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GLOBAL SMOOTHNESS AND APPROXIMATION BY GENERALIZED DISCRETE SINGULAR OPERATORS

GEORGE A. ANASTASSIOU* and MERVE KESTER*

Abstract. In this article we continue with the study of generalized discrete singular operators over the real line regarding their simultaneous global smoothness preservation property with respect to L_p norm for $1 \leq p \leq \infty$, by involving higher order moduli of smoothness. Additionally we study their simultaneous approximation to the unit operator with rates involving the modulus of smoothness. The Jackson type inequalities that produced in this article are almost sharp, containing neat constants, and they reflect the high order of differentiability of involved function.

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1. INTRODUCTION

This article is motivated mainly by [3], [6], Chapter 18, and [8], where J. Favard in 1944 introduced the discrete version of Gauss-Weierstrass operator

$$(F_n f)(x) = \tfrac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\left(\tfrac{\nu}{n}\right) \exp\left(-n\left(\tfrac{\nu}{n} - x\right)^2\right), \tag{1}$$

$n \in \mathbb{N}$, which has the property that $(F_n f)(x)$ converges to $f(x)$ pointwise for each $x \in \mathbb{R}$, and uniformly on any compact subinterval of $\mathbb{R}$, for each continuous function f ($f \in C(\mathbb{R})$) that fulfills $|f(t)| \leq Ae^{Bt^2}$, $t \in \mathbb{R}$, where A, B are positive constants.

We are also greatly motivated by [1] and [2].

Furthermore, we are inspired by [4] and [5] where the authors studied pointwise, uniform, and L_p, $p \geq 1$, approximation properties of generalized discrete singular operators of Picard, Gauss-Weierstrass, and Poisson-Cauchy type and their non-unitary analogs.

In this article, we study the discrete operators mentioned above regarding their global smoothness preservation properties, additionally we study their simultaneous global smoothness and approximation properties in L_p norm for $1 \leq p \leq \infty$.

*Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, U.S.A., e-mail: ganastss@memphis.edu, mkester@memphis.edu.

2. BACKGROUND

In [6], the authors studied the global smoothness preservation properties and differentiability, also approximations, of smooth general singular integral operators $\Theta_{r,\xi}(f;x)$, defined as follows.

Let $\xi > 0$ and μ_ξ be Borel probability measures on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ they defined

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum\limits_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-n}, & j = 0, \end{cases} \tag{2}$$

that is $\sum\limits_{j=0}^{r} \alpha_j = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Borel measurable, they defined for $x \in \mathbb{R}$,

$$\Theta_{r,\xi}(f;x) := \int_{-\infty}^{\infty} \Big(\sum_{j=0}^{r} \alpha_j f(x + jt) \Big) d\mu_\xi(t). \tag{3}$$

They supposed $\Theta_{r,\xi}(f;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Let $f \in C(\mathbb{R})$, for $m \in \mathbb{N}$ the m-th modulus of smoothness for $1 \le p \le \infty$, is given by

$$\omega_m(f,h)_p := \sup_{0 \le t \le h} \|\Delta_t^m f(x)\|_{p,x}, \tag{4}$$

where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt), \tag{5}$$

see also [7, p. 44].

Denote

$$\omega_m(f,h)_\infty = \omega_m(f,h). \tag{6}$$

Notice that

$$\omega_m(cf,h)_p = |c|\, \omega_m(f,h)_p \tag{7}$$

where c is a real constant.

They gave the main global smoothness preservation result in [6] as follows:

THEOREM 1. *Let $h > 0$, $f \in C(\mathbb{R})$.*

i) *Suppose $\Theta_{r,\xi}(f;x) \in \mathbb{R}$, $\xi > 0$, $\forall x \in \mathbb{R}$ and $\omega_m(f,h) < \infty$. Then*

$$\omega_m(\Theta_{r,\xi} f, h) \le \Big(\sum_{j=0}^{r} |\alpha_j| \Big) \omega_m(f,h). \tag{8}$$

ii) *Suppose $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p \ge 1$. Then*

$$\omega_m(\Theta_{r,\xi} f, h)_p \le \Big(\sum_{j=0}^{r} |\alpha_j| \Big) \omega_m(f,h)_p. \tag{9}$$

Next, in [6], the authors discussed about the derivatives of $\Theta_{r,\xi}(f;x)$ and their impact to simultaneous global smoothness preservation and convergence of these operators.

