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THIRD ORDER CONVERGENCE THEOREM FOR A FAMILY OF NEWTON LIKE METHODS IN BANACH SPACE

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Abstract. In this paper, we propose a family of Newton-like methods in Banach space which includes some well known third-order methods as particular cases. We establish the Newton-Kantorovich type convergence theorem for a proposed family and get an error estimate.

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1. INTRODUCTION

Recently, many third order iterative methods free from second derivative have been derived and studied for nonlinear systems [1]-[10]. In particular, in [1] were suggested two Chebyshev-like (CL1, CL2) methods, while in [10] are considered two families of modifications of Chebyshev method (MOD1, MOD2). In [6] was also presented a new family of Chebyshev-type methods with a real parameter θ ($A2_\theta$). All the above mentioned methods are obtained using different approximations of second derivative in Chebyshev method.

In [9] it was proposed a family of third-order methods given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\left(1+\frac{1}{2a}\right)f'(x_n) - \frac{1}{2a}f'\left(x_n + a\frac{f(x_n)}{f'(x_n)}\right)}, \qquad a \in \mathbb{R}\setminus\{0\} \tag{1}$$

for solving nonlinear scalar equations $f(x) = 0$. In this study, we consider a generalization of methods (1) in Banach space, which is used to solve the nonlinear operator equation

$$F(x) = 0. \tag{2}$$

Suppose that F is defined on an open convex domain Ω of a Banach space X with values in a Banach space Y, $F'(x)$ is a Frechet derivative in Ω, and

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$F'(x)^{-1}$ exists. The generalization of methods (1) is

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$
$$z_n = (1+a)x_n - ay_n, \qquad a \neq 0$$
$$x_{n+1} = x_n - \left[\left(1+\tfrac{1}{2a}\right)F'(x_n) - \tfrac{1}{2a}F'(z_n)\right]^{-1}F(x_n). \tag{3}$$

Thus we have a family of methods (3) for solving nonlinear equation (2). We consider some particular cases of (3). Let $a = -1$. Then (3) leads to

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \tfrac{1}{2}\left[F'(x_n) + F'(y_n)\right]^{-1}F(x_n), \end{aligned} \tag{4}$$

which was proposed by Q.Wu and Y.Zhao in [7]. They established third-order convergence of this method by using majorizing function and obtained the error estimate. It should be mentioned that the iteration (4) for scalar equation was given also in [5]. Let $a = -\frac{1}{2}$. Then (3) leads to

$$\begin{aligned} y_n &= x_n - \tfrac{1}{2}F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - F'(y_n)^{-1}F(x_n). \end{aligned}$$

This is a generalization of the third order method proposed by the Frontini and Sormani [2] for scalar case. Thus, the proposed iteration (3) can be considered as a generalization of well known iterations.

We prove Newton-Kantorovich type convergence theorem for the family of methods (3) to show that it has third order convergence by using recurrent relations [3]-[4] and get the error bounds. Finally, some examples are provided to show the application of the proposed method.

2. PRELIMINARIES

Let us assume that $F'(x_0)^{-1} \in L(Y,X)$ exists for some $x_0 \in \Omega$, where $L(Y,X)$ is a set of bounded linear operators from Y into X. Moreover, we suppose that (see [4])

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \beta, \tag{c$_1$}$$
$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta, \tag{c$_2$}$$
$$\|F''(x)\| \leq M, \qquad x \in \Omega, \tag{c$_3$}$$
$$\|F''(x) - F''(y)\| \leq K\|x-y\|, \quad x,y \in \Omega, \quad K > 0. \tag{c$_4$}$$

Let F be a nonlinear twice Frechet differentiable operator in an open convex domain Ω. We denote $\Gamma_n = F'(x_n)^{-1}$,

$$\begin{aligned} a_0 &= M\beta\eta, \\ f(x) &= \tfrac{2-x}{2-3x}, \qquad 0 < x < \tfrac{2}{3}, \\ g(x) &= \tfrac{x^2}{(2-x)^2}d, \end{aligned} \tag{5}$$

where

$$d = \tfrac{3}{7} + \tfrac{\omega}{4}(1+|a|), \qquad (6)$$

$$\omega = \tfrac{K}{M^2 m},$$

$$m = \min_n \|\Gamma_n\| > 0$$

and define the sequence

$$a_{n+1} = f(a_n)^2 g(a_n) a_n.$$

We need following technical lemmas, whose proofs are trivial [4].

