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CERTAIN PROPERTIES FOR A CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract. In this particular paper, we investigate coefficient inequalities, closure theorems, convolution properties for the functions belonging to the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Further, integral transforms of functions in the same class are also discussed.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k; \qquad z \in (\mathbb{U} = \{z \in C : |z| < 1\}) \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of *univalent functions*, and $\mathcal{S}(\alpha)$, $\mathcal{C}(\alpha)$ $(0 < \alpha \leq 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are starlike of order α and convex of order α in $\mathbb{U}$, respectively.

For two analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ in the open unit disc $\mathbb{U} = \{z \in C : |z| < 1\}$, the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{2}$$

For complex parameters $\alpha_1, ...\alpha_r$ and $\beta_1, ...\beta_s$ $(\beta_j \neq 0, -1, -2, ...; j = 1...s)$, Dziok and Srivastava [1] defined the generalized hypergeometric function

$$_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z)$$

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by

$$(3) \qquad {}_rF_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \tfrac{(\alpha_1)_k ... (\alpha_r)_k}{(\beta_1)_k, ..., (\beta_s)_k} \tfrac{z^k}{k!};$$

$$(4) \qquad (r \leq s+1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup 0; z \in \mathcal{U}),$$

where $(x)_k$ is the Pochhammer symbol defined, in terms of Gamma function Γ, by

$$(x)_k = \tfrac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & \text{if } k = 0, \\ x(x+1)...(x+k-1), & \text{if } k \in \mathbb{N}. \end{cases}$$

Dziok and Srivastava [1] defined also the linear operator

$$(5) \qquad H(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s) f(z) = z + \sum_{k=2}^{\infty} \Gamma_k a_k z^k$$

where

$$(6) \qquad \Gamma_k = \frac{(\alpha_1)_{k-1} ... (\alpha_r)_{k-1}}{(\beta_1)_{k-1}, ..., (\beta_s)_{k-1} (k-1)!}.$$

Al-Abbadi and Darus [2] defined the analytic function

$$(7) \qquad \Phi^m_{\lambda_1, \lambda_2} = z + \sum_{k=2}^{\infty} \tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} z^k,$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

Using the Hadamard product (2), we can derive the generalized derivative operator $\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}$ as follows

$$(8) \qquad \mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m} \Gamma_k a_k z^k$$

where Γ_k is as given in (6).

Remark 1. When $(\lambda_1 = \lambda_2 = 0)$, $(\lambda_1 = m = 0)$ or $(\lambda_2 = 0$ and $m = 1)$ we get Dziok-Srivastava operator [1].

Also there are three cases to get the Hohlov operator [3], by giving $(\lambda_1 = \lambda_2 = 0, \alpha_i = 0, \beta_j = 0)$, $(\lambda_1 = m = 0, \alpha_i = 0, \beta_j = 0)$ or $(\lambda_2 = 0, m = 1, \alpha_i = 0, \beta_j = 0)$ where $(i = 1...r$ and $j = 1...s)$.

Putting $(\lambda_1 = \lambda_2 = 0, \alpha_2 = 1, \alpha_3 = ... = \alpha_r = 0, \beta_2 = ... = \beta_s = 0)$, $(\lambda_1 = m = 0, \alpha_2 = 1, \alpha_3 = ... = \alpha_r = 0, \beta_2 = ... = \beta_s = 0)$ or $(\lambda_2 = 0, m = 1, \alpha_2 = 1, \alpha_3 = ... = \alpha_r = 0, \beta_2 = ... = \beta_s = 0)$, we obtain the Carlson-Shaffer operator [4].

There are six cases to get the Ruscheweyh operator [5] as follows: $(\lambda_1 = \lambda_2 = 0, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$, $(\lambda_1 = m = 0, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$, $(\lambda_2 = 0, m = 1, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$,$(\lambda_1 = \lambda_2 = 0, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_2 =$

$... = \beta_s = 0)$, $(\lambda_1 = m = 0, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_2 = ... = \beta_s = 0)$ or $(\lambda_2 = 0, m = 1, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_2 = ... = \beta_s = 0)$.

If$(\lambda_2 = 0, m = 2, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$, we get the generalized Ruscheweyh derivative operator as well [6] .

Moreover, if we put $(\alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$ or $(\alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$, we can get operator given by Al-Abbadi and Darus [2].