In [6], they obtained also the next differentiation result

THEOREM 2. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n, r \in \mathbb{N}$. Furthermore assume that for each $x \in \mathbb{R}$ the function $f^{(i)}(x + jt) \in L_1(\mathbb{R}, \mu_\xi)$ as a function of t, for all $i = 0, 1, \dots, n-1$; $j = 1, \dots, r$. Suppose that there exist $g_{i,j} \geq 0$, $i = 1, \dots, n$; $j = 1, \dots, r$, with $g_{i,j} \in L_1(\mathbb{R}, \mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$|f^{(i)}(x + jt)| \leq g_{i,j}(t), \tag{10}$$

for μ_ξ-almost all $t \in \mathbb{R}$, all $i = 1, \dots, n$; $j = 1, 2, \dots, r$. Then $f^{(i)}(x+jt)$ defines a μ_ξ-integrable function with respect to t for each $x \in \mathbb{R}$, all $i = 1, \dots, n$; $j = 1, \dots, r$, and

$$(\Theta_{r,\xi}(f;x))^{(i)} = \Theta_{r,\xi}\left(f^{(i)};x\right), \tag{11}$$

for all $x \in \mathbb{R}$, all $i = 1, \dots, n$.

On the other hand, in [4], the authors defined important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ as follows:

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{12}$$

they defined the generalized discrete Picard operators as

$$P^*_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{13}$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{14}$$

they defined the generalized discrete Gauss-Weierstrass operators as

$$W^*_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{15}$$

iii) Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}, \tag{16}$$

they defined the generalized discrete Poisson-Cauchy operators as

$$Q^*_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+j\nu)\right)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{17}$$

They observed that for c constant they have

$$P^*_{r,\xi}(c;x) = W^*_{r,\xi}(c;x) = Q^*_{r,\xi}(c;x) = c. \tag{18}$$

They assumed that the operators $P^*_{r,\xi}(f;x)$, $W^*_{r,\xi}(f;x)$, and $Q^*_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}} < \infty$.

iv) When

$$\mu_\xi(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}, \tag{19}$$

they defined the generalized discrete non-unitary Picard operators as

$$P_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}. \tag{20}$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$m_{\xi,P} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}. \tag{21}$$

They observed that

$$\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{22}$$

which is the probability measure (12) defining the operators $P^*_{r,\xi}$.

v) When

$$\mu_\xi(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}, \tag{23}$$

with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, $\operatorname{erf}(\infty) = 1$, they defined the generalized discrete non-unitary Gauss-Weierstrass operators as

$$W_{r,\xi}(f;x) := \frac{\sum_{\nu=-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+j\nu)\right)e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{24}$$

Here $\mu_{\xi,W}(\nu)$ has mass

(25) $$m_{\xi,W} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}.$$

They observed that

(26) $$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}},$$

which is the probability measure (14) defining the operators $W^*_{r,\xi}$.

The authors observed that $P_{r,\xi}(f;x)$, $W_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$.

We notice that

(27) $$P_{r,\xi}(f;x) = \lambda_1(\xi)\, P^*_{r,\xi}(f;x),$$

where

(28) $$\lambda_1(\xi) = \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}},$$

and

(29) $$W_{r,\xi}(f;x) = \lambda_2(\xi)\, W^*_{r,\xi}(f;x),$$

where

(30) $$\lambda_2(\xi) = \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}.$$

In [4], for $k = 1, ..., n$, the authors defined the ratios of sums

(31) $$c^*_{k,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}},$$

(32) $$p^*_{k,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}},$$

and for $\alpha \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, they introduced

(33) $$q^*_{k,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}.$$

Furthermore, they proved that these ratios of sums $c^*_{k,\xi}$, $p^*_{k,\xi}$, and $q^*_{k,\xi}$ are finite for all $\xi \in (0,1]$.

