure and, in this sense, we confirm and expand what already addressed in [7, Remark 2.6].

The paper is organized as follows: in Section 2 we study the eigenstructure of our operators, indicating the eigenvalues and the corresponding eigenpolynomials by quite simple techniques, which should however be compared with those employed in [8] and [11].

The same analysis is carried out with respect to the differential operator W quoted in (2.1) and to the strongly continuous semigroup $(T(t))_{t\geq 0}$ written as limit of iterates of L_n as in (2.2).

In Section 3, proceeding along the lines illustrated in [7, Section 3] and sketched, though inside a simpler context, in [6, Theorem 2.2], we focus our attention upon the asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$, namely upon the limit $\lim_{t\to+\infty} T(t)f$ $(f \in C([0,1]))$ and to its possible interplay with the limits

$$\lim_{n\to+\infty}\lim_{k\to+\infty} L_n^k f, \quad \lim_{n\to+\infty} L_n^{k(n)} f, \tag{1.1}$$

$(k(n))_{n\geq 1}$ being a sequence of positive integers satisfying $k(n)/n \to +\infty$ as $n \to +\infty$ and L_n^k denoting the iterate of L_n of order k $(n, k \geq 1)$. The limits in (1.1) are involved in an overiteration procedure, introduced and developed by the authors in [7] and which will be our leading mark in Section 3.

Occasionally, we also touch upon the convergence of the series

$$\sum_{k=0}^{+\infty} L_n^k f$$

for suitable functions f.

As a significant application, we recapture, as particular cases, some results about the limits of the semigroups expressed in terms of iterates of the Beta, Stancu and Bernstein operators.

The notation used throughout the paper are quite standard in approximation theory and needs no particular or preparatory indication.

Therefore we shall confine ourselves to list only the most important ones: for any integer $r \geq 0$ let us set $e_r(x) := x^r,\ x \in [0,1]$.

In the sequel Π will denote the subalgebra of all polynomials on $[0,1]$: more specifically, we shall often deal with the space Π_r of all polynomials on $[0,1]$ of degree at most $r = 0, 1, \ldots$ If $p \in \Pi$, $\deg p$ is the degree of p. As usual, if $k \geq 1$ is an integer, $C^k([0,1])$ is the vector space of all real-valued k-times continuously differentiable functions on $[0,1]$. Finally, if x is a real number, then the integer part of x will be denoted by $[x]$.

Other notation which are not encompassed above, shall be specified at each occurrence.

2. EIGENVALUES AND EIGENPOLYNOMIALS

Throughout this section we shall deal with positive linear operators $L_n : C([0,1]) \to C([0,1])$ $(n \geq 1)$ acting on the Banach space $(C([0,1]), \|\cdot\|_\infty)$ and satisfying the following:

(1) For each $n \geq 1$ and $r \geq 1$ $L_n e_r$ is a polynomial of degree r with positive leading coefficient, i.e., $L_n e_r = a_{n,r} e_r + \dots$, with $a_{n,r} \geq 0$; moreover $L_n e_0 = a_{n,0} e_0$ with $a_{n,0} > 0$.

(2) The limit

$$l_r := \lim_{n\to+\infty} (a_{n,r})^n, \quad r \geq 0,$$

exists and is finite.

(3) (Voronovskaja-type result) For any $u \in C^2([0,1])$ we have

$$\lim_{n\to+\infty} n\big(L_n u - u\big) = Wu \quad \text{in } C([0,1]), \tag{2.1}$$

where explicitly $Wu(x) := a(x)u''(x) + b(x)u'(x),\ \ a, b \in C([0,1]),\ x \in [0,1]$.

(4) For every $f \in C([0,1])$ and $t \geq 0$ the limit

$$T(t)f := \lim_{n\to+\infty} L_n^{[nt]} f \tag{2.2}$$

exists in $C([0,1])$, and $(T(t))_{t\geq 0}$ is a C_0-semigroup on $C([0,1])$ with infinitesimal generator $(A, D(A))$ such that $C^2([0,1]) \subset D(A)$ and $Au = Wu$ for any $u \in C^2([0,1])$.

From (1) it clearly follows that each L_n is bounded with $\|L_n\| = \|L_n e_0\|_\infty = a_{n,0}$; moreover each Π_r is invariant under L_n and consequently the same happens under W and the semigroup $(T(t))_{t\geq 0}$ as well, due to (2.1), (2.2) and the closedness of Π_r itself. In particular $We_1 = b \in \Pi_1$ and $We_2 = 2a + 2be_1$, whence $a \in \Pi_2$.

As outlined in the introduction, besides these general assumptions we have to impose some further conditions over the coefficients $a_{n,j},\ l_j$ as well as over the degree of the polynomials $L_n e_r$, gathered together into three groups, namely Case I, Case II and Case III.