After that, if$(\lambda_2 = 0, m = m+1, \alpha_2 = \alpha_3 = ... = \alpha_r = 0, \beta_1 = \beta_2 = ... = \beta_s = 0)$, we get the generalized derivative operator by Al-Shaqsi and Darus [7]. □

Definition 2. *Let $f \in \mathcal{A}$. Then $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ if and only if*

$$\Re\left\{\frac{z[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2} f(z)}\right\} > \eta, \quad 0 \leq \eta < 1, z \in \mathcal{U}. \tag{9}$$

In this present paper, we obtain the coefficient inequalities, closure theorems, convolution properties for the functions belonging to the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Finally, the preserving integral operators of the form

$$G_c(z) = c\int_0^1 u^{c-2} f(uz)du; \qquad (c > 0) \tag{10}$$

for the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ is considered. We employ techniques similar to those used earlier by [8].

2. COEFFICIENT ESTIMATE FOR THE CLASS $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$

Theorem 3. *Let $f(z) \in \mathcal{A}$. If*

$$\sum_{k=2}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \leq 1-\eta, \qquad 0 \leq \eta \leq 1 \tag{11}$$

then $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. The result (11) *is sharp.*

Proof. Suppose that (11) holds. Since

$$\begin{aligned} 1-\eta \quad &\geq \quad \sum_{k=2}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \\ &\geq \quad \sum_{k=2}^{\infty}\eta\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| - \sum_{k=2}^{\infty}k\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k| \end{aligned}$$

we deduce that

$$\frac{1+\sum\limits_{k=2}^{\infty}k\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k|}{1+\sum\limits_{k=2}^{\infty}\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_k|} > \eta.$$

Thus

$$\Re\left\{\frac{z\left[\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)\right]'}{\mathcal{K}^{m,r,s}_{\lambda_1,\lambda_2}f(z)}\right\} > \eta, \qquad 0 \le \eta < 1, z \in \mathcal{U}. \qquad \square$$

We note that the assertion (11) is sharp, moreover, the extremal function can be given by

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k} z^k.$$

Corollary 4. *If the hypotheses of Theorem* 3 *is satisfied, then*

$$|a_k| \le \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}, \qquad \forall k \ge 2. \tag{12}$$

3. CLOSURE THEOREMS

Let the functions $f_j(z)$ be defined by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k; \qquad (a_{k,j} \ge 0, z \in \mathcal{U}). \tag{13}$$

Theorem 5. *Let the functions $f_j(z)$ defined by* (13) *be in the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ for every $j = 1, 2, ..., l$. Then the function $G(z)$ defined by*

$$G(z) = z + \sum_{k=2}^{\infty} b_k z^k; \qquad (b_k \ge 0, z \in \mathcal{U})$$

is a member of the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$, where

$$b_k = \tfrac{1}{l}\sum_{j=1}^{l} a_{k,j}; \qquad (k \ge 2).$$

Proof. Since $f_j(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$, from Theorem 3 we can write

$$\sum_{k=2}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_{k,j}| \le 1-\eta, \qquad 0 \le \eta \le 1$$

for every $j = 1, 2, ..., l$. Thus

$$\sum_{k=2}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|b_k| =$$

$$= \sum_{k=2}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k\left|\tfrac{1}{l}\sum_{j=1}^{l} a_{k,j}\right|$$

$$\leq \tfrac{1}{l}\sum_{j=1}^{l}\left(\sum_{k=n}\infty(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_{k,j}|\right)$$

$$= \tfrac{1}{l}\sum_{j=1}^{l}(1-\eta) = (1-\eta).$$

In view of Theorem 3, we conclude that $G(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. □

THEOREM 6. *The class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ is closed under convex linear combination.*

Proof. Let $f_j(z)$ defined by (13) be belonged to $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ for every $j = 1, 2, ..., l$, it is sufficient to prove that the function

$$h(z) = \mu f_1(z) + (1-\mu)f_2(z)$$

is also in the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$.

Let us write, for $0 \leq \mu \leq 1$,

$$h(z) = z + \sum_{k=n}^{\infty}\{\mu a_{k,1} + (1-\mu)a_{k,2}\}z^k,$$

we note that

$$\sum_{k=n}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k\left|\mu a_{k,1} + (1-\mu)a_{k,2}\right| =$$

$$\leq \sum_{k=n}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|\mu a_{k,1}|+$$

$$+ \sum_{k=n}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|(1-\mu)a_{k,2}|$$

$$= \mu\sum_{k=n}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_{k,1}|$$

$$+ (1-\mu)\sum_{k=n}^{\infty}(k-\eta)\left[\tfrac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k|a_{k,2}|$$

$$\leq \mu(1-\eta) + (1-\mu)(1-\eta) = (1-\eta).$$

It follows from Theorem 3 that $h(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$, which completes the proof. □

THEOREM 7. *Let*

$$f_0(z) = z$$

and

$$f_k(z) = z + \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k} z^k.$$

Then $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \rho_k f_k(z) \tag{14}$$

where $\rho_k \geq 0$ *and* $\sum_{k=0}^{\infty} \rho_k = 1$.