In [4], the authors also proved

$$m_{\xi,P} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \to 1, \quad \text{as } \xi \to 0^+ \tag{34}$$

and

$$m_{\xi,W} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}}{1+\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)} \to 1, \quad \text{as } \xi \to 0^+. \tag{35}$$

The authors introduced also

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{36}$$

Additionally, in [4], the authors defined the following error quantities:

$$\begin{aligned} E_{0,P}(f,x) &:= P_{r,\xi}(f;x) - f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+j\nu)\right) e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} - f(x), \end{aligned} \tag{37}$$

$$\begin{aligned} E_{0,W}(f,x) &:= W_{r,\xi}(f;x) - f(x) \\ &= \frac{\sum_{\nu=-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+j\nu)\right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} - f(x). \end{aligned} \tag{38}$$

Furthermore, they introduced the errors ($n \in \mathbb{N}$):

$$E_{n,P}(f,x) := P_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \tag{39}$$

and

$$E_{n,W}(f,x) := W_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{40}$$

Next, they obtained the inequalities

$$|E_{0,P}(f,x)| \le m_{\xi,P} \left|P^*_{r,\xi}(f;x) - f(x)\right| + |f(x)| \, |m_{\xi,P} - 1|, \tag{41}$$

$$|E_{0,W}(f,x)| \le m_{\xi,W} \left|W^*_{r,\xi}(f;x) - f(x)\right| + |f(x)| \, |m_{\xi,W} - 1|, \tag{42}$$

and

$$|E_{n,P}(f,x)| \leq \tag{43}$$
$$\leq m_{\xi,P}\left|P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c^*_{k,\xi}\right| + |f(x)|\,|m_{\xi,P}-1|,$$

with

$$|E_{n,W}(f,x)| \leq \tag{44}$$
$$\leq m_{\xi,W}\left|W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k p^*_{k,\xi}\right| + |f(x)|\,|m_{\xi,W}-1|.$$

In [4], they first gave the following simultaneous approximation results for unitary operators. They showed

Theorem 3. *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$ (uniformly continuous functions on $\mathbb{R}$).*
i) *For $n \in \mathbb{N}$,*

$$\left\|P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c^*_{k,\xi}\right\|_{\infty,x} \leq \tag{45}$$
$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!}\left(\frac{\sum_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}\right),$$

and

$$\left\|W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k p^*_{k,\xi}\right\|_{\infty,x} \leq \tag{46}$$
$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!}\left(\frac{\sum_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}\right).$$

ii) *For $n = 0$,*

$$\left\|P^*_{r,\xi}(f;x) - f(x)\right\|_{\infty,x} \leq \omega_r(f,\xi)\left(\frac{\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}\right), \tag{47}$$

and

$$\left\|W^*_{r,\xi}(f;x) - f(x)\right\|_{\infty,x} \leq \omega_r(f,\xi)\left(\frac{\sum_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}\right). \tag{48}$$

In the above inequalities (45)–(48)*, the ratios of sums in their right hand sides* (*R.H.S.*) *are uniformly bounded with respect to* $\xi \in (0,1]$.

In [4], they had also

THEOREM 4. *Let* $f \in C^n(\mathbb{R})$ *with* $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, *and* $\beta > \frac{n+r+1}{2\alpha}$.
i) *For* $n \in \mathbb{N}$,

$$\left\| Q^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q^*_{k,\xi} \right\|_{\infty,x} \leq \tag{49}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \right).$$

ii) *For* $n = 0$,

$$\left\| Q^*_{r,\xi}(f;x) - f(x) \right\|_{\infty,x} \leq \omega_r(f,\xi) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \right). \tag{50}$$

In the above inequalities (49)–(50)*, the ratios of sums in their R.H.S. are uniformly bounded with respect to* $\xi \in (0,1]$.