Case I

(5) $1 = a_{n,0} > a_{n,1} > \dots > 0,\ n \geq 1$.

(6) $1 = l_0 > l_1 > \dots > 0$.

Case II

(7) $\deg L_n e_r = \min\{n, r\},\ n \geq 1,\ r \geq 0$.

(8) $1 = a_{n,0} > a_{n,1} > \dots > a_{n,r} > 0,\ n \geq r \geq 0$.

(9) $1 = l_0 > l_1 > \dots > 0$.

Case III

(10) $\deg L_n e_r = \min\{n, r\},\ n \geq 1,\ r \geq 0$.

(11) $1 = a_{n,0} = a_{n,1} > a_{n,2} > \cdots > a_{n,r} > 0,\ n \geq r \geq 0.$
(12) $1 = l_0 = l_1 > l_2 > \cdots > 0.$

Remark 2.1. (i) All the assumptions (1)-(6) are satisfied by the classical Beta operators $\mathcal{B}_n$, introduced by Lupaş in [13] and studied for instance, as far as our investigation is concerned, in [4], [5] and [7]: in particular, in [5, Example 3.1], the explicit expression of the coefficients $a_{n,r}$ and of the limits l_r may be found.
In addition, the related differential operator W defined in (3) and its interplay with a strongly continuous semigroup has been completely investigated in [4, Theorem 2.10].
Note that each $\mathcal{B}_n$ maps Π_r into itself for any $r \geq 0$, hence $\mathcal{B}_n(\Pi) \subset \Pi$, even if its whole range $R(\mathcal{B}_n)$ is different from Π.
The classical Stancu operators S_n fulfill all the assumptions (1)-(4) and the ones listed in Case II : we refer the reader, e.g., to [7, Section 4], where a result about the related Voronovskaja-type formula and the existence of a strongly continuous semigroup expressed in terms of iterates of the S_n's has been stated. For the reader's convenience we recall that explicitly, for fixed $\alpha \geq 1/2$ and $\beta \geq \alpha + 1/2$, the n-th Stancu operator S_n is given by

$$S_n f := \sum_{i=0}^{n} b_{ni} f\left(\tfrac{i+\alpha}{n+\beta}\right)$$

for all $f \in C([0,1])$, where $b_{ni}(x) := \binom{n}{i} x^i (1-x)^{n-i},\ x \in [0,1]$.

Accordingly, it is not a difficult task to show that $a_{n,0} = 1$ and

$$a_{n,r} = \tfrac{n(n-1)\times\cdots\times(n-(r-1))}{(n+\beta)^r},\ \ n \geq r \geq 1, \quad l_r = e^{-\frac{r(r+2\beta-1)}{2}},\ \ r \geq 0.$$

Finally, we remark how the last Case III is of particular interest since all the conditions (10)-(12) enclosed herein, together with (1)-(4), hold true for the Bernstein operators B_n. In this particular situation one easily computes $a_{n,0} = 1$ and

$$a_{n,r} = \tfrac{n(n-1)\times\cdots\times(n-(r-1))}{n^r},\ \ n \geq r \geq 1, \quad l_r = e^{-\frac{r(r-1)}{2}},\ \ r \geq 0.$$

For a rather complete analysis about the related Voronovskaja formula and the existence of a strongly continuous semigroup expressed in terms of the iterates of the B_n's see, for instance, [1]-[3], [9], [10], [14] and [15].

(ii) Under the assumptions quoted in Cases II and III, each L_n maps continuous functions into polynomials in Π_n; indeed, choose $f \in C([0,1])$ and a sequence $(p_r)_{r\geq 1}$ in Π such that $\lim\limits_{r\to+\infty} p_r = f$. For a fixed $n \geq 1$ we get

$$L_n f = L_n(\lim_{r\to+\infty} p_r) = \lim_{r\to+\infty} L_n p_r,$$

which gives $L_n f \in \Pi_n$ since Π_n is closed and $L_n p_r \in \Pi_n$ for r large enough by virtue of (7) or (10).

(iii) In Case III the additional information $L_n e_1 = e_1$ is available, too; indeed, for any given $n \geq 1$, by assumption (11) we already know that $L_n e_1 = e_1 + \alpha_n e_0$ for some $\alpha_n \in \mathbb{R}$. On the other hand $0 \leq e_1 \leq e_0$ and therefore $0 \leq e_1 + \alpha_n e_0 \leq e_0$ because obviously $L_n e_0 = e_0$. Evaluating the last inequality in $x = 0$ and $x = 1$ gives $\alpha_n = 0$, as desired.
In turn, according to (4), the condition $L_n e_1 = e_1$ gives $T(t)e_1 = e_1$ for all $t \geq 0$ whence $We_1 \equiv 0$, i.e., $b \equiv 0$ (see the discussion straight after (2.2)).
(iv) Incidentally observe that in any case $L_n e_0 = e_0$ and therefore each L_n has norm 1. □

Now we may establish our first result.