LEMMA 1. *Let f and g be two real functions given in* (5). *Then*
(i) $f(x)$ and $g(x)$ are increasing and $f(x) > 1$ for $x \in (0, \frac{2}{3})$
(ii) $f(\gamma x) < f(x)$ and $g(\gamma x) < \gamma^2 g(x)$ for $\gamma \in (0,1)$.

LEMMA 2. *Let $f^2(a_0)g(a_0) < 1$. Then the sequence $\{a_n\}$ is decreasing.*

LEMMA 3. *If $0 < a_0 < \frac{2}{3+\sqrt{d}}$, then $f^2(a_0)g(a_0) < 1$.*

LEMMA 4. *Let $0 < a_0 < \frac{2}{3+\sqrt{d}}$ and define $\gamma = a_1/a_0$. Then*
(i) $\gamma = f^2(a_0)g(a_0) \in (0,1)$;
(ii_n) $a_n \le \gamma^{3^n-1} a_{n-1} \le \gamma^{\frac{3^n-1}{2}} a_0$;
(iii_n) $f(a_n)g(a_n) \le \frac{\gamma^{3^n}}{f(a_0)} = \Delta\gamma^{3^n}$, $\qquad \Delta = \frac{1}{f(a_0)} < 1$.

3. CONVERGENCE STUDY

According to (3), we have

$$x_1 - x_0 = -A_0^{-1} F(x_0), \qquad (7)$$

where

$$A_0 = (1 + \tfrac{1}{2a})F'(x_0) - \tfrac{1}{2a}F'(z_0). \qquad (8)$$

Using the following formula

$$F'(z_0) = F'(x_0) + \int_{x_0}^{z_0} F''(x)\mathrm{d}x$$

in (8), we obtain

$$A_0 = F'(x_0)(I - P_0),$$

where

$$P_0 = \tfrac{1}{2a}\Gamma_0 \int_{x_0}^{z_0} F''(x)\mathrm{d}x.$$

If we notice that $M||\Gamma_0||||\Gamma_0 F(x_0)|| \le a_0 < \frac{2}{3}$, then follows

$$||P_0|| \le \tfrac{1}{2|a|}||\Gamma_0||M|a|||\Gamma_0 F(x_0)|| \le \tfrac{a_0}{2} < \tfrac{1}{3},$$

which shows the existence of A_0^{-1}

$$A_0^{-1} = (I - P_0)^{-1}\Gamma_0,$$

where $P_0 = \frac{1}{2}\Gamma_0 F''(\xi_0)\Gamma_0 F(x_0)$. So, from (7) we get

$$\|x_1 - x_0\| \le \frac{1}{1-\frac{a_0}{2}}\|\Gamma_0 F(x_0)\| \le \frac{\eta}{1-\frac{a_0}{2}} < \frac{\eta}{(1-\frac{a_0}{2})(1-\gamma\Delta)} = R\eta,$$

where $R = \frac{1}{(1-\frac{a_0}{2})(1-\gamma\Delta)}$. This means that $y_0, x_1 \in B(x_0, R\eta) = \{x \in X : \|x - x_0\| < R\eta\}$.

In these conditions, we prove the following statements for $n \ge 1$:

(I_n) $$\|\Gamma_n\| \le f(a_{n-1})\|\Gamma_{n-1}\|,$$

(II_n) $$\|\Gamma_n F(x_n)\| \le f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|,$$

(III_n) $$M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \le a_n,$$

(IV_n) $$\|x_{n+1} - x_n\| \le \frac{1}{1-a_n/2}\|\Gamma_n F(x_n)\|,$$

(V_n) $$y_n, x_{n+1} \in B(x_0, R\eta).$$

Assuming $\frac{a_0}{1-a_0/2} < 1$ which is valid for $a_0 < 2/3$ and $x_1 \in \Omega$ we have

$$\|I - \Gamma_0 F'(x_1)\| \le \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \le \|\Gamma_0\|M\|x_0 - x_1\| \le \frac{a_0}{1-\frac{a_0}{2}} < 1.$$