Next, they stated their results in [4] for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$. They had

COROLLARY 5. *Let* $f \in C_u(\mathbb{R})$. *Then*
i)

$$|E_{0,P}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) + |f(x)| \cdot |m_{\xi,P} - 1|, \tag{51}$$

ii)

$$|E_{0,W}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + |f(x)| \cdot |m_{\xi,W} - 1|. \tag{52}$$

In [4], for $E_{n,P}$ and $E_{n,W}$, the authors presented

THEOREM 6. *Let* $f \in C^n(\mathbb{R})$ *with* $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$,*and* $\|f\|_{\infty,\mathbb{R}} < \infty$. *Then*
i)

$$\|E_{n,P}(f,x)\|_{\infty,x} \leq \tag{53}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) + \|f\|_{\infty,\mathbb{R}} |m_{\xi,P} - 1|,$$

ii)

(54) $$\|E_{n,W}(f,x)\|_{\infty,x} \leq$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + \|f\|_{\infty,\mathbb{R}} |m_{\xi,W}-1| .$$

In the above inequalities (53)–(54)*, the ratios of sums in their R.H.S. are uniformly bounded with respect to* $\xi \in (0,1]$ *.*

In [5], the authors represented simultaneous L_p approximation results. They started with

THEOREM 7. **i**) *Let* $f \in C^n(\mathbb{R})$*, with* $f^{(n)} \in L_p(\mathbb{R})$*,* $n \in \mathbb{N}$*,* $p,q > 1$*:* $\frac{1}{p}+\frac{1}{q}=1$*, and rest as above in this section.Then*

(55) $$\left\| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c^*_{k,\xi} \right\|_p \leq$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(M^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p$$

where

(56) $$M^*_{p,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}-1\right)|\nu|^{np-1} e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}$$

which is uniformly bounded for all $\xi \in (0,1]$ *.*

Additionally, as $\xi \to 0^+$ *we obtain that R.H.S. of* (55) *goes to zero.*

ii) *When* $p=1$*, let* $f \in C^n(\mathbb{R})$*,* $f^{(n)} \in L_1(\mathbb{R})$*, and* $n \in \mathbb{N}-\{1\}$*. Then*

(57) $$\left\| P^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c^*_{k,\xi} \right\|_1 \leq$$

$$\leq \frac{1}{(n-1)!(r+1)} M^*_{1,\xi}\,\xi\,\omega_r(f^{(n)},\xi)_1$$

holds where $M^*_{1,\xi}$ *is defined as in* (56)*. Hence, as* $\xi \to 0^+$*, we obtain that R.H.S. of* (57) *goes to zero.*

iii) *When* $n=0$*, let* $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$ *,* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$ *and the rest as above in this section. Then*

(58) $$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_p \leq \left(\bar{M}^*_{p,\xi}\right)^{1/p} \omega_r(f,\xi)_p$$

where

(59) $$\bar{M}^*_{p,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}$$

which is uniformly bounded for all $\xi \in (0,1]$ *.*

Hence, as $\xi \to 0^+$, we obtain that $P^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

iv) *When $n = 0$ and $p = 1$, let $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$ and the rest as above in this section. Then the inequality*

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_1 \leq \bar{M}^*_{1,\xi}\omega_r(f,\xi)_1 \tag{60}$$

holds where $\bar{M}^*_{1,\xi}$ is defined as in (59). Furthermore, we get $P^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

Next, the authors presented their quantitative results for the Gauss-Weierstrass operators, see [5]. They started with

THEOREM 8. **i**) *Let $f \in C^n(\mathbb{R})$, with $f^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, and the rest as above in this section. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k p^*_{k,\xi} \right\|_p \leq \tag{61}$$
$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(N^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p$$

where

$$N^*_{p,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \tag{62}$$

which is uniformly bounded for all $\xi \in (0,1]$.