THEOREM 2.2. *Let $n \geq 1$ and $r \geq 0$ be fixed; in Cases II and III we further assume $n \geq r$. Then $L_n : \Pi_r \to \Pi_r$ has $r + 1$ eigenvalues*

$$a_{n,0},\ a_{n,1}, \ldots, a_{n,r}. \tag{2.3}$$

To each $a_{n,j}$ there corresponds a monic eigenpolynomial $p_{n,j}$ with $\deg p_{n,j} = j$, i.e., $L_n p_{n,j} = a_{n,j} p_{n,j},\ j = 0, \ldots, r$.

Proof. The assumptions over n and r guarantee that in any case $L_n(\Pi_r) \subset \Pi_r$ so that on account of (1), (7) and (10) we may rightly write $L_n : \Pi_r \to \Pi_r$; with respect to the basis $\{e_0, \ldots, e_r\}$ the matrix of L_n is upper triangular and is given by

$$M_{n,r} = \begin{pmatrix} a_{n,0} & \cdots & \cdots \\ 0 & a_{n,1} & \cdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,r} \end{pmatrix}. \tag{2.4}$$

Clearly the corresponding eigenvalues are those indicated in (2.3). Moreover, they are distinct in Cases I and II on account of (5) and (8), and an easy computation allows to determine uniquely the related eigenpolynomials $p_{n,j},\ j = 0, \ldots, r$. In Case III the eigenpolynomials corresponding to the eigenvalues $a_{n,0} = a_{n,1} = 1$ are given respectively by $p_{n,0} = e_0$ and $p_{n,1} = e_1$ by virtue of (iii) and (iv) in Remark 2.1, whereas the remaining ones $p_{n,j},\ j = 2, \ldots, r$ may be found in the usual way. □

The following theorem is devoted to the analysis of the eigenstructure of the differential operator W.

THEOREM 2.3. *The differential operator $W : \Pi \to \Pi$ defined in (2.1) has eigenvalues $\log l_0,\ \log l_1, \ldots$ and corresponding monic eigenpolynomials $p_0,\ p_1, \ldots$ with $\deg p_j = j$, i.e., $Wp_j = (\log l_j)p_j,\ j = 0, 1, \ldots$ In addition, for any j we have*

$$\lim_{n \to +\infty} p_{n,j} = p_j \text{ uniformly on } [0,1], \tag{2.5}$$

$p_{n,j}$ being defined in the previous theorem.

Proof. Since $\Pi_0 \subset \Pi_1 \subset \cdots \subset \Pi$, it is enough to show that, for an arbitrary but fixed $r \geq 0$, the operator $W : \Pi_r \to \Pi_r$ has eigenvalues $\log l_0,\ \log l_1, \ldots, \log l_r$ with monic eigenpolynomials $p_0,\ p_1, \ldots, p_r$ $(\deg p_j = j)$ satisfying (2.5) for any $j = 0, 1, \ldots, r$.

To this purpose, let us fix, once and for all, an integer $r \geq 0$ and denote by U_r the matrix of the operator $W : \Pi_r \to \Pi_r$ with respect to the basis $\{e_0, \ldots, e_r\}$. Writing down (2.1) for $e_0, \ldots, e_r$ and denoting by I the unit matrix, we deduce a matricial version of the Voronovskaja formula, namely

$$\lim_{n\to+\infty} n(M_{n,r} - I) = U_r \quad \text{coordinatewise,} \tag{2.6}$$

which soon implies $\lim_{n\to+\infty} M_{n,r} = I$, i.e.,

$$\lim_{n\to+\infty} a_{n,j} = 1, \quad j = 0, 1, \ldots, r \tag{2.7}$$

on account of (2.4). But then, recalling (2), we get

$$l_j = \lim_{n\to+\infty} \big(1 + (a_{n,j} - 1)\big)^n = e^{\lim_{n\to+\infty} n(a_{n,j} - 1)},$$

and consequently

$$\lim_{n\to+\infty} n(a_{n,j} - 1) = \log l_j, \quad j = 0, 1, \ldots, r, \tag{2.8}$$

which, together with (2.4) and (2.6), allows to conclude that the matrix U_r is upper triangular and is given by

$$U_r = \begin{pmatrix} \log l_0 & \ldots & \ldots \\ 0 & \log l_1 & \ldots \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \log l_r \end{pmatrix}. \tag{2.9}$$

Thus U_r (or equivalently W) has $\log l_0,\ \log l_1, \ldots, \log l_r$ as its eigenvalues: they are distinct in Cases I and II and to each of them there corresponds a unique eigenpolynomial $p_j,\ j = 0, 1, \ldots, r$. In Case III we find $\log l_0 = \log l_1 = 0$ with corresponding eigenpolynomials $p_0 = e_0$ and $p_1 = e_1$ and a standard computation allows to determine uniquely the remaining ones $p_j,\ j = 2, \ldots, r$.