Proof. Firstly, suppose that

$$f(z) = \sum_{k=0}^{\infty} \rho_k f_k(z) \tag{15}$$

where $\rho_k \geq 0$ and $\sum_{k=0}^{\infty} \rho_k = 1$. Then

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \rho_k f_k(z) = \rho_0 f_0(z) + \sum_{k=1}^{\infty} \rho_k f_k(z) \\ &= z + \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k} \rho_k z^k. \end{aligned}$$

We observe that

$$\sum_{k=n}^{\infty} (k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k \cdot \left[\frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}\right]\rho_k =$$

$$= (1-\eta)\sum_{k=1}^{\infty} \rho_k = (1-\eta)(1-\rho_0) \leq (1-\eta).$$

In view of Theorem 3, we conclude that $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$.

Conversely, let us suppose that $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Since

$$a_k \leq \frac{(1-\eta)}{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}, \qquad \forall k \geq 2. \tag{16}$$

Then by Corollary 4, we set

$$\rho_k = \frac{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta)} a_k,$$

and

$$\rho_0 = 1 - \sum_{k=1}^{\infty} \rho_k.$$

We thus conclude that $f(z)=\sum_{k=0}^{\infty}\rho_k f_k(z)$. This completes the proof of the theorem. □

4. CONVOLUTION PROPERTIES

For functions $f_j(z)\in\mathcal{A}; (j=1,2,...,m)$ given by

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k; \qquad (z\in\mathcal{U}),$$

the Hadamard product (or convolution) of $f_1(z), f_2(z), ..., f_m(z)$ is defined by

$$G_m(z)=(f_1*f_2*...*f_m)(z)=z+\sum_{k=2}^{\infty}\Big(\prod_{j=1}^{m}a_{k,j}z^k\Big).$$

THEOREM 8. *If $f_j(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ for each $(j=1,2,...,m)$, then $G_m(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ with*

$$\eta^*=\frac{\prod\limits_{j=1}^{m}(1-\eta_j)}{\prod\limits_{j=1}^{m}(2-\eta_j)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k-\prod\limits_{j=1}^{m}(1-\eta_j)}.$$

Proof. We use the mathematical induction to get to the required result. Firstly, we have to show that $G_2(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$ for $f_1(z)$ and $f_2(z)$ belonging to $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_1)$, $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_1)$ respectively. We can write

$$\sum_{k=2}^{\infty}\frac{(k-\eta_j)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_j)}|a_{k,j}|\le 1; \qquad (j=1,2).$$

Applying the Schwarz inequality, we have the following inequality

$$\sum_{k=2}^{\infty}\sqrt{\frac{(k-\eta_1)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_1)}\frac{(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_2)}}\sqrt{|a_{k,1}|\cdot|a_{k,2}|}\le 1.$$

Then, we will determine the largest η^* such that

$$\sum_{k=2}^{\infty}\frac{(k-\eta^*)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta^*)}|a_{k,1}|\cdot|a_{k,2}|\le 1.$$

That is

$$\sum_{k=2}^{\infty}\frac{(k-\eta^*)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta^*)}|a_{k,1}|\cdot|a_{k,2}|\le$$

$$\le\sum_{k=2}^{\infty}\sqrt{\frac{(k-\eta_1)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_1)}\frac{(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_2)}}\sqrt{|a_{k,1}||a_{k,2}|}.$$

Therefore, we need to find the largest η^* such that

$$\frac{(k-\eta^*)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta^*)}\sqrt{|a_{k,1}|\cdot|a_{k,2}|}\leq$$
$$\leq\sqrt{\frac{(k-\eta_1)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_1)}\frac{(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_2)}}$$

for all $k\geq 2$. Thus we can write

$$\frac{(k-\eta^*)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta^*)}\leq$$
$$\leq\left\{\frac{(k-\eta_1)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_1)}\right\}\left\{\frac{(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta_2)}\right\}.$$

After some calculations, we get

$$\eta^*\leq 1-\frac{(k-1)(1-\eta_1)(1-\eta_2)}{(k-\eta_1)(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k-(1-\eta_1)(1-\eta_2)}.$$

We note that the right hand side of the above inequality is an increasing function for all $k\geq 2$. This implies that

$$\begin{aligned}\eta^*&=\min_{k\geq 2}\left\{\frac{(k-1)(1-\eta_1)(1-\eta_2)}{(k-\eta_1)(k-\eta_2)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k-(1-\eta_1)(1-\eta_2)}\right\}\\&=\frac{(1-\eta_1)(1-\eta_2)}{(2-\eta_1)(2-\eta_2)\left[\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m}\right]\Gamma_2-(1-\eta_1)(1-\eta_2)}.\end{aligned}\tag{17}$$

