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ON APPROXIMATING THE SOLUTIONS OF NONLINEAR EQUATIONS BY A METHOD OF AITKEN-STEFFENSEN TYPE

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Abstract. This paper completes the results that are presented in [14]. Using as a starting point the abstract method of the chord, in the mentioned paper we have presented an iterative method of approximation for the solutions of an equation. This method uses auxiliary sequences, and aims to improve the convergence order. The used method generalizes the method of Aitken-Steffensen. In the paper [14] we have given the statement of the main theorem and the statement and the proof of an auxiliary proposition concerning the convergence of some recurrence sequences of real numbers. In the present paper we give the proof of the main result and at the same time we discuss an interesting special case.

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1. INTRODUCTION

We will take again the main elements that are presented in detail in the papers [13]-[14] and that constitute the basis of our result.

Let us consider X, Y two linear normed spaces. We note by $\|\cdot\|_X : X \to \mathbb{R}$ and $\|\cdot\|_Y : Y \to \mathbb{R}$ their norms respectively, and by θ_X and θ_Y their null elements. Through $(X,Y)^*$ we note the set of the linear and continuous mappings defined from X to Y. The set $(X,Y)^*$ is a linear normed space as well, if we define the norm $\|\cdot\| : (X,Y)^* \to [0,+\infty[$, for any $U \in (X,Y)^*$ having $\|U\| = \sup\{\, \|U(h)\| : h \in X,\ \|h\|_X = 1\}$. For the case of $Y = \mathbb{R}$ we denote by X^* the set $(X,\mathbb{R})^*$, this set representing the space of real, linear and continuous functionals defined on the linear normed space X.

Let us consider now a set $D \subseteq X$ and a nonlinear mapping $f : D \to Y$. Using this mapping we have the equation:

$$f(x) = \theta_Y. \tag{1}$$

We will study the approximation of its solutions.

In order to clarify the aforementioned notions we have the following definition:

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Definition 1. *If we fix the nonlinear mapping $f : D \to Y$ together with the points $x, y \in D$, $x \neq y$, any mapping $\Gamma_{f;x,y} \in (X, Y)^*$ that verifies the equality:*

$$\Gamma_{f;x,y}(x - y) = f(x) - f(y) \tag{2}$$

is called ***generalized abstract divided difference of the function*** $f : D \to Y$ ***at the points*** x, y.

In connection with the previous definition we have the following remark:

Remark 2. **a)** We consider the theorem according to which in every linear normed space $(X, \|\cdot\|_X)$, for any $a \in X \setminus \{\theta_X\}$ there exists a linear and continuous functional $u \in X^*$ such that $\|u\| = 1$ and $u(a) = \|a\|_X$. Then, for any $x, y \in X$ with $x \neq y$ there exists the functional $U_{xy} \in X^*$ such that $\|U_{xy}\| = 1$ and $U_{xy}(x - y) = \|x - y\|_X$. At the same time there exists the functional $U_{yx} \in X^*$ such that $\|U_{yx}\| = 1$ and $U_{yx}(x - y) = \|y - x\|_X$ as well. In the paper [13] there appears the mapping $[x, y; f] \in (X, Y)^*$, defined by the equality:

$$[x, y; f]\, h = \frac{U_{xy}(h) f(x) + U_{yx}(h) f(y)}{\|x-y\|_X} \tag{3}$$

for any $h \in X$.

This mapping verifies the equality (2) and it is called **abstract divided difference** of the nonlinear mapping $f : D \to Y$ at the points $x, y \in D$ with $x \neq y$. This mapping is a special case of generalized abstract divided difference. Therefore we can choose $\Gamma_{f;x,y} = [x, y; f]$.

b) Let us suppose now that the space X is a space with a scalar product $\langle\cdot|\cdot\rangle : X \times X \to \mathbb{R}$. Defining $\|\cdot\|_X : X \to \mathbb{R}$ by $\|x\|_X = \sqrt{\langle x|x\rangle}$, the space $(X, \|\cdot\|_X)$ is a linear normed space.

For any $x, y \in X$ with $x \neq y$ the functional $U_{xy} \in X^*$ from **a)** will be defined by:

$$U_{xy}(h) = \frac{\langle h|x-y\rangle}{\|x-y\|_X}$$

for any $h \in X$. So, for the same elements $x, y \in X$ with $x \neq y$ we have that the abstract divided difference $[x, y; f] \in (X, Y)^*$ is defined by:

$$[x, y; f]\, h = \frac{\langle x-y|h\rangle (f(x) - f(y))}{\|x-y\|_X^2}$$

for any $h \in X$. □

Let us consider a initial element $x_0 \in D$. Besides the main sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ we also use two auxiliary sequences $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D$.

For these auxiliary sequences we request the existence of the numbers $K_1, K_2, p, q > 0$ such that for any $n \in \mathbb{N}^*$ the following inequalities are verified:

$$\begin{aligned} \|f(y_n)\|_Y &\leq K_1 \|f(x_n)\|_Y^p, \\ \|f(z_n)\|_Y &\leq K_2 \|f(x_n)\|_Y^q. \end{aligned} \tag{4}$$

Then, if for a number $n \in \mathbb{N}^*$ we have available the elements $y_n, z_n \in D$ starting from $x_n \in D$, we will build the new iterate $x_{n+1} \in D$ for the verification of the equality:

$$\Gamma_{f;y_n,z_n}(x_{n+1} - y_n) + f(y_n) = \theta_Y. \tag{5}$$

On account of the property of definition of the mapping $\Gamma_{f;y_n,z_n} \in (X,Y)^*$ the equality (5) is equivalent to:

$$\Gamma_{f;y_n,z_n}(x_{n+1} - z_n) + f(z_n) = \theta_Y. \tag{6}$$

If for any $n \in \mathbb{N}^*$ there exists the mapping $\Gamma^{-1}_{f;y_n,z_n} \in (Y,X)^*$ we have:

$$x_{n+1} = y_n - \Gamma^{-1}_{f;y_n,z_n} f(y_n) = z_n - \Gamma^{-1}_{f;y_n,z_n} f(z_n). \tag{7}$$

In connection with the main sequence $(x_n)_{n\in\mathbb{N}} \subseteq D$ and also with the auxiliary sequences $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}} \subseteq D$, one can see the remarks 2.1– 2.2 from the paper [14].

Remark 3. It is clear that if the first of the inequalities (4) is verified for any $n \in \mathbb{N}$ with a certain $K_1 > 0$, this inequality is verified with any number $K \geq \max\{1, K_1\}$. The situation is identical regarding the second inequality from (4). In conclusion we can suppose that in these relations we have $K_1 = K_2 = K \geq 1$.

Identically, we can suppose that in the inequalities:

$$\|y_n - x_n\|_X \leq a \|f(x_n)\|_Y,$$
$$\|z_n - x_n\|_X \leq b \|f(x_n)\|_Y,$$

that are true for any $n \in \mathbb{N}$, we can have $b = a \geq 1$.

In conclusion, for the main sequence $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ together with the auxiliary sequences $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D$ we can suppose that for any $n \in \mathbb{N}^*$ we have $y_n \neq z_n$ and there exist the numbers $K, a \geq 1$ such that for any $n \in \mathbb{N}^*$ the following inequalities are verified:

$$\begin{aligned} \|f(y_n)\|_Y &\leq K \|f(x_n)\|_Y^p, \\ \|f(z_n)\|_Y &\leq K \|f(x_n)\|_Y^q, \\ \|y_n - x_n\|_X &\leq a \|f(x_n)\|_Y, \\ \|z_n - x_n\|_X &\leq a \|f(x_n)\|_Y. \end{aligned} \tag{8}$$

□

In connection with the stated problem we consider, for the real numbers $p, q \geq 1$, the following equation in x on the interval $[0, +\infty[$:

$$x^{p+q-1} + 2x^2 + 2x - 1 = 0. \tag{9}$$

We have the following remarks:

Remark 4. **a)** The equation (9) has an unique root $\alpha \in]0,1[$.

b) If $\alpha \in]0,1[$ is the root of the equation (9) one verifies the following inequalities as well:

$$\alpha^2 + \alpha - 1 < 0, \qquad \alpha^2 + 2\alpha - 1 < 0, \qquad 2\alpha^2 + 2\alpha - 1 < 0 \tag{10}$$

and these inequalities are equivalent to the following inequalities respectively:

$$(11) \qquad 0 < \tfrac{\alpha^2}{1-\alpha} < 1, \qquad 0 < \tfrac{\alpha}{1-\alpha-\alpha^2} < 1, \qquad 0 < \tfrac{\alpha^2}{1-2\alpha-\alpha^2} < 1. \qquad \square$$