In order to prove (2.5), for the above fixed r let $n \geq r$. By Theorem 2.2 we already know that

$$n(L_n - I)p_{n,j} = n(a_{n,j} - 1)p_{n,j}, \quad j = 0, 1, \ldots, \ r, \tag{2.10}$$

which may be rephrased by saying that each $p_{n,j}$ is an eigenpolynomial of $n(L_n - I)$ corresponding to the eigenvalue $n(a_{n,j} - 1)$.

For each $j = 0, 1, \ldots, r$ let $p_{n,j} = e_j + x^n_{j-1}e_{j-1} + \cdots + x^n_0 e_0$. According to (2.10) the unknowns $x^n_{j-1}, \ldots, x^n_0$ may be uniquely determined by solving the $(r + 1) \times (r + 1)$ system

$$\Big(n(M_{n,r} - I) - n(a_{n,j} - 1)I\Big) \cdot (x^n_0, \ldots, x^n_{j-1}, 1, 0, \ldots, 0)^T = (0, \ldots, 0)^T, \tag{2.11}$$

which obviously reduces to a $(j+1)\times(j+1)$ system

(2.12)

$$\begin{pmatrix} n(a_{n,0}-a_{n,j}) & \dots & \dots & \dots & \dots \\ 0 & n(a_{n,1}-a_{n,j}) & \dots & \dots & \dots \\ 0 & 0 & \ddots & \dots & \dots \\ \vdots & \vdots & \dots & n(a_{n,j-1}-a_{n,j}) & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_0^n \\ \vdots \\ \vdots \\ x_{j-1}^n \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

On the other hand, if we consider an eigenpolynomial $p_j = e_j + y_{j-1} + \dots + y_0 e_0$ of U_r, then the unknowns $y_{j-1}, \dots, y_0$ are the solution of the $(r+1)\times(r+1)$ system

$$\Big(U_r - (\log l_j) I\Big) \cdot (y_0, \dots, y_{j-1}, 1, 0, \dots, 0)^T = (0, \dots, 0)^T, \tag{2.13}$$

which obviously reduces to a $(j+1)\times(j+1)$ system

(2.14)

$$\begin{pmatrix} \log l_0 - \log l_j & \dots & \dots & \dots & \dots \\ 0 & \log l_1 - \log l_j & \dots & \dots & \dots \\ 0 & 0 & \ddots & \dots & \dots \\ \vdots & \vdots & \dots & \log l_{j-1} - \log l_j & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_{j-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

From (2.6) and (2.8) we infer that the matrix of the coefficients of the system (2.11) tends coordinatewise as $n \to +\infty$ to the analogous matrix of the system (2.13) and therefore the same happens for the matrices of the coefficients in the systems (2.12) and (2.14), respectively. It immediately follows that $\lim\limits_{n\to+\infty} x_i^n = y_i$ for any $i = 0, \dots, j-1$ and consequently

$$\lim_{n\to+\infty} p_{n,j} = p_j, \text{ uniformly on } [0,1], \quad j = 0, 1, \dots, r\ , \tag{2.15}$$

which concludes the proof. □

The next corollary deals with the eigenstructure of the strongly continuous semigroup $(T(t))_{t\geq 0}$ quoted in (2.2).

Corollary 2.4. *For any* $t \geq 0$ $T(t) : \Pi \to \Pi$ *has eigenvalues* $l_0^t, l_1^t, \dots$ *with the same eigenpolynomials* $p_0, p_1, \dots$ *from Theorem* 2.3.

Proof. Simply observe that, on account of Theorem 2.2, for any $n \geq 1$, $j \geq 0$ and $t \geq 0$ one gets $L_n^{[nt]} p_{n,j} = a_{n,j}^{[nt]} p_{n,j}$; passing to the limit as $n \to +\infty$ yields

$$T(t) p_j = l_j^t p_j \tag{2.16}$$

by virtue of (2), (2.2), (2.5) and the boundedness of each L_n. □

It seems useful to display the situation described so far about eigenpolynomials in the following tables, where we adopt the same notation used in Theorems 2.2 and 2.3.