Then, by the Banach lemma, Γ_1 is defined and satisfies

$$\|\Gamma_1\| \le \frac{\|\Gamma_0\|}{1-\|\Gamma_0\|\|F'(x_0)-F'(x_1)\|} \le \frac{a_0}{1-\frac{a_0}{1-\frac{a_0}{2}}}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|.$$

Taking into account (3) and the Taylor formula of $x_n, y_n \in \Omega$, we have
(9)

$$F(x_{n+1}) = F(y_n) + F'(y_n)(x_{n+1} - y_n) + \int_{y_n}^{x_{n+1}} F''(x)(x_{n+1} - x)\mathrm{d}x, \; n = 0, 1, \ldots$$

Also

$$\begin{aligned} F(y_n) &= F(x_n) + F'(x_n)(y_n - x_n) + \int_{x_n}^{y_n} F''(x)(y_n - x)\mathrm{d}x \\ &= \int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)^2(1-t)\mathrm{d}t \end{aligned}$$

$$F'(y_n)(x_{n+1} - y_n) = \frac{1}{2}\int_0^1 F''(x_n - at(y_n - x_n))(x_n - y_n)(x_{n+1} - x_n)\mathrm{d}t.$$

Substituting the last two expressions into (9), we get

$$\begin{aligned} F(x_{n+1}) \;\; = \;\; & \int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)^2(1-t)\mathrm{d}t + \\ & +\frac{1}{2}\int_0^1 F''(x_n - at(y_n - x_n))(x_n - y_n)(x_{n+1} - y_n)\mathrm{d}t \\ & +\int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)(x_{n+1} - y_n)\mathrm{d}t \\ & +\int_0^1 F''(y_n + t(x_{n+1} - y_n))(x_{n+1} - y_n)^2(1-t)\mathrm{d}t. \end{aligned}$$

From (3) we also obtain

$$x_{n+1}-y_n=\tfrac{\Gamma_n}{2a}\int_{x_n}^{z_n}F''(x)(x_{n+1}-x_n)\mathrm{d}x$$

and

$$x_{n+1}-x_n=(I-P_n)^{-1}(y_n-x_n)=y_n-x_n+P_n(I-P_n)(y_n-x_n),$$

where

$$P_n=\tfrac{\Gamma_n}{2a}\int_{x_n}^{z_n}F''(x)\mathrm{d}x.$$

Taking into account

$$\begin{aligned}&\int_0^1 F''(x_n-at(y_n-x_n))(x_{n+1}-y_n)(x_{n+1}-x_n)\mathrm{d}t=\\&=-\int_0^1 F''(x_n-at(y_n-x_n))(y_n-x_n)^2\mathrm{d}t\\&\quad-\int_0^1 F''(x_n-at(y_n-x_n))(y_n-x_n)P_n(I-P_n)^{-1}(y_n-x_n)\mathrm{d}t\end{aligned}$$

we obtain

$$\begin{aligned}F(x_{n+1})\ =\ &\int_0^1[F''(x_n+t(y_n-x_n))-F''(x_n)](y_n-x_n)^2(1-t)\mathrm{d}t\\&+\tfrac12\int_0^1[F''(x_n)-F''(x_n-at(y_n-x_n))](y_n-x_n)^2\mathrm{d}t\\&-\tfrac12\int_0^1 F''(x_n-at(y_n-x_n))(y_n-x_n)P_n(I-P_n)^{-1}(y_n-x_n)\mathrm{d}t\\&+\int_0^1 F''(x_n+t(y_n-x_n))(y_n-x_n)(x_{n+1}-y_n)\mathrm{d}t\\&+\int_0^1 F''(y_n+t(x_{n+1}-y_n))(x_{n+1}-y_n)^2(1-t)\mathrm{d}t.\end{aligned}\tag{10}$$