Let us consider now the numbers $a, K, L, B_0, R_0 > 0$ and the numbers $p, q \geq 1$ and using these numbers we build the real number sequences $(u_n)_{n\in\mathbb{N}^*}$, $(s_n)_{n\in\mathbb{N}^*}$, $(v_n)_{n\in\mathbb{N}^*}$, $(w_n)_{n\in\mathbb{N}^*}$, $(t_n)_{n\in\mathbb{N}^*}$, $(B_n)_{n\in\mathbb{N}^*}$ and $(R_n)_{n\in\mathbb{N}^*}$ using the following recurrence relations:

$$(12) \qquad \begin{aligned} u_n &= LKB_n^2R_n^p, \\ s_n &= LKB_n^2R_n^q, \\ v_n &= aL^2K^2\cdot\tfrac{B_n^3R_n^{p+q}}{1-u_n}, \\ w_n &= \tfrac{LKB_n^2R_n^q}{(1-u_n)(1-v_n)}, \\ t_n &= \tfrac{aL^2K^2B_n^3R_n^{p+q}}{(1-u_n)(1-v_n)(1-w_n)}, \\ B_{n+1} &= \tfrac{B_n}{(1-u_n)(1-v_n)(1-w_n)(1-t_n)}, \\ R_{n+1} &= LK^2B_n^2R_n^{p+q}. \end{aligned}$$

It is obvious that this construction has a meaning if for any $n \in \mathbb{N}^*$ we have that $u_n,\ v_n,\ w_n,\ t_n \in \mathbb{R}\setminus\{1\}$ and $B_n, R_n > 0$.

It is clear that for any $n \in \mathbb{N}^*$ we have:

$$(13) \qquad \begin{aligned} v_n &= \tfrac{a}{B_n}\cdot\tfrac{u_ns_n}{1-u_n}, \\ w_n &= \tfrac{s_n}{(1-u_n)(1-v_n)}, \\ t_n &= \tfrac{v_n}{(1-v_n)(1-w_n)}, \\ R_{n+1} &= \tfrac{u_ns_n}{LB_n^2}, \end{aligned}$$

as well.

Referring to the sequences that are defined by the relations (12) we have the following proposition:

Proposition 5. *If the following inequalities are verified:*

$$(14) \qquad a \leq B_0 \leq \tfrac{1}{\sqrt{L}}\cdot\min\left\{K^{\frac{p-q+1}{2(q-1)}}, K^{\frac{q-p+1}{2(p-1)}}\right\}$$

(with the specification that for $q = 1$ the expression that has $q-1$ in its denominator is $+\infty$, and the same for the expression that has $p-1$ in its denominator) and:

$$(15) \qquad d = \tfrac{LKB_0^2}{\alpha^2}\cdot\max{}^{\frac{1}{p+q-1}}\left\{\frac{R_0^{p(p+q-1)}K^{p-q+1}}{\left(LB_0^2\right)^{q-1}}, \frac{R_0^{q(p+q-1)}K^{q-p+1}}{\left(LB_0^2\right)^{p-1}}\right\} < 1$$

where $\alpha \in]0,1[$ *is the unique root of the equation* (9)*, then for any* $n \in \mathbb{N}^*$ *we have the following inequalities:*

$$
\begin{aligned}
u_n &\leq \alpha d^{(p+q)^n} < \alpha < 1, \\
s_n &\leq \alpha d^{(p+q)^n} < \alpha < 1, \\
v_n &\leq \tfrac{\alpha^2}{\alpha-1} \cdot d^{2(p+q)^n} < \tfrac{\alpha^2}{\alpha-1} < 1, \\
w_n &\leq \tfrac{\alpha}{1-\alpha-\alpha^2} \cdot d^{(p+q)^n} < \tfrac{\alpha}{1-\alpha-\alpha^2} < 1, \\
t_n &\leq \tfrac{\alpha^2}{1-2\alpha-\alpha^2} d^{2(p+q)^n} < \tfrac{\alpha^2}{1-2\alpha-\alpha^2} < 1, \\
B_{n+1} &\leq \tfrac{B_n}{1-2\alpha-2\alpha^2}, \\
R_{n+1} &\leq \tfrac{\alpha^2}{LB_0^2} d^{2(p+q)^n}.
\end{aligned} \tag{16}
$$

The proof of this proposition has been given in the paper [14].

2. THE MAIN RESULT

In this section we present the statement and the proof of the main theorem regarding the convergence of the sequences $(x_n)_{n\in\mathbb{N}^*}$, $(y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*} \subseteq D \subseteq X$.

Theorem 6. *We suppose that the following conditions hold:*

i) *The linear normed space* $(X, \|\cdot\|_X)$ *is a Banach space;*

ii) *The mapping* $f : D \to Y$ *admits for any* $x, y \in D$ *with* $x \neq y$ *a generalized abstract divided difference* $\Gamma_{f;x,y} \in (X,Y)^*$ *and there exists a number* $L > 0$ *such that for any* $x, y, z \in D$ *with* $x \neq y,\ y \neq z$ *we have the inequality:*

$$\|\Gamma_{f;x,y} - \Gamma_{f;y,z}\| \leq L \|x - z\|_X ;$$

iii) *The main approximant sequence* $(x_n)_{n\in\mathbb{N}^*}$ *together with the secondary sequences* $(y_n)_{n\in\mathbb{N}^*}$ *and* $(z_n)_{n\in\mathbb{N}^*}$ *are such that for any* $n \in \mathbb{N}^*$ *the following equality is fulfilled:*

$$\Gamma_{f;y_n,z_n}(x_{n+1} - y_n) + f(y_n) = \theta_Y \tag{17}$$

together with the inequalities (8) with the constants $a, K > 1$ *and* $p, q \geq 1$*. We also have that* $f(y_n), f(z_n) \in Y \setminus \{\theta_Y\}$, $y_n \neq z_n$ *and we are in one of the following situations:*

iii$_1$) $x_n \neq y_n$ *and* $y_{n+1} \neq z_n$,
or
iii$_2$) $x_n \neq z_n$ *and* $z_{n+1} \neq y_n$.

iv) *The mapping* $\Gamma_{f;y_0,z_0} \in (X,Y)^*$ *is invertible and* $\Gamma^{-1}_{f;y_0,z_0} \in (Y,X)^*$.

v) *If we note:*

$$
\begin{aligned}
&B_0 = \max\left\{a, \|\Gamma^{-1}_{f;y_0,z_0}\|\right\},\\
&R_0 = \|f(x_0)\|_Y,\\
&\overline{K} = \max\left\{K,\ (B_0\sqrt{L})^{\frac{2(q-1)}{p-q+1}}, (B_0\sqrt{L})^{\frac{2(p-1)}{q-p+1}}\right\},\\
&d = \tfrac{L\overline{K}B_0^2}{\alpha^2}\cdot\max\left\{\frac{R_0^{p(p+q-1)}\overline{K}^{p-q+1}}{\left(LB_0^2\right)^{q-1}}, \frac{R_0^{q(p+q-1)}\overline{K}^{q-p+1}}{\left(LB_0^2\right)^{p-1}}\right\},\\
&\delta = 2aR_0 + \tfrac{a\alpha^2}{LB_0^2}\cdot\tfrac{d^2}{1-d^{2(p+q-1)}} + \tfrac{2\alpha}{L\overline{K}B_0}\cdot\tfrac{d}{1-d^{p+q-1}},
\end{aligned}
$$

where $\alpha\in\,]0,1[$ *is the unique root of the equation* (9), *the conditions* $d<1$ *and* $S(x_0,\delta)=\{x\in X/\,\|x-x_0\|_X\le\delta\}$ *are fulfilled.*