Thus $G_2(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Therefore the theorem is true for $m=2$. Now, we suppose that $G_{m-1}(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_0)$ and $f_m(z)\in\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta_m)$, where

$$\eta_0=\frac{\prod_{j=1}^{m-1}(1-\eta_j)}{\prod_{j=1}^{m-1}(2-\eta_j)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^{m-1}}\right]\Gamma_k-\prod_{j=1}^{m-1}(1-\eta_j)}.$$

Replacing η_1 by η_0, and η_2 by η_m in the inequality (17), we get

$$\begin{aligned}\eta^*&=\frac{(1-\eta_0)(1-\eta_m)}{(2-\eta_0)(2-\eta_m)\left[\frac{(1+\lambda_1)^{m-1}}{(1+\lambda_2)^m}\right]\Gamma_2-(1-\eta_0)(1-\eta_m)}\\&=\frac{\prod_{j=1}^{m}(1-\eta_j)}{\prod_{j=1}^{m}(2-\eta_j)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k-\prod_{j=1}^{m}(1-\eta_j)}.\end{aligned}$$

For the integer m the theorem ia also true. By the mathematical induction, the proof of the theorem is complete. □

5. INTEGRAL OPERATOR

In this section we consider integral transforms of functions in the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$.

THEOREM 9. *Let the function $f(z)$ defined by* (1) *be in the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$. Then the integral transforms*

$$G_c(z) = c\int_0^1 u^{c-2} f(uz)du; \qquad (c>0) \tag{18}$$

are in the class $\mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\gamma)$, where

$$\gamma = 1 - \tfrac{c(1-\eta)}{(2-\eta)(c+1)-c(1-\eta)}. \tag{19}$$

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Then we have

$$G_c(z) = c\int_0^1 u^{c-2} f(uz)du = z + \sum_{k=2}^{\infty}\left(\tfrac{c}{c+k-1}\right)a_k z^k. \tag{20}$$

Since $f(z) \in \mathcal{S}^{m,r,s}_{\lambda_1,\lambda_2}(\eta)$, we have

$$\sum_{k=2}^{\infty} \frac{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta)}|a_k| \le 1. \tag{21}$$

In view of Theorem 3, we shall find the largest γ for which

$$\sum_{k=2}^{\infty} \frac{(k-\gamma)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\gamma)\left(\frac{c}{c+k-1}\right)}|a_k| \le 1. \tag{22}$$

Let us find the range of values of γ for which

$$\frac{(k-\gamma)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\gamma)\left(\frac{c}{c+k-1}\right)} \le \frac{(k-\eta)\left[\frac{(1+\lambda_1(k-1))^{m-1}}{(1+\lambda_2(k-1))^m}\right]\Gamma_k}{(1-\eta)}, \qquad (k\ge 2).$$

After some calculations, we obtain from the above inequality that

$$\gamma \le 1 - \tfrac{c(k-1)(1-\eta)}{(k-\eta)(c+k-1)-c(1-\eta)}.$$

We note that the right hand side of the above inequality is an increasing function for all $k \ge 2$. This implies that

$$\begin{aligned}\gamma &= \min_{k\ge 2}\left\{1 - \tfrac{c(k-1)(1-\eta)}{(k-\eta)(c+k-1)-c(1-\eta)}\right\}\\ &= 1 - \tfrac{c(1-\eta)}{(2-\eta)(c+1)-c(1-\eta)}.\end{aligned}$$

The proof is complete. □

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REFERENCES

[1] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103**(1) (1999), pp. 1–13.

[2] M. H. Al-Abbadi and M. Darus, *Differential subordination for new generalized derivative operator*, Acta Univ. Apulensis, **20** (2009), pp. 265–280.

[3] Yu. E. Hohlov, *Operators and operations in the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat., **10**(10) (1978), pp. 83–89.

[4] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric function*, SIAM J. Math. Anal., **15**(4) (1984), pp. 737–745, 1984.

[5] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), pp. 109–115.

[6] K. Al-Shaqsi and M. Darus, *On univalent functions with respect to the k-symmetric points defined by a generalization Ruscheweyh derivative operators*, Jour. Anal. Appl., **7**(1) (2009), pp. 53–61.

[7] K. Al-Shaqsi and M. Darus, *Differential Subordination with generalized derivative operator*, Int. J. Comp. Math. Sci., **2**(2) (2008), pp. 75–78.

[8] J. Nishiwaki and S. Owa, *Convolutions for certain analytic functions*, Gen. Math., **15** (2007), no. 2–3, pp. 38–51.

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