Case I

$$\begin{array}{ccccc} p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} & \cdots \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} & \cdots \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} & \cdots \\ p_{4,0} & p_{4,1} & p_{4,2} & p_{4,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \quad (n \to +\infty) \\ p_0 & p_1 & p_2 & p_3 & \cdots \end{array} \tag{2.17}$$

Cases II and III

$$\begin{array}{cccccc} p_{1,0} & p_{1,1} & & & & \\ p_{2,0} & p_{2,1} & p_{2,2} & & & \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} & & \\ p_{4,0} & p_{4,1} & p_{4,2} & p_{4,3} & p_{4,4} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \quad (n \to +\infty) \\ p_0 & p_1 & p_2 & p_3 & p_4 & \cdots \end{array} \tag{2.18}$$

3. ASYMPTOTIC BEHAVIOUR OF THE SEMIGROUP $(T(t))_{t\geq 0}$ AND OVERITERATION

Let us open this section with the following two general results, which shall be useful in the sequel, covering, perhaps, an interest on their own.

PROPOSITION 3.1. *Let T be an arbitrary bounded positive linear operator on a Banach space $(X, \|\cdot\|)$ and suppose that, for a given $x \in X$, there exists $Px := \lim_{k\to+\infty} T^k x \in X$. Then we have:*

(i) $P^2x = Px$.

(ii) *There exists $PTx \in X$ and $TPx = Px = PTx$.*

Proof. If $k \geq 1$, then

$$T^k Px = T^k\big(\lim_{j\to+\infty} T^j x\big) = \lim_{j\to+\infty} T^{k+j}x = Px, \tag{3.1}$$

which, for $k = 1$ and $k \to +\infty$, gives $TPx = Px$ and $P^2x = Px$, respectively.

Lastly, $Px = \lim_{k\to+\infty} T^{k+1}x = \lim_{k\to+\infty} T^k(Tx)$, i.e., PTx exists in X and is equal to Px. The proof is now complete. □

PROPOSITION 3.2. *Under the same assumptions and notation of Proposition* 3.1*, the following are equivalent:*

(a) *The series* $\sum_{k=0}^{+\infty} T^k x$ *is convergent in* X.
(b) *There exists* $y \in X$ *with* $Py = 0$ *such that* $x = y - Ty$.
(c) *There exists* $z \in X$ *such that* Pz *exists in* X *and* $x = z - Tz$.

Proof. $(a) \implies (b)$: Let us set $y = \sum_{k=0}^{+\infty} T^k x \in X$; then for all $j \geq 1$ one has

$$T^j y = \sum_{k=0}^{+\infty} T^{k+j} x = \sum_{i=0}^{+\infty} T^i x - \sum_{i=0}^{j-1} T^i x = y - \sum_{i=0}^{j-1} T^i x,$$

and therefore $\lim_{j \to +\infty} T^j y = y - y = 0$, i.e., $Py = 0$. Moreover, for any $k \geq 1$

$$(I - T)(I + T + \cdots + T^k)x = (I - T^{k+1})x, \tag{3.2}$$

and letting $k \to +\infty$ immediately gives $y - Ty = x$, since clearly $\lim_{k \to +\infty} T^{k+1} x = 0$.

Since $(b) \implies (c)$ is obvious, let us pass to show that $(c) \implies (a)$. To this aim, replacing x in (3.2) with the z given in (c) soon yields

$$(I + T + \cdots + T^k)x = z - T^{k+1} z \underset{k \to +\infty}{\longrightarrow} z - Pz \in X, \tag{3.3}$$

so that the series $\sum_{k=0}^{+\infty} T^k x$ is convergent, as desired. □

Remark 3.3. As a deeper insight, let us consider, as in (c), $x = z - Tz$ such that Pz exists in X. If we put $y := z - Pz$, then one readily gets $Py = Pz - P^2 z = 0$ and $y - Ty = x$ as a direct application of Proposition 3.1. □

Now let us pass to the main objective of this section, i.e., the study of the asymptotic behaviour of the semigroup $(T(t))_{t \geq 0}$ by means of the overiteration procedure involving limits in (1.1). To attain our main goals, we have to assume that henceforth each L_n has a totally positive kernel in the sense of Karlin, as described, in great details, in [12].

As pointed out in [5], we are dealing with a quite natural hypothesis, by no means breaking the generality of our investigation, since commonly fulfilled in concrete cases by most of the classical positive linear operators occurring in approximation theory and, however, intimately connected to the issue about the preservation of higher order convexity and Lipschitz classes: for a rather complete analysis in this direction, we refer the reader to [12] and [5].

Throughout the remaining of this section the discussion will be split up into two parts, the first concerning the Case I and the latter the Cases II and III.