If the previous hypotheses are true, then the following conclusions are true as well:

j) *for any* $n\in\mathbb{N}^*$ *we have that* $x_n, y_n, z_n\in S(x_0,\delta)$, *there exists the mapping* $\Gamma^{-1}_{f;y_n,z_n}\in(Y,X)^*$ *and:*

$$x_{n+1} = y_n - \Gamma^{-1}_{f;y_n,z_n}f(y_n) = z_n - \Gamma^{-1}_{f;y_n,z_n}f(z_n); \tag{18}$$

jj) *the sequences* $(x_n)_{n\in\mathbb{N}^*}, (y_n)_{n\in\mathbb{N}^*}, (z_n)_{n\in\mathbb{N}^*}\subseteq X$ *are convergent to the limit* $\overline{x}\in S(x_0,\delta)$ *and* $f(\overline{x})=\theta_Y$;

jjj) *for any* $n\in\mathbb{N}^*$ *the following inequalities are fulfilled:*

$$\|x_{n+1}-x_n\|_X \le \tfrac{a\alpha^2}{LB_0^2}\cdot d^{2(p+q)^{n-1}} + \tfrac{\alpha}{L\overline{K}B_0}\cdot d^{(p+q)^n}; \tag{19}$$

$$\|x_n-\overline{x}\|_X \le \tfrac{a\alpha^2}{LB_0^2}\cdot\tfrac{d^{2(p+q)^{n-1}}}{1-d^{2(p+q)^{n-1}(p+q-1)}} + \tfrac{\alpha}{L\overline{K}B_0}\cdot\tfrac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}}; \tag{20}$$

$$
\begin{aligned}
&\max\left\{\|y_n-\overline{x}\|_X, \|z_n-\overline{x}\|_X\right\}\le\\
&\le \tfrac{a\alpha^2}{LB_0^2}\cdot d^{2(p+q)^{n-1}}\cdot\tfrac{2-d^{2(p+q)^{n-1}(p+q-1)}}{1-d^{2(p+q)^{n-1}(p+q-1)}} + \tfrac{\alpha}{L\overline{K}B_0}\cdot\tfrac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}}.
\end{aligned}
\tag{21}
$$

Proof. For more clarity in the case of the hypothesis **iii)** we suppose that the situation **iii**$_1$**)** is fulfilled, namely for any $n\in\mathbb{N}^*$ we will suppose that we have:

$$f(y_n), f(z_n)\in Y\setminus\{\theta_Y\},\ y_n\ne z_n,\ x_n\ne y_n,\ y_{n+1}\ne z_n,$$

while the equality (17) is verified.

Using the constants $L, B_0, \overline{K}, R_0>0$ we generate the real number sequences $(u_n)_{n\in\mathbb{N}^*}, (s_n)_{n\in\mathbb{N}^*}, (v_n)_{n\in\mathbb{N}^*}, (w_n)_{n\in\mathbb{N}^*}, (t_n)_{n\in\mathbb{N}^*}, (B_n)_{n\in\mathbb{N}^*}, (R_n)_{n\in\mathbb{N}^*}$ on the basis of the relations (12) in which the constant $K>0$ is replaced by $\overline{K}>0$.

From the expression of $\overline{K}$ we immediately deduce that:

$$\overline{K}^{\frac{p-q+1}{2(q-1)}}\ge B_0\sqrt{L},\ \overline{K}^{\frac{q-p+1}{2(p-1)}}\ge B_0\sqrt{L},$$

therefore:

$$B_0\le\tfrac{1}{\sqrt{L}}\cdot\min\left\{\overline{K}^{\frac{p-q+1}{2(q-1)}}, \overline{K}^{\frac{q-p+1}{2(p-1)}}\right\},$$

so the double inequality (14) is true. The condition imposed to d, expressed by the inequality (15), from the hypothesis of the Proposition 5 is fulfilled on the basis of the hypotheses of the theorem.

On account of the Proposition 5 we deduce that for any $n \in \mathbb{N}^*$ the relations (16) are true.

We now prove that for any $n \in \mathbb{N}^*$ the following relations are true:

a) $x_n, y_n, z_n \in S(x_0, \delta)$;

b) $\|f(x_n)\|_Y \leq R_n$;

c) there exists the mapping $\Gamma^{-1}_{f;y_n,z_n} \in (Y,X)^*$, $\|\Gamma^{-1}_{f;y_n,z_n}\| \leq B_n$ and:

$$x_{n+1} = y_n - \Gamma^{-1}_{f;y_n,z_n} f(y_n) = z_n - \Gamma^{-1}_{f;y_n,z_n} f(z_n). \tag{22}$$

In order to prove these relations we will use the method of the mathematical induction.

For $n = 0$ we notice the following statements:

a) Evidently $x_0 \in S(x_0, \delta)$ and:

$$\|y_0 - x_0\|_X \leq a\|f(x_0)\|_Y = aR_0 \leq \delta,$$

therefore $y_0 \in S(x_0, \delta)$.

Similarly:

$$\|z_0 - x_0\|_X \leq a\|f(x_0)\|_Y = aR_0 \leq \delta$$

so we have that $z_0 \in S(x_0, \delta)$ as well.

b) Evidently $\|f(x_0)\|_Y \leq R_0$;

c) The existence of the mapping $\Gamma^{-1}_{f;y_0,z_0} \in (Y,X)^*$ is assured from the hypotheses of the theorem and the inequality $\|\Gamma^{-1}_{f;y_0,z_0}\| \leq B_0$ is assured from the definition of the number B_0.

We suppose that the inequalities **a) - c)** are true for any number $n \in \mathbb{N}^*$, $n \leq k$ and we prove them for $n = k+1$.

a) For any $j \in \{0, 1, ..., k\}$ we have that:

$$\begin{aligned} \|x_{j+1} - x_j\|_X &\leq \|y_j - x_j\|_X + \|\Gamma^{-1}_{f;y_j,z_j}\| \cdot \|f(y_j)\|_Y \\ &\leq a\|f(x_j)\|_Y + B_j\|f(x_j)\|_Y^p \\ &= aR_j + B_j R_j^p \\ &= aR_j + \tfrac{B_j^2 R_j^p}{B_j} \\ &\leq aR_j + \tfrac{u_j}{LKB_0}. \end{aligned}$$

For $j = 0$ we obtain:

$$\|x_1 - x_0\|_X \leq aR_0 + \tfrac{u_0}{LKB_0} = aR_0 + \tfrac{\alpha}{LKB_0} \cdot d.$$

For $j \in \{1, 2, ..., k\}$ we obtain:

$$\|x_{j+1} - x_j\|_X \leq a \cdot \tfrac{\alpha^2}{LB_0^2} \cdot d^{2(p+q)^{j-1}} + \tfrac{\alpha}{LKB_0} \cdot d^{(p+q)^j}. \tag{23}$$

Therefore:

$$\begin{aligned}\|x_{k+1}-x_0\|_X &\le \left\|\sum_{j=0}^{k}(x_{j+1}-x_j)\right\|_X\\ &\le \sum_{j=0}^{k}\|x_{j+1}-x_j\|_X\\ &\le aR_0+\tfrac{\alpha}{L\overline{K}B_0}\cdot d+\tfrac{a\alpha^2}{LB_0^2}\sum_{j=0}^{k-1}d^{2(p+q)^j}+\tfrac{\alpha}{L\overline{K}B_0}\sum_{j=0}^{k}d^{(p+q)^j}.\end{aligned}$$

In the last expression we have:

$$\sum_{j=0}^{k-1}d^{2(p+q)^j}=d^2+d^{2(p+q)}+...+d^{2(p+q)^{k-1}}=d^2\sum_{j=0}^{k-1}d^{2(p+q)^j-2}.$$

As $p,q\ge 1$ we have that:

$$\begin{aligned}2(p+q)^j-2&=2(p+q-1)\left[1+(p+q)+...+(p+q)^{j-1}\right]\\ &\ge 2j(p+q-1),\end{aligned}$$

and as $d<1$ we deduce that $d^{2(p+q)^j-2}\le\left[d^{2(p+q-1)}\right]^j$, therefore:

$$\sum_{j=0}^{k-1}d^{2(p+q)^j}\le d^2\sum_{j=0}^{\infty}\left[d^{2(p+q-1)}\right]^j=\tfrac{d^2}{1-d^{2(p+q-1)}}.$$

Similarly

$$\sum_{j=0}^{k}d^{(p+q)^j}\le\tfrac{d}{1-d^{p+q-1}}.$$

Therefore

$$\begin{aligned}\|x_{k+1}-x_0\|_X&\le aR_0+\tfrac{a\alpha^2}{LB_0^2}\left(d^2+\tfrac{d^2}{1-d^{2(p+q-1)}}\right)+\tfrac{\alpha}{L\overline{K}B_0}\cdot\tfrac{d}{1-d^{p+q-1}}\\ &\le\delta,\end{aligned}$$

so $x_{k+1}\in S(x_0,\delta)$.

From here it is clear that:

$$\begin{aligned}\|y_{k+1}-x_0\|_X&\le\|y_{k+1}-x_{k+1}\|_X+\|x_{k+1}-x_0\|_X\\ &\le a\|f(x_{k+1})\|_Y+\|x_{k+1}-x_0\|_X\\ &\le aR_{k+1}+\|x_{k+1}-x_0\|_X\\ &\le\tfrac{a\alpha^2}{LB_0^2}\cdot d^{2(p+q)^k}+\|x_{k+1}-x_0\|_X.\end{aligned}$$

As $d<1$ and $(p+q)^k \geq 1$ it is clear that $d^{2(p+q)^K} \leq d^2$, therefore:

$$\begin{aligned}\|y_{k+1}-x_0\|_X &\leq 2aR_0 + \tfrac{a\alpha^2}{LB_0^2}\left(d^2 + \tfrac{d^2}{1-d^{2(p+q-1)}}\right) + \tfrac{\alpha}{L\overline{K}B_0}\cdot\tfrac{d}{1-d^{p+q-1}}\\ &= \delta,\end{aligned}$$

therefore $y_{k+1} \in S(x_0, \delta)$.