From (10) for $n=0$, we obtain $||\Gamma_0 F(x_n)||$; therefore

$$||\Gamma_1F(x_1)||\le||\Gamma_1F'(x_0)||||\Gamma_0F(x_1)||\le f(a_0)g(a_0)||\Gamma_0F(x_0)||.$$

So, (II_1) is true. To prove (III_1) and (IV_1), notice that

$$M\|\Gamma_1\|\|\Gamma_1F(x_1)\|\le Mf^2(a_0)g(a_0)\eta\beta=f^2(a_0)g(a_0)a_0=a_1$$

and

$$\|x_2-x_1\|\le\|A_1^{-1}F(x_1)\|$$

where

$$A_1=F'(x_1)+\tfrac{1}{2a}(F'(x_1)-F'(z_1))=F'(x_1)[I-P_1].$$

Since

$$\|P_1\|=\tfrac12\|\Gamma_1F''(\eta_1)\Gamma_1F(x_1)\|\le\tfrac{a_1}{2}<1,$$

there exists $A_1^{-1}=(I-P_1)^{-1}\Gamma_1$, thereby we get

$$\|x_2-x_1\|\le\tfrac{1}{1-a_1/2}\|\Gamma_1F(x_1)\|\le\tfrac{f(a_0)g(a_0)}{1-a_0/2}\eta=\tfrac{\Delta\gamma}{1-a_0/2}\eta.$$

Consequently, we obtain

$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le \tfrac{\Delta\gamma}{1-a_0/2}\eta + \tfrac{\eta}{1-a_0/2} = \tfrac{1+\Delta\gamma}{1-a_0/2}\eta < R\eta.$$

Analogously, we get

$$\|y_1 - x_0\| \le \|y_1 - x_1\| + \|x_1 - x_0\| \le f(a_0)g(a_0)\eta + \tfrac{1}{1-a_0/2}\eta < R\eta$$

i.e. (IV_1) and (V_1) are proved. Now, following an inductive procedure and assuming

$$y_n, x_{n+1} \in \Omega \quad \text{and} \quad \tfrac{a_n}{1-a_n/2} < 1 \quad \forall n \in N, \tag{11}$$

the items $(I_n) - (V_n)$ can be proved. Notice that $\Gamma_n > 0$ for all $n = 0, 1, \ldots$. Indeed if $\Gamma_k = 0$ for some k, then due to statement (I_n), we have $\|\Gamma_n\| = 0$ for all $n \ge k$. As a consequence, the iteration (3) terminated after k-th step, i.e. the convergence of iteration does not hold. To establish the convergence of $\{x_n\}$ we only have to prove that it is a Cauchy sequence and that the above assumptions (11) are true. Note that

$$\begin{aligned}\tfrac{1}{1-a_n/2}\|\Gamma_n F(x_n)\| &\le \tfrac{1}{1-a_0/2} f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\| \\ &\le \ldots \le \tfrac{1}{1-a_0/2}\|\Gamma_0 F(x_0)\| \prod_{k=0}^{n-1} f(a_k)g(a_k).\end{aligned}$$

As a consequence of Lemma 4, it follows that

$$\prod_{k=0}^{n-1} f(a_k)g(a_k) \le \prod_{k=0}^{n-1} \Delta\gamma^{3^k} = \Delta^n \gamma^{\frac{3^n-1}{2}}.$$

So, from $\Delta < 1$ and $\gamma < 1$, we deduce that $\prod_{k=0}^{n-1} f(a_k)g(a_k)$ converges to zero by letting $n \to \infty$. We are now ready to state the main result on convergence for (3).