Case I

As a preparatory material, let us start by choosing $p \in \Pi$; if $\deg p = r,\ r \geq 0$, then surely by Proposition 2.2 for every fixed $n \geq 1$ the polynomial p may be expanded as

$$p = c_{n,0}(p)p_{n,0} + c_{n,1}(p)p_{n,1} + \cdots + c_{n,r}(p)p_{n,r}, \tag{3.4}$$

where the coefficients $c_{n,j}(p),\ j = 0, 1, \ldots, r$, are uniquely determined and $c_{n,0}(p)$ does not depend on r. Then, for each $k \geq 1$, we easily compute

$$L_n^k p = c_{n,0}(p)e_0 + c_{n,1}(p)a_{n,1}^k p_{n,1} + \cdots + c_{n,r}(p)a_{n,r}^k p_{n,r}\ , \tag{3.5}$$

because $L_n p_{n,j} = a_{n,j}p_{n,j},\ j = 1, \ldots, r$ due to Theorem 2.2, $a_{n,0} = 1$ and $p_{n,0} = e_0$.

Note that $c_{n,0} : \Pi \to \mathbb{R}$ is a linear functional; moreover, if $p \geq 0$, then $L_n^k p \geq 0$ and letting $k \to +\infty$ in (3.5) yields $c_{n,0}(p) \geq 0$ since $a_{n,j} < 1$ for every $j = 1, \ldots r$ by assumption.

On the other hand, taking $p = e_0$ in (3.5) gives $c_{n,0}(e_0) = 1$.

Summing up, we have just shown that $c_{n,0} : \Pi \to \mathbb{R}$ is a positive linear functional on $(\Pi, \|\cdot\|_\infty)$ with $\|c_{n,0}\| = 1$.

The application of the classical Hahn-Banach Theorem allows to extend $c_{n,0}$ to a norm-one functional on the whole space $C([0,1])$; due to the density of Π, such extension, still denoted by $c_{n,0}$, is unique and positive, as well.

Now let us set $P_n : C([0,1]) \to C([0,1])$ as

$$P_n f = c_{n,0}(f)e_0 \quad \text{for every } f \in C([0,1]). \tag{3.6}$$

Of course P_n is a norm-one positive linear operator on $C([0,1])$ such that $\lim\limits_{k\to+\infty} L_n^k p = P_n p$ for all $p \in \Pi$ and this, by a density argument, leads soon to

$$\lim_{k\to+\infty} L_n^k f = P_n f \quad \text{for every } f \in C([0,1]). \tag{3.7}$$

Now we are in a position to state the following result.

THEOREM 3.4. *Under the above-mentioned assumptions and notation, there exists* $\lim\limits_{t\to+\infty} T(t)f := Sf$ *for every* $f \in C([0,1])$*; moreover*

$$\lim_{n\to+\infty} P_n f = Sf, \quad \lim_{n\to+\infty} L_n^{k(n)} f = Sf \tag{3.8}$$

for every $f \in C([0,1])$ *and for every sequence of positive integers* $(k(n))_{n\geq1}$ *satisfying* $k(n)/n \to +\infty$ *as* $n \to +\infty$.

Proof. The existence of the limit of the semigroup $(T(t))_{t\geq0}$ as $t \to +\infty$ follows from Theorem 2.2 in [6], indicating, in addition, that Sf is a constant function. For the remainder of the proof simply apply Corollary 2.3 and Theorem 2.2 in [7]. □

REMARK 3.5. (i) If, in particular, the L_n's are the Beta operators $\mathcal{B}_n$ (see (i) in Remark (2.1)), then S is explicitly described in [7, Theorem 3.1].

(ii) If for a given $n \geq 1$ $c_{n,0}(p) = 0$ in (3.5), then

$$\sum_{k=0}^{+\infty} L_n^k p = \tfrac{c_{n,1}(p)}{1-a_{n,1}} p_{n,1} + \cdots + \tfrac{c_{n,r}(p)}{1-a_{n,r}} p_{n,r} = \big(I - L_n\big)^{-1} p,$$

where $\big(I - L_n\big)^{-1} : \Pi \to \Pi$.

(iii) By using (3.5) an estimate of the speed of convergence in (3.7) for every fixed $n \geq 1$ as far as the polynomials are concerned may be obtained; indeed, since $1 = a_{n,0} > a_{n,1} > \cdots > 0$, $n \geq 1$, one has

$$\|L_n^k p - P_n p\| \leq \Big(|c_{n,1}(p)| \cdot \|p_{n,1}\| + \cdots + |c_{n,r}(p)| \cdot \|p_{n,r}\|\Big) a_{n,1}^k$$

for all $k \geq 1$ and $p \in \Pi$. □

Actually, something more can be said still in the framework of Case I, as stated in the two next propositions.

Proposition 3.6. *For a fixed $n \geq 1$ and $p \in \Pi$ we have*

$$\lim_{k \to +\infty} L_n^k p = 0 \iff p \in R(I - L_n), \tag{3.9}$$

$R(I - L_n)$ denoting the range of $I - L_n$.