As

$$\begin{aligned}\|z_{k+1}-x_0\|_X &\leq \|z_{k+1}-x_{k+1}\|_X + \|x_{k+1}-x_0\|_X\\ &\leq a\,\|f(x_{k+1})\|_Y + \|x_{k+1}-x_0\|_X,\end{aligned}$$

we obtain for $\|z_{k+1}-x_0\|_X$ the same delimitation as for $\|y_{k+1}-x_0\|_X$ and the expression of this delimitation is δ. Therefore we have that $z_{k+1} \in S(x_0, \delta)$.

b) We know that:

$$\begin{aligned}\|f(x_{k+1})\|_Y &= \|f(x_{k+1}) - \theta_Y\|_Y\\ &= \|f(x_{k+1}) - f(y_k) - \Gamma_{f;y_k,z_k}(x_{k+1}-y_k)\|_Y\\ &\leq \left\|\Gamma_{f;y_k,x_{k+1}} - \Gamma_{f;y_k,z_k}\right\| \cdot \|x_{k+1}-y_k\|_X\\ &\leq L\,\|x_{k+1}-y_k\|_X \cdot \|x_{k+1}-z_k\|_X.\end{aligned}$$

On account of the equalities (7) that are true for $n=k$ we have that:

$$\|f(x_{k+1})\|_Y \leq L\|\Gamma^{-1}_{f;y_k,z_k}\|^2 \cdot \|f(y_k)\|_Y \cdot \|f(z_k)\|_Y.$$

Using the hypothesis of the induction and the hypothesis of the verification of the inequalities (8) we obtain:

$$\|f(x_{k+1})\|_Y \leq LK^2B_k^2\,\|f(x_k)\|_Y^{p+q} \leq L\overline{K}^2B_k^2R_k^{p+q} = R_{k+1}.$$

c) From the existence of the mapping $\Gamma^{-1}_{f;y_k,z_k} \in (Y,X)^*$, of the fact that $f(y_k) \neq \theta_Y$, $f(z_k) \neq \theta_Y$ and using the equalities:

$$x_{k+1} = y_k - \Gamma^{-1}_{f;y_k,z_k} f(y_k) = z_k - \Gamma^{-1}_{f;y_k,z_k} f(z_k)$$

we deduce that $x_{k+1} \neq y_k$ and $z_{k+1} \neq z_k$.

So, the following mapping has a meaning:

$$U_k = \Gamma^{-1}_{f;y_k,z_k}\left(\Gamma_{f;y_k,z_k} - \Gamma_{f;x_{k+1},z_k}\right) \in (X,X)^*.$$

From here it is clear that:

$$\Gamma_{f;x_{k+1},z_k} = \Gamma_{f;y_k,z_k}(\mathbf{I}_X - U_k). \tag{24}$$

Considering the fact that $y_k, z_k \in S(x_0,\delta)$ and taking into account what we have proved at **a)** and as we have $x_{k+1} \in S(x_0,\delta)$, we deduce that:

$$\begin{aligned}\|U_k\| &\leq \|\Gamma^{-1}_{f;y_k,z_k}\| \cdot \left\|\Gamma_{f;y_k,z_k} - \Gamma_{f;x_{k+1},z_k}\right\|\\ &\leq B_kL\,\|x_{k+1}-y_k\|_X \leq\end{aligned}$$

$$\leq LB_k \|\Gamma^{-1}_{f;y_k,z_k}\| \cdot \|f(y_k)\|_Y$$
$$\leq LB_k^2 K \|f(x_k)\|_Y^p$$
$$= LKB_k^2 R_k^p$$
$$= u_k \leq \alpha < 1.$$

From here, using the well known theorem of Banach (on account of the fact that $(X, \|\cdot\|_X)$ is a Banach space), we deduce that there exists the mapping $(\mathbf{I}_X - U_k)^{-1} \in (X,X)^*$ and

$$\| (\mathbf{I}_X - U_k)^{-1} \| \leq \tfrac{1}{1-\|U_k\|} \leq \tfrac{1}{1-u_k}.$$

From the existence of the mappings $\Gamma^{-1}_{f;y_k,z_k} \in (Y,X)^*$ and $(\mathbf{I}_X - U_k)^{-1} \in (X,X)^*$ using the equality (24) we deduce the existence of the mapping $\Gamma^{-1}_{f;x_{k+1},z_k} \in (Y,X)^*$ and:

$$\Gamma^{-1}_{f;x_{k+1},z_k} = (\mathbf{I}_X - U_k)^{-1} \Gamma^{-1}_{f;y_k,z_k}. \tag{25}$$

From the equality (25) we have the inequality:

$$\|\Gamma^{-1}_{f;x_{k+1},z_k}\| \leq \| (\mathbf{I}_X - U_k)^{-1} \| \cdot \|\Gamma^{-1}_{f;y_k,z_k}\| \leq \tfrac{B_k}{1-u_k}$$

as well.

Also the mapping

$$S_k = \Gamma^{-1}_{f;y_k,z_k} \left(\Gamma_{f;y_k,z_k} - \Gamma_{f;y_k,x_{k+1}} \right) \in (X,X)^*$$

has a meaning. We can write that:

$$\Gamma_{f;y_k,x_{k+1}} = \Gamma_{f;y_k,z_k} (\mathbf{I}_X - S_k). \tag{26}$$

In the same way as in the case of the mapping U_k we have that:

$$\|S_k\| \leq LKB_k^2 R_k^q = s_k \leq \alpha < 1,$$

therefore there exists the mapping $(\mathbf{I}_X - S_k)^{-1} \in (X,X)^*$, so there exists the mapping $\Gamma^{-1}_{f;y_k,x_{k+1}} \in (Y,X)^*$ as well.

As $\Gamma^{-1}_{f;y_k,x_{k+1}} = (\mathbf{I}_X - S_k)^{-1} \Gamma^{-1}_{f;y_k,z_k}$ we have that:

$$\|\Gamma^{-1}_{f;y_k,x_{k+1}}\| \leq \| (\mathbf{I}_X - S_k)^{-1} \| \cdot \|\Gamma^{-1}_{f;y_k,z_k}\| \leq \tfrac{B_k}{1-s_k}.$$

As $y_{k+1} \neq z_k$ the following mapping has a meaning

$$V_k = \Gamma^{-1}_{f;x_{k+1},z_k} \left(\Gamma_{f;x_{k+1},z_k} - \Gamma_{f;y_{k+1},z_k} \right) \in (X,X)^*,$$

from where evidently

$$\Gamma_{f;y_{k+1},z_k} = \Gamma_{f;x_{k+1},z_k} (\mathbf{I}_X - V_K). \tag{27}$$

As $x_{k+1}, z_k, y_{k+1} \in S(x_0, \delta) \subseteq D$ we deduce that:

$$\begin{aligned}\|V_k\| &\leq \|\Gamma^{-1}_{f;x_{k+1},z_k}\| \cdot \left\|\Gamma_{f;x_{k+1},z_k} - \Gamma_{f;y_{k+1},z_k}\right\|_X \\ &\leq \|\Gamma^{-1}_{f;x_{k+1},z_k}\| L \|x_{k+1} - y_{k+1}\| \\ &\leq aL \cdot \tfrac{B_k}{1-u_k} \cdot \|f(x_{k+1})\|_Y \\ &= \tfrac{aLB_k}{1-u_k} \cdot L\overline{K}^2 B_k^2 R_k^{p+q} \\ &= \tfrac{aL^2\overline{K}^2 B_k^3 R_k^{p+q}}{1-u_k} \\ &= v_k < \tfrac{\alpha^2}{1-\alpha} < 1.\end{aligned}$$

From here we deduce that there exists the mapping $(\mathbf{I}_X - V_k)^{-1} \in (X, X)^*$ and $\|(\mathbf{I}_X - V_k)^{-1}\| \leq \frac{1}{1-v_k}$.