Theorem 5. *Let us assume that* $\Gamma_0 = F'(x_0)^{-1} \in L(Y, X)$ *exists at some* $x_0 \in \Omega$ *and* $(c_1) - (c_4)$ *are satisfied. Suppose that*

$$0 < a_0 < \tfrac{2}{3+\sqrt{d}} \quad \text{with } d \text{ given by } (6) \tag{12}$$

Then, if $\overline{B(x_0, R\eta)} = \{x \in X : \|x - x_0\| \le R\eta\} \subseteq \Omega$, *the sequence* $\{x_n\}$ *defined in* (3) *and starting at* x_0 *has, at least, R-order three and converges to a solution* x^* *of the equation* (2). *In that case, the solution* x^* *and the iterates* y_n, x_n *belong to* $\overline{B(x_0, R\eta)}$ *and* x^* *is the only solution of* (2) *in* $B(x_0, 2/M\beta - R\eta) \cap \Omega$. *Furthermore, we have the following error estimates:*

$$\|x_n - x^*\| \le \frac{1}{1-\frac{a_0}{2}\gamma^{\frac{3^n-1}{2}}} \gamma^{\frac{3^n-1}{2}} \frac{\Delta^n}{1-\Delta\gamma^{3^n}}\eta. \tag{13}$$

Proof. Let us now prove (12). From $a_0 \in (0; \frac{2}{3+\sqrt{d}})$ follows

$$\frac{a_n}{1-\frac{a_n}{2}} < \frac{a_0}{1-\frac{a_0}{2}} < 1.$$

In addition, as $y_n, x_n \in B(x_0, R\eta)$ for all $n \in N$, then $y_n, x_n \in \Omega$, $\forall n \in N$. Hence (12) follows. Now we prove that $\{x_n\}$ is a Cauchy sequence. To do this, we consider $n, m \geq 1$:

$$\begin{aligned}(14)\quad &\|x_{n+m} - x_n\| \leq \\ &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \ldots + \|x_{n+1} - x_n\| \\ &\leq \tfrac{1}{1-\frac{a_n}{2}}\eta \left(\prod_{k=0}^{n+m-2} f(a_k)g(a_k) + \prod_{k=0}^{n+m-3} f(a_k)g(a_k) + \ldots + \prod_{k=0}^{n-1} f(a_k)g(a_k) \right) \\ &\leq \tfrac{\eta}{1-\frac{a_n}{2}} \left(\Delta^{n+m-1}\gamma^{\frac{3^{n+m-1}-1}{2}} + \Delta^{n+m-2}\gamma^{\frac{3^{n+m-2}-1}{2}} + \ldots + \Delta^n\gamma^{\frac{3^n-1}{2}} \right) \\ &< \frac{\eta}{1-\frac{a_0}{2}\gamma^{\frac{3^n-1}{2}}}\gamma^{\frac{3^n-1}{2}}\Delta^n \tfrac{1-\gamma^{3^n m}\Delta^m}{1-\gamma^{3^n}\Delta},\end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence. By letting $m \to \infty$ in (14), we obtain (13). To prove that $F(x^*) = 0$, notice that $\|\Gamma_n F(x_n)\| \to 0$ by letting $n \to \infty$. As $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is a bounded sequence, we deduce $\|F(x_n)\| \to 0$, this means $F(x^*) = 0$ by the continuity of F.

Now to show the uniqueness, suppose that $y^* \in B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$ is another solution of (2). Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t(y^* - x^*).$$

Using the estimate

$$\begin{aligned}&\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\mathrm{d}t \leq \\ &\leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|\mathrm{d}t \\ &\leq M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)\mathrm{d}t \\ &< \tfrac{M\beta}{2}(R\eta + \tfrac{2}{M\beta} - R\eta) = 1,\end{aligned}$$

we have that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\mathrm{d}t$ has an inverse and consequently, $y^* = x^*$. □

It should be mentioned that in [7] the convergence of iteration (4) was proved under conditions $(c_1) - (c_4)$ and

$$a_0 \leq \tfrac{1}{2(1+\frac{5K}{3M^2\beta})},$$

whereas convergence of iteration (3) holds under condition (12).In [10] was found the convergence domain

$$0 < a_0 < \tfrac{1}{2d}, \tag{15}$$

where

$$d = \begin{cases} 1+2w, & \text{for Chebyshev method (CM)} \\ 1+5w, & \text{for the first modification of CM} \\ 1+4w, & \text{for the second modification of CM.} \end{cases}$$

The comparison of (12) and (15) shows that the convergence domain of (3) is larger than that of CM and its modifications, when $|a| < 7$.