Proof. According to (3.5) and the subsequent discussion, if $\deg p = r$, $r \geq 0$, then $\lim\limits_{k \to +\infty} L_n^k p = 0$ if and only if $c_{n,0}(p) = 0$ which, on account of (3.4), implies

$$p = c_{n,1}(p) p_{n,1} + \cdots + c_{n,r}(p) p_{n,r},$$

i.e., $p = (I - L_n) z$ where, by definition,

$$z := \tfrac{c_{n,1}(p)}{1-a_{n,1}} p_{n,1} + \cdots + \tfrac{c_{n,r}(p)}{1-a_{n,r}} p_{n,r}. \tag{3.10}$$

□

Proposition 3.7. *For a fixed $n \geq 1$ and $p \in \Pi$ we have*

$$\sum_{k=0}^{+\infty} L_n^k p \textit{ is convergent} \iff p \in R\big((I - L_n)^2\big). \tag{3.11}$$

Proof. By virtue of Proposition 3.2, the series $\sum\limits_{k=0}^{+\infty} L_n^k p$ is convergent if and only if $p = y - L_n y$ for some $y \in \Pi$ such that $\lim\limits_{k \to +\infty} L_n^k y = 0$, which, by virtue of (3.9), equals to $y = s - L_n s$ for some $s \in R(I - L_n)$. But then $p \in R\big((I - L_n)^2\big)$ and the proof is now fully performed. □

Now we have to consider the Cases II and III which will be treated simultaneously.

Cases II and III

Also in these cases a result word-for-word identical to Theorem 3.4 (with $P_n f$ defined below in (3.15) or (3.16)) may be achieved. Therefore we do not restate the assertion, passing soon to show the proof which runs similar, in fact simpler.

Indeed, due to [6, Theorem 2.2], in both cases there exists

$$\lim_{t\to+\infty} T(t)f := Sf \text{ for every } f \in C([0,1]), \tag{3.12}$$

Sf being a constant function in Case II (when $l_1 < 1,\ l_2 < 1$), a polynomial in Π_1 in Case III (when $l_2 < 1$ and $b \equiv 0$; see (iii) in Remark 2.1).

Now consider $f \in C([0,1])$ and $n \geq 1$; on account of (ii) in Remark 2.1 we already know that $L_n \in \Pi_n$ and therefore an expansion analogous to (3.4)

$$L_n f = c_{n,0}(f)p_{n,0} + c_{n,1}(f)p_{n,1} + \cdots + c_{n,n}(f)p_{n,n} \tag{3.13}$$

is just available; consequently, for any $k \geq 2$, keeping in mind Theorem 2.2 one has

$$L_n^k f = c_{n,0}(f)a_{n,0}^{k-1}p_{n,0} + c_{n,1}(f)a_{n,1}^{k-1}p_{n,1} + \cdots + c_{n,n}(f)a_{n,n}^{k-1}p_{n,n} \tag{3.14}$$

so that we may easily investigate the limit as $k \to +\infty$. More precisely, in Case II we have

$$P_n f := \lim_{k\to+\infty} L_n^k f = c_{n,0}(f)e_0 \tag{3.15}$$

since $a_{n,0} = 1$ and $p_{n,0} = e_0$, whereas in Case III

$$P_n f := \lim_{k\to+\infty} L_n^k f = c_{n,0}(f)e_0 + c_{n,1}(f)e_1 \tag{3.16}$$

since now $a_{n,0} = a_{n,1} = 1$ and $p_{n,0} = e_0,\ p_{n,1} = e_1$.

Note that the limit Sf of the semigroup in (3.12) is invariant under each L_n in both cases (simply recall (iii) in Remark 2.1); therefore, exactly as in Case I, we are in a position to apply Corollary 2.3 and Theorem 2.2 in [7] and hence

$$\lim_{n\to+\infty} P_n f = Sf, \quad \lim_{n\to+\infty} L_n^{k(n)} f = Sf \tag{3.17}$$

hold true for every $f \in C([0,1])$ and for every sequence of positive integers $(k(n))_{n\geq 1}$ satisfying $k(n)/n \to +\infty$ as $n \to +\infty$.

Remark 3.8. We also point out that, if $f \in C([0,1])$ and $n \geq 1$, then from (3.14) it easily follows that

$$\sum_{k=0}^{+\infty} L_n^k f \text{ is convergent} \iff P_n f = 0 \left(= \lim_{k\to+\infty} L_n^k f\right),$$

where $P_n f$ is defined in (3.15) or (3.16), accordingly. □

Remark 3.9. As already mentioned, the Stancu operators S_n fall within Case II; an explicit expression of Sf in (3.12) and (3.17) in this particular case may be found in [7, Formula (4.1) and Theorem 4.1].