Taking into account of the existence of the mappings $\Gamma^{-1}_{f;x_{k+1},z_k} \in (Y, X)^*$ and $(\mathbf{I}_X - V_k)^{-1} \in (X, X)^*$ we deduce the existence of the mapping:

$$\Gamma^{-1}_{f;y_{k+1},z_k} = (\mathbf{I}_X - V_k)^{-1} \Gamma^{-1}_{f;x_{k+1},z_k} \in (Y, X)^*$$

and the relations:

$$\|\Gamma^{-1}_{f;y_{k+1},z_k}\| \leq \|(\mathbf{I}_X - V_k)^{-1}\| \cdot \|\Gamma^{-1}_{f;x_{k+1},z_k}\| \leq \tfrac{B_k}{(1-u_k)(1-v_k)}. \tag{28}$$

At the same time from the hypotheses we have $y_{k+1} \neq x_{k+1}$, so the mapping:

$$W_k = \Gamma^{-1}_{f;y_{k+1},z_k}\left(\Gamma_{f;y_{k+1},z_k} - \Gamma_{f;y_{k+1},x_{k+1}}\right) \in (X, X)^* \tag{29}$$

has a meaning.

Obviously we have:

$$\Gamma_{f;y_{k+1},x_{k+1}} = \Gamma_{f;y_{k+1},z_k}(\mathbf{I}_X - W_k). \tag{30}$$

As $z_k, y_{k+1}, x_{k+1} \in S(x_0, \delta) \subseteq D$ we deduce that

$$\begin{aligned}\|W_k\| &\leq \|\Gamma^{-1}_{f;y_{k+1},z_k}\| \cdot \left\|\Gamma_{f;y_{k+1},z_k} - \Gamma_{f;y_{k+1},x_{k+1}}\right\| \\ &\leq \|\Gamma^{-1}_{f;y_{k+1},z_k}\| \cdot L\|x_{k+1} - z_k\|_X \\ &\leq L \cdot \tfrac{B_k}{(1-u_k)(1-v_k)} \cdot \|\Gamma^{-1}_{f;y_{k+1},z_k}\| \cdot \|f(z_k)\|_Y^q \\ &= \tfrac{LB_k^2}{(1-u_k)(1-v_k)} \cdot \overline{K}\|f(x_k)\|_Y^q \\ &= L\overline{K} \cdot \tfrac{B_k^2 R_k^q}{(1-u_k)(1-v_k)} \\ &= w_k < \tfrac{\alpha}{1-\alpha-\alpha^2} < 1,\end{aligned}$$

therefore there exists the mapping $(\mathbf{I}_X - W_k)^{-1} \in (X, X)^*$ and we have the inequality $\|(\mathbf{I}_X - W_k)^{-1}\| \leq \frac{1}{1-\|W_k\|} \leq \frac{1}{1-w_k}$.

Considering the existence of the mapping $\Gamma^{-1}_{f;y_{k+1},z_k} \in (Y, X)^*$ from the previous evaluation we deduce on the basis of the equality (30) the existence of the mapping $\Gamma^{-1}_{f;y_{k+1},x_{k+1}} \in (Y,X)^*$ such that

$$\Gamma^{-1}_{f;y_{k+1},x_{k+1}} = (\mathbf{I}_X - W_k)^{-1}\,\Gamma^{-1}_{f;y_{k+1},z_k}$$

and

$$\begin{aligned}\|\Gamma^{-1}_{f;y_{k+1},x_{k+1}}\| &\leq \|\,(\mathbf{I}_X - W_k)^{-1}\,\| \cdot \|\Gamma^{-1}_{f;y_{k+1},z_k}\| \\ &\leq \tfrac{B_k}{(1-u_k)(1-v_k)(1-w_k)}.\end{aligned}$$

Evidently $y_{k+1} \neq z_{k+1}$ therefore the mapping:

(31) $$T_k = \Gamma^{-1}_{f;y_{k+1},x_{k+1}}\left(\Gamma_{f;y_{k+1},x_{k+1}} - \Gamma_{f;y_{k+1},z_{k+1}}\right) \in (X,X)^*$$

has a meaning.

From (31) we obviously have that:

(32) $$\Gamma_{f;y_{k+1},z_{k+1}} = \Gamma_{f;y_{k+1},x_{k+1}}\,(\mathbf{I}_X - T_k)\,.$$

As $y_{k+1}, x_{k+1}, z_{k+1} \in S\,(x_0,\delta) \subseteq D$ we deduce that

$$\begin{aligned}\|T_k\| &\leq \|\Gamma^{-1}_{f;y_{k+1},x_{k+1}}\| \cdot \left\|\Gamma_{f;y_{k+1},x_{k+1}} - \Gamma_{f;y_{k+1},z_{k+1}}\right\| \\ &\leq \|\Gamma^{-1}_{f;y_{k+1},x_{k+1}}\| \cdot L\,\|x_{k+1} - z_{k+1}\| \\ &\leq \tfrac{aLB_k\|f(x_{k+1})\|_Y}{(1-u_k)(1-v_k)(1-w_k)} \\ &= \tfrac{aLB_kR_{k+1}}{(1-u_k)(1-v_k)(1-w_k)} \\ &= \tfrac{aLB_k}{(1-u_k)(1-v_k)(1-w_k)} L\overline{K}^2 B_k^2 R_k^{p+q} \\ &= \tfrac{aL^2\overline{K}^2 B_k^3 R_k^{p+q}}{(1-u_k)(1-v_k)(1-w_k)} \\ &= t_k < \tfrac{\alpha^2}{1-2\alpha-\alpha^2} < 1.\end{aligned}$$

Therefore, based on the same theorem of Banach, there exists the mapping $(\mathbf{I}_X - T_k)^{-1} \in (X,X)^*$ and $\|\,(\mathbf{I}_X - T_k)^{-1}\,\| \leq \frac{1}{1-\|T_k\|} \leq \frac{1}{1-t_k}$.

Adding to the last information the fact that there exists the mapping $\Gamma^{-1}_{f;y_{k+1},x_{k+1}} \in (Y,X)^*$, we deduce on the basis of the equality (32) the existence of the mapping:

$$\Gamma^{-1}_{f;y_{k+1},z_{k+1}} = (\mathbf{I}_X - T_k)^{-1}\,\Gamma^{-1}_{f;y_{k+1},x_{k+1}} \in (Y,X)^*$$

together with the inequality:

$$\begin{aligned}\|\Gamma^{-1}_{f;y_{k+1},z_{k+1}}\| &\leq \|\,(\mathbf{I}_X - T_k)^{-1}\,\| \cdot \|\Gamma^{-1}_{f;y_{k+1},x_{k+1}}\| \\ &\leq \tfrac{B_k}{(1-u_k)(1-v_k)(1-w_k)(1-t_k)} = B_{k+1}.\end{aligned}$$

The existence of the mapping $\Gamma^{-1}_{f;y_{k+1},z_{k+1}} \in (Y,X)^*$ and the inequality $\|\Gamma^{-1}_{f;y_{k+1},z_{k+1}}\| \leq B_{k+1}$ express the fact that the proposition **c)** is true for $n = k+1$.

Therefore the propositions **a) - c)** are true for $n = k+1$.

On account of the principle of mathematical induction these relations are true for any number $n \in \mathbb{N}$.

We will now show that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Taking into account of the inequality (23) we deduce that for any $n, p \in \mathbb{N}$ we have the following inequality:

$$\begin{aligned}\|x_{n+p} - x_n\|_X &= \left\| \sum_{j=n}^{n+p-1} (x_{j+1} - x_j) \right\|_X \\ &\leq \sum_{j=n}^{n+p-1} \|x_{j+1} - x_j\|_X \\ &\leq \frac{a\alpha^2}{LB_0^2} \sum_{j=n}^{n+p-1} d^{2(p+q)^{j-1}} + \frac{\alpha}{L\overline{K}B_0} \sum_{j=n}^{n+p-1} d^{(p+q)^j}.\end{aligned}$$

But, as $p, q \geq 1$, we deduce that for any $s \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}2(p+q)^{n-1+s} - 2(p+q)^{n-1} &= 2(p+q)^{n-1}\left[(p+q)^s - 1\right] \\ &> 2s(p+q)^{n-1}(p+q-1),\end{aligned}$$

therefore as $d < 1$ we deduce that:

$$\begin{aligned}\sum_{j=n}^{n+p-1} d^{2(p+q)^{j-1}} &< d^{2(p+q)^{n-1}} \sum_{s=0}^{p-1} \left[d^{2(p+q)^{n-1}(p+q-1)}\right]^s \\ &< \frac{d^{2(p+q)^{n-1}}}{1-d^{2(p+q)^{n-1}(p+q-1)}}.\end{aligned}$$

Similarly

$$\sum_{j=n}^{n+p-1} d^{(p+q)^j} < \frac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}},$$

therefore

$$\|x_{n+p} - x_n\|_X \leq \frac{a\alpha^2}{LB_0^2} \cdot \frac{d^{2(p+q)^{n-1}}}{1-d^{2(p+q)^{n-1}(p+q-1)}} + \frac{\alpha}{L\overline{K}B_0} \cdot \frac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}}. \tag{33}$$

From the fact that $d < 1$ we deduce that:

$$\lim_{n\to\infty} \frac{d^{2(p+q)^{n-1}}}{1-d^{2(p+q)^{n-1}(p+q-1)}} = \lim_{n\to\infty} \frac{d^{(p+q)^n}}{1-d^{(p+q)^n(p+q-1)}} = 0,$$

therefore $\lim\limits_{n\to\infty} \operatorname*{unif}\limits_{p\in\mathbb{N}} \|x_{n+p} - x_n\|_X = 0$ and this equality expresses the fact that $(x_n)_{n\in\mathbb{N}} \subseteq D \subseteq X$ is a Cauchy sequence.