4. NUMERICAL RESULTS

Now we present some numerical test results for the various third order, free from second derivative methods. Tests were done with a double arithmetic precision and the numbers of iterations such that $\|x_n - x_{n-1}\| \le 1.0e-15$ are shown below. Compared were

MOD1 [10]:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ z_n &= (1-\theta)x_n + \theta y_n, \qquad \theta \in (0,1] \\ x_{n+1} &= y_n - \tfrac{1}{2\theta}\Gamma_n(F'(z_n) - F'(x_n))(y_n - x_n) \end{aligned}$$

MOD2 [10]:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ z_n &= x_n - \Gamma_n F(x_n) \\ x_{n+1} &= y_n - \Gamma_n \left((1+\tfrac{b}{2})F(y_n) + F(x_n) - \tfrac{b}{2}F(z_n)\right) \qquad b \in [-2, 0] \end{aligned}$$

CL1 [1]:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ x_{n+1} &= y_n - \tfrac{1}{2}\Gamma_n(F'(y_n) - F'(x_n))(y_n - x_n) \end{aligned}$$

CL2 [1]:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ x_{n+1} &= y_n - \Gamma_n F(y_n) \end{aligned}$$

$A2_\theta$ [6]:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n) \\ x_n^p &= x_n - \theta\Gamma_n F(x_n) \\ y_n^p &= -\tfrac{1}{2}\Gamma_n(F'(x_n^p) - F'(x_n))(x_n^p - x_n) \\ x_n^c &= x_n^p + y_n^p \\ x_{n+1} &= y_n - \tfrac{1}{2}\Gamma_n(F'(x_n^c) - F'(x_n))(x_n^c - x_n) \end{aligned}$$

and the proposed iteration (3).

As a test we take the following systems of equations:

$$I. \quad \begin{aligned} &x_1^2 - x_2 + 1 = 0 \\ &x_1 + \cos(\tfrac{\pi}{2}x_2) = 0 \qquad x^0 = (0; 0.1) \end{aligned}$$

$$II. \quad \begin{aligned} &x_1x_3 + x_2x_4 + x_3x_5 + x_4x_6 = 0 \\ &x_1x_5 + x_2x_6 = 0 \\ &x_1 + x_3 + x_5 = 1 \\ &-x_1 + x_2 - x_3 + x_4 - x_5 + x_6 = 0 \\ &-3x_1 - 2x_2 - x_3 + x_5 + 2x_6 = 0 \\ &3x_1 - 2x_2 + x_3 - x_5 + 2x_6 = 0 \qquad x^0 = (0; 0; 0; 1; 1; 0) \end{aligned}$$

$$III. \quad \begin{aligned} &x_1^2 + x_2^2 = 1 \\ &x_1^2 - x_2^2 = -0.5 \qquad x^0 = (0.3; 0.7) \end{aligned}$$

$$IV. \quad \begin{aligned} &x_1^2 - x_1 - x_2^2 = 1 \\ &sin(x_1) - x_2 = 0 \qquad x^0 = (0.1; 0) \end{aligned}$$

$$V. \quad \begin{aligned} &x_1^2 + x_2^2 = 4 \\ &e^{x_1} + x_2 = 1 \qquad x^0 = (0.5; -1) \end{aligned}$$

	$MOD1$	$MOD2$	$CL1$	$CL2$	$A2_\theta$		(3)			
Ex.	$\theta=0.5$	$b=-1$			$\theta=-1$	$\theta=1$	$a=-1$	$a=-0.5$	$a=0.5$	$a=1$
I	7	7	8	7	6	-	6	6	6	5
II	6	6	6	6	5	18	6	6	6	6
III	6	6	6	6	5	7	5	5	5	5
IV	6	6	6	6	6	8	6	6	7	8
V	7	6	7	7	6	15	6	6	6	6

Table 1. Numerical results.

5. CONCLUSION

In this work we proposed a family of Newton type methods which is free from second derivative and includes some known third order methods as particular case. Also, we proved Newton-Kantorovich type convergence theorem using recurrent relations to show that it has a R-order three convergence and obtained an error estimate. The proposed method was compared to previously known third order methods to show that it has an equivalent performance.

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