An application of [7, Theorem 2.2] supplying a relationship analogous to (3.17) for the Bernstein operators is indicated in [15, Theorem 2.3]. However in this context, which corresponds to Case III, something more can be said in general about Sf. Indeed, we already know that $S : C([0,1]) \to \Pi_1$ is a positive linear projection.

Furthermore, $T(t)e_0 = e_0$ and $T(t)e_1 = e_1$ for all $t \geq 0$ (see (2.2) and (iii) in Remark 2.1) so that $Se_0 = e_0$ and $Se_1 = e_1$ by (3.12).

For any given $x \in [0,1]$ let us set $\eta_x(f) := Sf(x)$ for any $f \in C([0,1])$. Then η_x is a probability Radon measure with barycenter x; in particular, $\eta_0 = \delta_0$ and $\eta_1 = \delta_1$, where δ_0 and δ_1 denote the Dirac measures at 0 and 1, respectively.

Now, for an arbitrary $f \in C([0,1])$, we may write down $Sf = \alpha e_0 + \beta e_1$ for suitable reals α and β which entails $Sf(0) = \alpha$ and $Sf(1) = \alpha+\beta$; on the other hand, $Sf(0) = \delta_0(f) = f(0)$ and $Sf(1) = \delta_1(f) = f(1)$. In conclusion, we get $\alpha = f(0)$, $\beta = f(1) - f(0)$ so that the projection S is uniquely determined by

$$Sf(x) = (1-x)f(0) + xf(1) \quad (f \in C[0,1]),\ x \in [0,1]),$$

recapturing, in this way, a well-known result, scattered in the literature, about the limit, as $t \to +\infty$, of the semigroup $(T(t))_{t\geq 0}$ expressed through iterates of the classical Bernstein operators: we refer, for instance, to [3, Remark 3.11.1] and [14, Theorem 3.10]. □

REFERENCES

[1] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, W. de Gruyter, Berlin-New York, 1994.

[2] F. Altomare and I. Raşa, *On some classes of diffusion equations and related approximation problems,* in: Trends and Applications in Constructive Approximation, M. G. de Bruin, D. H. Mache and J. Szabados (Eds.), ISNM Vol. **151** (2005), 13-26, Birkhäuser-Verlag, Basel.

[3] F. Altomare, V. Leonessa and I. Raşa, *On Bernstein-Schabl operators on the unit interval*, Zeit. Anal. Anwend., **27** (2008), pp. 353–379.

[4] A. Attalienti, *Generalized Bernstein-Durrmeyer operators and the associated limit semigroup*, J. Approximation Theory, **99** (1999), pp. 289–309.

[5] A. Attalienti and I. Raşa, *Total Positivity: an application to positive linear operators and to their limiting semigroup*, Anal. Numér. Théor. Approx., **36** (2007), pp. 51–66.

[6] A. Attalienti and I. Raşa, *Asymptotic behaviour of C_0-semigroups*, in: Proceedings of the International Conference on Numerical Analysis and Approximation Theory, Cluj-Napoca, Romania, July 5-8, 2006, ISBN 973-686-961-X, 127-130.

[7] A. Attalienti and I. Raşa, *Overiterated linear operators and asymptotic behaviour of semigroups*, Mediterr. J. Math., **5** (2008), pp. 315–324.

[8] S. Cooper and S. Waldron, *The eigenstructure of the Bernstein operator*, J. Approx. Theory, **105** (2000), no. 1, pp. 133–165.

[9] H. Gonska, P. Piţul and I. Raşa, *Over-iterates of Bernstein-Stancu operators*, Calcolo, **44** (2007), pp. 117–125.
[10] H. Gonska and I. Raşa, *The limiting semigroup of the Bernstein iterates: degree of convergence*, Acta Math. Hungar., **111** (2006), pp. 119–130.
[11] H. Gonska, I. Raşa and E. D. Stănilă, *The eigenstructure of operators linking the Bernstein and the genuine Bernstein-Durrmeyer operators*, Mediterr. J. Math, **11** (2014), no. 2, pp. 561–576.
[12] S. Karlin, *Total Positivity*, Stanf. University Press, Stanford, 1968.
[13] A. Lupaş, *Die Folge der Beta Operatoren*, Dissertation Universität Stuttgart, 1972.
[14] I. Raşa, *Asymptotic behaviour of iterates of positive linear operators*, Jaen J. Approx., **1** (2009), pp. 195–204.
[15] I. Raşa, *Estimates for the semigroup associated with Bernstein-Schnabl operators*, Carpathian J. Math., **28** (2012), no.1, pp. 157–162.

Received by the editors: February 24, 2014.