From the quality of the linear normed space $(X, \|\cdot\|_X)$ of being a Banach space we deduce that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a convergent sequence, therefore there exists an element $\overline{x} \in X$ such that $\overline{x} = \lim\limits_{n\to\infty} x_n$.

If in the inequality (33) we tend to the limit as $p \to \infty$, we obtain the inequality (20) form the conclusion of the theorem.

From the inequality (20) with $n = 0$ we deduce that:

$$\begin{aligned}\|\overline{x} - x_0\|_X &\le \|\overline{x} - x_1\|_X + \|x_1 - x_0\|_X \\ &\le \tfrac{a\alpha^2}{LB_0^2} \cdot \tfrac{d^2}{1-d^{2(p+q-1)}} + \tfrac{\alpha}{L\overline{K}B_0} \cdot \tfrac{d^{p+q}}{1-d^{(p+q)(p+q-1)}} + aR_0 + \tfrac{\alpha d}{L\overline{K}B_0} \\ &\le \delta,\end{aligned}$$

and this inequality expresses the fact that $\overline{x} \in S(x_0, \delta)$.

The existence of the mapping $\Gamma^{-1}_{f;y_n,z_n} \in (Y, X)^*$ allows for the expressions of the element x_{n+1} under the forms (18).

Obviously, for any $n \in \mathbb{N}$ we have the inequalities:

$$\begin{aligned}\|y_n - \overline{x}\|_X &\le \|y_n - x_n\|_X + \|x_n - \overline{x}\|_X \\ &\le a\,\|f(x_n)\|_Y + \|x_n - \overline{x}\|_X \\ &\le aR_n + \|x_n - \overline{x}\|_X \\ &\le \tfrac{a\alpha^2}{LB_0^2} \cdot d^{2(p+q)^{n-1}} + \|x_n - \overline{x}\|_X .\end{aligned} \tag{34}$$

We have a similar estimate for $\|z_n - \overline{x}\|_X$ as well. We obtain in this way the estimates (21). The inequality (21) expresses in fact that:

$$\lim_{n\to\infty} \|y_n - \overline{x}\|_X = \lim_{n\to\infty} \|z_n - \overline{x}\|_X = 0.$$

We still need to show that $\overline{x} \in D$ is the solution of the equation $f(x) = \theta_Y$.

We will first remark that from the fact that for any $n \in \mathbb{N}$ we have that $\|f(x_n)\|_Y \le R_n \le \frac{\alpha^2}{LB_0^2} \cdot d^{2(p+q)^{n-1}}$ and from $d < 1$ we deduce that $\lim\limits_{n\to\infty} \|f(x_n)\|_Y = 0$.

Let us consider a number $n \in \mathbb{N}$ arbitrarily. As $f(y_n) \ne \theta_Y$ we deduce that $y_n \ne \overline{x}$, therefore as $x_n \ne y_n$ we will have that:

$$\begin{aligned}0 &\le \|f(\overline{x})\|_Y \\ &\le \|f(y_n) - f(\overline{x})\|_Y + \|f(y_n)\|_Y \\ &\le \|\Gamma_{f;\overline{x},y_n}\| \cdot \|y_n - \overline{x}\|_X + K\,\|f(x_n)\|_Y^p .\end{aligned}$$

Here, we have:

$$\begin{aligned}\|\Gamma_{f;\overline{x},y_n}\| &\le \|\Gamma_{f;\overline{x},y_n} - \Gamma_{f;\overline{x},y_0}\| + \|\Gamma_{f;\overline{x},y_0}\| \\ &\le L\,\|y_n - y_0\|_X + \|\Gamma_{f;\overline{x},y_0}\| \\ &\le L\left(\|y_n - x_0\|_X + \|y_0 - x_0\|_X\right) + \|\Gamma_{f;\overline{x},y_0}\| \\ &\le L\left(\delta + \|y_0 - x_0\|_X\right) + \|\Gamma_{f;\overline{x},y_0}\| ,\end{aligned}$$

therefore

$$0 \le \|f(\overline{x})\|_Y \le \left[L\left(\delta + \|y_0 - x_0\|_X\right) + \|\Gamma_{f;\overline{x},y_0}\|\right] \cdot \|y_n - \overline{x}\|_X + K\,\|f(x_n)\|_Y^p . \tag{35}$$

As $\lim\limits_{n\to\infty} \|y_n - \overline{x}\|_X = 0 = \lim\limits_{n\to\infty} \|f(x_n)\|_Y$ and $p > 1$ we deduce that:

$$\lim_{n\to\infty} \left\{\left[L\left(\delta + \|y_0 - x_0\|_X\right) + \|\Gamma_{f;\overline{x},y_0}\|\right] \cdot \|y_n - \overline{x}\|_X + K\,\|f(x_n)\|_Y^p\right\} = 0,$$

from where on account of the relation (35) we deduce that $\|f(\overline{x})\|_Y = 0$, namely $f(\overline{x}) = \theta_Y$.

With that the theorem is proven. $\square$

3. REMARKS ON THE MARGIN OF THE MAIN RESULT

Remark 7. With the hypotheses of the Theorem 6 the sequence:

$$(\|\Gamma^{-1}_{f;y_n,z_n}\|)_{n\in\mathbb{N}^*}$$

is bounded, and for any $n \in \mathbb{N}^*$ the following inequality takes place:

$$\|\Gamma^{-1}_{f;y_n,z_n}\| \le B_0 \mathbf{e}^{G(d,\alpha)} \tag{36}$$

where:

$$\begin{aligned} G(d,\alpha) = {} & \tfrac{\alpha\left(2-3\alpha-\alpha^2\right)}{(1-\alpha)(1-2\alpha-\alpha^2)} \cdot \tfrac{d}{1-d^{p+q-1}} \\ & + \tfrac{\alpha^2\left(2-3\alpha-3\alpha^2\right)}{(1-\alpha-\alpha^2)(1-2\alpha-2\alpha^2)} \cdot \tfrac{d^2}{1-d^{2(p+q-1)}}. \end{aligned} \tag{37}$$

Also the sequence $(\|\Gamma_{f;y_n,z_n}\|)_{n\in\mathbb{N}^*}$ is bounded and for any $n \in \mathbb{N}^*$ we have the inequality:

$$\|\Gamma_{f;y_n,z_n}\| \le \|\Gamma_{f;y_0,z_0}\| + 2\alpha\left(K + \tfrac{a\alpha}{B_0^2}\right) \cdot \tfrac{d}{1-d^{p+q-1}}. \quad \square \tag{38}$$

Indeed, from the recurrence relation of the sequence $(B_n)_{n\in\mathbb{N}^*}$ we deduce that:

$$B_n \le \frac{B_0}{\prod\limits_{j=0}^{n-1}\left[(1-u_j)(1-v_j)(1-w_j)(1-t_j)\right]}.$$

Obviously

$$\begin{aligned} & \frac{1}{\prod\limits_{j=0}^{n-1}\left[(1-u_j)(1-v_j)(1-w_j)(1-t_j)\right]} \le \\ & \le \left[\tfrac{1}{4n}\sum_{j=0}^{n-1}\left(\tfrac{1}{1-u_j} + \tfrac{1}{1-v_j} + \tfrac{1}{1-w_j} + \tfrac{1}{1-t_j}\right)\right]^{4n} \\ & = \left[1 + \tfrac{1}{4n}\sum_{j=0}^{n-1}\left(\tfrac{u_j}{1-u_j} + \tfrac{v_j}{1-v_j} + \tfrac{w_j}{1-w_j} + \tfrac{t_j}{1-t_j}\right)\right]^{4n} \\ & \le \left[1 + \tfrac{1}{4n}\left(Q_1(\alpha)\sum_{j=0}^{n-1}u_j + Q_2(\alpha)\sum_{j=0}^{n-1}v_j + Q_3(\alpha)\sum_{j=0}^{n-1}w_j + Q_4(\alpha)\sum_{j=0}^{n-1}t_j\right)\right]^{4n}, \end{aligned}$$

where

$$Q_1(\alpha) = \tfrac{1}{1-\alpha},$$
$$Q_2(\alpha) = \frac{1}{1-\frac{\alpha^2}{1-\alpha}} = \tfrac{1-\alpha}{1-\alpha-\alpha^2},$$
$$Q_3(\alpha) = \frac{1}{1-\frac{\alpha}{1-\alpha-\alpha^2}} = \tfrac{1-\alpha-\alpha^2}{1-2\alpha-\alpha^2},$$
$$Q_4(\alpha) = \frac{1}{1-\frac{\alpha^2}{1-2\alpha-\alpha^2}} = \tfrac{1-2\alpha-\alpha^2}{1-2\alpha-2\alpha^2}.$$

It is obvious that

$$\sum_{j=0}^{n-1} u_j \le \alpha \sum_{j=0}^{n-1} d^{(p+q)^j} \le \tfrac{\alpha d}{1-d^{p+q-1}},$$
$$\sum_{j=0}^{n-1} v_j \le \tfrac{\alpha^2}{1-\alpha} \sum_{j=0}^{n-1} d^{2(p+q)^j} \le \tfrac{\alpha^2 d^2}{(1-\alpha)\left(1-d^{2(p+q-1)}\right)},$$
$$\sum_{j=0}^{n-1} w_j \le \tfrac{\alpha}{1-\alpha-\alpha^2} \sum_{j=0}^{n-1} d^{(p+q)^j} \le \tfrac{\alpha d}{(1-\alpha-\alpha^2)(1-d^{p+q-1})},$$
$$\sum_{j=0}^{n-1} t_j \le \tfrac{\alpha^2}{1-2\alpha-\alpha^2} \sum_{j=0}^{n-1} d^{2(p+q)^j} \le \tfrac{\alpha^2 d^2}{(1-2\alpha-\alpha^2)\left(1-d^{2(p+q-1)}\right)}.$$

Therefore

$$\begin{aligned} & Q_1(\alpha) \sum_{j=0}^{n-1} u_j + Q_2(\alpha) \sum_{j=0}^{n-1} v_j + Q_3(\alpha) \sum_{j=0}^{n-1} w_j + Q_4(\alpha) \sum_{j=0}^{n-1} t_j \le \\ & \le \tfrac{\alpha d}{(1-\alpha)(1-d^{p+q-1})} + \tfrac{\alpha^2 d^2}{(1-\alpha-\alpha^2)\left(1-d^{2(p+q-1)}\right)} \\ & \quad + \tfrac{\alpha d}{(1-2\alpha-\alpha^2)(1-d^{p+q-1})} + \tfrac{\alpha^2 d^2}{(1-2\alpha-2\alpha^2)\left(1-d^{2(p+q-1)}\right)} \\ & = G(d,\alpha). \end{aligned}$$

As

$$B_n \le B_0 \left[1 + \tfrac{1}{4n} G(d,\alpha)\right]^{4n} \le B_0 \mathbf{e}^{G(d,\alpha)},$$

the inequality (36) is proven.

In order to establish the inequality (38), we note that for any $n \in \mathbb{N}$ we have:

$$\Gamma_{f;y_n,z_n} = \Gamma_{f;y_0,z_0} + \sum_{j=0}^{n-1} \left(\Gamma_{f;y_{j+1},z_{j+1}} - \Gamma_{f;y_j,z_j}\right),$$

from where:

$$\|\Gamma_{f;y_n,z_n}\| \le \|\Gamma_{f;y_0,z_0}\| + \sum_{j=0}^{n-1} \left\|\Gamma_{f;y_{j+1},z_{j+1}} - \Gamma_{f;y_j,z_j}\right\|.$$

For any $j \in \mathbb{N}$ we have

$$\begin{aligned} \left\| \Gamma_{f;y_{j+1},z_{j+1}} - \Gamma_{f;y_j,z_j} \right\| &\le \left\| \Gamma_{f;y_{j+1},z_{j+1}} - \Gamma_{f;y_{j+1},z_j} \right\| + \left\| \Gamma_{f;y_{j+1},z_j} - \Gamma_{f;y_j,z_j} \right\| \\ &\le L \left(\|y_{j+1} - y_j\|_X + \|z_{j+1} - z_j\|_X \right). \end{aligned}$$

But

$$\begin{aligned} \|y_{j+1} - y_j\|_X &\le \|y_{j+1} - x_{j+1}\|_X + \|x_{j+1} - y_j\|_X \\ &\le a \|f(x_{j+1})\|_Y + \left\| \Gamma^{-1}_{f;y_j,z_j} \right\| \cdot \|f(x_j)\|_Y^p \\ &= aR_{j+1} + KB_j R_j^p \\ &\le \tfrac{a\alpha^2}{LB_0^2} \cdot d^{2(p+q)^j} + \tfrac{K\alpha}{B_0} \cdot d^{(p+q)^j} \\ &< \tfrac{\alpha\left(KB_0^2 + a\alpha\right)}{LB_0^2} d^{(p+q)^j}. \end{aligned}$$

For $\|z_{j+1} - z_j\|_X$ we obtain the same estimate.
Therefore, for any $n \in \mathbb{N}$ we obtain:

$$\|\Gamma_{f;y_n,z_n}\| \le \|\Gamma_{f;y_0,z_0}\| + \tfrac{2\alpha\left(KB_0^2 + a\alpha\right)}{LB_0^2} \sum_{j=0}^{n-1} d^{(p+q)^j},$$

from where the inequality (38) derives directly.

REMARK 8. For any $n \in \mathbb{N}$ we have the following inequality:

$$\|x_n - \overline{x}\|_X \le \tfrac{\alpha}{LB_0\left[1 - d^{2(p+q-1)}\right]} \cdot \left(\tfrac{\alpha}{B_0} + \tfrac{1}{K} \right) d^{2(p+q)^{n-1}}, \tag{39}$$

and this inequality show that the convergence order of the studied iterative method is $p + q$.

Indeed, from the fact that $p, q \ge 1$ and $d < 1$, we deduce that $(p+q)^n \ge$ $\ge 2(p+q)^{n-1}$, therefore $d^{(p+q)^n} \le d^{2(p+q)^{n-1}}$ and

$$d^{(p+q)^n(p+q-1)} \le d^{2(p+q)^{n-1}(p+q-1)}.$$

So

$$\tfrac{d^{(p+q)^n}}{1 - d^{(p+q)^n(p+q-1)}} \le \tfrac{d^{2(p+q)^{n-1}}}{1 - d^{2(p+q)^{n-1}(p+q-1)}},$$

therefore:

$$\|x_n - \overline{x}\|_X \le \tfrac{\alpha}{LB_0} \cdot \left(\tfrac{\alpha}{B_0} + \tfrac{1}{K} \right) \cdot \tfrac{d^{2(p+q)^{n-1}}}{1 - d^{2(p+q)^{n-1}(p+q-1)}}.$$

But $\overline{K} \ge K$, and as $n \ge 1$ we have that $d^{2(p+q)^{n-1}(p+q-1)} \le d^{2(p+q-1)}$ and in this way we obtain the inequality (39). □

4. NOTICEABLE SPECIAL CASES

REMARK 9. An important special case is that in which for any $n \in \mathbb{N}$ we have $x_n = z_n$. This case is admitted from the hypotheses, and it is necessary for the following conditions to hold:

$$f\left(x_n\right), f\left(y_n\right) \in Y \setminus \left\{\theta_Y\right\},\ x_n \neq y_n,\ x_n \neq y_{n+1}$$

and:

$$\Gamma_{f;x_n,y_n}\left(x_{n+1} - x_n\right) + f\left(x_n\right) = \theta_Y,$$

using for this aim the property that is expressed by the equality (6).

For any $n \in \mathbb{N}$ there exists the mapping $\Gamma^{-1}_{f;x_n,y_n} \in (Y, X)^*$, therefore the recurrence relation of the sequence $(x_n)_{n\in\mathbb{N}^*}$ will be:

$$x_{n+1} = x_n - \Gamma^{-1}_{f;x_n,y_n} f\left(x_n\right).$$

the inequality (39) in this case becomes:

$$\|x_n - \overline{x}\|_X \le \tfrac{\alpha}{LB_0(1-d^{2p})} \cdot \left(\tfrac{\alpha}{B_0} + \tfrac{1}{K}\right) \cdot d^{2(p+1)^{n-1}}, \tag{40}$$

therefore the convergence order of the method is $p + 1$. □

One verifies the hypotheses of the Theorem 6 in the case of $q = 1$, therefore we have the conclusions of this theorem in this case.

REMARK 10. We will now consider an even more special case. Let be a mapping $U \in (X, Y)^* \setminus \{\Theta\}$ (Θ being the null mapping) and we will choose the sequence $(y_n)_{n\in\mathbb{N}^*}$ by the relation $y_n = x_n - Uf\left(x_n\right)$ for any $n \in \mathbb{N}^*$. At the same time we will choose $z_n = x_n$. This case comes in the framework of the previous more general case.

Therefore the sequence $(x_n)_{n\in\mathbb{N}^*}$ will be chosen such that for any $n \in \mathbb{N}^*$ the following equality is verified:

$$\Gamma_{f;x_n,x_n-Uf(x_n)}\left(x_{n+1} - x_n\right) + f\left(x_n\right) = \theta_Y, \tag{41}$$

or if for any $n \in \mathbb{N}^*$ there exists the mapping $\Gamma^{-1}_{f;x_n,x_n-Uf(x_n)} \in (Y, X)^*$ we have the equality:

$$x_{n+1} = x_n - \Gamma^{-1}_{f;x_n,x_n-Uf(x_n)} f\left(x_n\right). \quad \square \tag{42}$$

For the convergence of the iterative method generated by the relation (41) we have the following corollary:

COROLLARY 11. *If the following hypotheses are fulfilled:*

i) $(X, \|\cdot\|_X)$ *is a Banach space;*

ii) *the mapping* $f : D \to Y$ *verifies the hypothesis* ***ii)*** *of the Theorem* 6 *with a constant* $L > 0$ *and there exists a number* $\lambda > 0$ *such that for any* $x, y \in D$ *the following inequality is true:*

$$\|\Gamma_{f;x,y}\| \le \lambda;$$

iii) *the sequence* $(x_n)_{n\in\mathbb{N}^*} \subseteq D$ *verifies for any* $n \in \mathbb{N}^*$ *the equality* (41) *with a mapping* $U \in (X,Y)^* \setminus \{\Theta\}$;

iv) *the mapping* $\Gamma_{f;x_0,x_0-Uf(x_0)} \in (X,Y)^*$ *is invertible and* $\Gamma^{-1}_{f;x_0,x_0-Uf(x_0)} \in$ $\in (Y,X)^*$;

v) *denoting:*

$$\begin{aligned} B_0 &= \max\left\{\|U\|, \|\Gamma^{-1}_{f;x_0,x_0-Uf(x_0)}\|\right\}, \\ R_0 &= \|f(x_0)\|_Y, \\ d &= \tfrac{LB_0^2R_0(1+\lambda\|U\|)^2}{\alpha}, \\ \delta &= 2\|U\|R_0 + \tfrac{\alpha}{LB_0}\cdot\left(\alpha+\tfrac{2}{1+\lambda\|U\|}\right)\cdot\tfrac{d}{1-d} \end{aligned}$$

with $\alpha = \frac{\sqrt{17}-3}{4}$, *the relations* $d<1$ *and* $S(x_0,\delta)\subseteq D$ *are true.*

Then we have the following conclusions:

j) *for any* $n\in\mathbb{N}^*$ *we have that* $x_n, x_n - Uf(x_n) \in S(x_0,\delta)$, *there exists* $\Gamma^{-1}_{f;x_n,x_n-Uf(x_n)} \in (Y,X)^*$ *and the equality* (42) *is true;*

jj) *the sequences* $(x_n)_{n\in\mathbb{N}^*}$, $(x_n - Uf(x_n))_{n\in\mathbb{N}^*}$ *are convergent to the same limit* $\overline{x}\in S(x_0,\delta)$ *for which* $f(\overline{x}) = \theta_Y$;

jjj) *for any* $n\in\mathbb{N}^*$ *the following inequalities are true:*

$$\|x_{n+1}-x_n\|_X \le \tfrac{\alpha}{LB_0}\left(\tfrac{a}{B_0}+\tfrac{\alpha}{1+\lambda\|U\|}\right)d^{2^n}, \tag{43}$$

$$\|x_n-\overline{x}\|_X \le \tfrac{\alpha}{LB_0}\left(\tfrac{a}{B_0}+\tfrac{\alpha}{1+\lambda\|U\|}\right)\tfrac{d^{2^n}}{1-d^{2^n}}. \tag{44}$$

Proof. For any $n\in\mathbb{N}^*$ we will note $y_n = x_n - Uf(x_n)$, $z_n = x_n$.

For the verification of the hypothesis **iii)** of the Theorem 6 we have the following relations:

$$\begin{aligned} \|y_n-x_n\|_X &\le \|U\|\cdot\|f(y_n)\|_Y, \\ \|f(y_n)\|_Y &\le \|f(x_n)\|_Y + \|f(y_n)-f(x_n)\|_Y \\ &\le \|f(x_n)\|_Y + \|\Gamma_{f;x_n,y_n}\|\cdot\|y_n-x_n\| \\ &\le (1+\lambda\|U\|)\|f(x_n)\|_Y, \end{aligned}$$

therefore the inequalities (8) will be verified with $a=\|U\|$, $K = 1+\lambda\|U\|$, if we take into account the conditions on the sequence $(z_n)_{n\in\mathbb{N}^*} = (x_n)_{n\in\mathbb{N}^*}$ as well.

Obviously, from $p=q=1$ we deduce that $\overline{K}=K$ and the others constants from the statement of Theorem 6, have the values from the present corollary.

By the application in this case of Theorem 6 we can deduce the conclusions j)-jjj) from the previous statement. □

An other special case is the case in which we obtain the sequences $(y_n)_{n\in\mathbb{N}^*}$, $(z_n)_{n\in\mathbb{N}^*}\subseteq D$ with the help of the iterative operators $Q_1, Q_2 : X\to X$ that verify $Q_i(D)\subseteq D$ for any $i\in\{1,2\}$. In this case we can choose $y_n = Q_1(x_n)$

and $z_n = Q_2(x_n)$, obtaining the sequence $(x_n)_{n\in\mathbb{N}^*}$ by the verification for any $n \in \mathbb{N}^*$ of the equality:

$$\Gamma_{f;Q_1(x_n),Q_2(x_n)}(x_{n+1} - Q_1(x_n)) + f(Q_1(x_n)) = \theta_Y. \tag{45}$$

In this equality the roles of the operators Q_1 and Q_2 can be inverted.

If for any $n \in \mathbb{N}^*$ there exists the mapping $\Gamma^{-1}_{f;Q_1(x_n),Q_2(x_n)} \in (Y, X)^*$ the relation (45) is equivalent to:

$$x_{n+1} = Q_1(x_n) - \Gamma^{-1}_{f;Q_1(x_n),Q_2(x_n)} f(Q_1(x_n)). \tag{46}$$

If for any $x \in D$ and $i \in \{1, 2\}$ we have the following relations:

$$\begin{cases} \|f(Q_i(x))\|_Y \le K \|f(x)\|_Y^{p_i}, \\ \|Q_i(x) - x\|_X \le a \|f(x)\|_Y, \end{cases} \tag{47}$$

for the sequences that are chosen in the manner showed the hypotheses of the Theorem 6 are fulfilled, therefore we have the conclusions of this theorem. One obtains the convergence order $p_1 + p_2$.

The convergence order can be increased if we replace the mapping Q_2 with the mapping $Q_2 \circ Q_1 : X \to X$.

In this case:

$$\begin{aligned} \|f(Q_2 \circ Q_1)(x)\|_Y &= \|f(Q_2(Q_1(x)))\|_Y \\ &\le K \|f(Q_1(x))\|_Y^{p_2} \\ &\le K^{1+p_2} \|f(x)\|_Y^{p_1+p_2}, \\ \|(Q_2 \circ Q_1)(x) - x\|_X &\le \|Q_2(Q_1(x)) - Q_1(x)\|_X + \|Q_1(x) - x\|_X \\ &\le a \|f(Q_1(x))\|_Y + a \|f(x)\|_Y \\ &\le a (K \|f(x)\|_Y^p + \|f(x)\|_Y). \end{aligned}$$

Usually we are interested in the set of points from around the solutions, for which $\|f(x)\|_Y \le 1$, therefore as $p \ge 1$ we have $\|f(x)\|_Y^p \le \|f(x)\|_Y$ and so:

$$\|(Q_2 \circ Q_1)(x) - x\|_X \le a(1 + K) \|f(x)\|_Y.$$

Therefore using the sequences $(y_n)_{n\in\mathbb{N}^*}$, $(z_n)_{n\in\mathbb{N}^*}$ that are defined by $y_n = Q_1(x_n)$, $z_n = (Q_2 \circ Q_1)(x_n)$ for the main sequence $(x_n)_{n\in\mathbb{N}^*}$ we obtain the convergence order $p_1 + p_1p_2$.

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