Additionally, as $\xi \to 0^+$ we obtain that R.H.S. of (61) goes to zero.

ii) *For $p = 1$, let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k p^*_{k,\xi} \right\|_1 \leq \tag{63}$$
$$\leq \frac{1}{(n-1)!(r+1)} N^*_{1,\xi}\xi\omega_r(f^{(n)},\xi)_1$$

holds where $N^*_{1,\xi}$ is defined as in (62). Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (63) goes to zero.

iii) *For $n = 0$, let $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above in this section. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) \right\|_p \leq \left(\bar{N}^*_{p,\xi}\right)^{1/p} \omega_r(f,\xi)_p \tag{64}$$

where

$$\bar{N}^*_{p,\xi} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \tag{65}$$

which is uniformly bounded for all $\xi \in (0,1]$.

Hence, as $\xi \to 0^+$, we obtain that $W^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

iv) *For $n = 0$ and $p = 1$, let $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$ and the rest as above in this section. Then the inequality*

$$\left\| W^*_{r,\xi}(f;x) - f(x) \right\|_1 \le \bar{N}^*_{1,\xi}\omega_r(f,\xi)_1 \tag{66}$$

holds where $\bar{N}^*_{1,\xi}$ is defined as in (65). Furthermore, we get $W^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

For the Poisson-Cauchy operators, in [5], the authors showed

Theorem 9. **i**) *Let $f \in C^n(\mathbb{R})$, with $f^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+n)+1}{2\alpha}$, $\alpha \in \mathbb{N}$, and the rest as above in this section. Then*

$$\left\| Q^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k q^*_{k,\xi} \right\|_p \le \tag{67}$$

$$\le \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(S^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p$$

where

$$S^*_{p,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1\right)|\nu|^{np-1}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{68}$$

is uniformly bounded for all $\xi \in (0,1]$.

Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (67) goes to zero.

ii) *When $p = 1$, let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, $\beta > \frac{r+n+1}{2\alpha}$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| Q^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k q^*_{k,\xi} \right\|_1 \le \tag{69}$$

$$\le \frac{1}{(n-1)!(r+1)} S^*_{1,\xi}\xi\omega_r(f^{(n)},\xi)_1$$

holds where $S^*_{1,\xi}$ is defined as in (68). Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (69) goes to zero.

iii) *When $n = 0$, let $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+2)+1}{2\alpha}$, and the rest as above in this section. Then*

$$\left\| Q^*_{r,\xi}(f;x) - f(x) \right\|_p \le \left(\bar{S}^*_{p,\xi}\right)^{1/p} \omega_r(f,\xi)_p \tag{70}$$

where

$$\bar{S}^*_{p,\xi} := \frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \tag{71}$$

which is uniformly bounded for all $\xi \in (0,1]$.

Hence, as $\xi \to 0^+$, *we obtain that* $Q^*_{r,\xi} \to$ *unit operator* I *in the* L_p *norm for* $p > 1$.

iv) *When* $n = 0$ *and* $p = 1$, *let* $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $\beta > \frac{r+3}{2\alpha}$ *and the rest as above in this section. The inequality*

$$\left\| Q^*_{r,\xi}(f;x) - f(x) \right\|_1 \le \bar{S}^*_{1,\xi}\omega_r(f,\xi)_1 \tag{72}$$

holds where $\bar{S}^*_{1,\xi}$ *is defined as in* (71). *Furthermore, we get* $Q^*_{r,\xi} \to I$ *in the* L_1 *norm as* $\xi \to 0^+$.

Next in [5], they stated their results for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$ as follows

THEOREM 10. **i**) *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ *such that* $np \neq 1$, $f \in L_p(\mathbb{R})$, *and the rest as above in this section. Then*

$$\|E_{n,P}(f,x)\|_p \le \frac{\xi^{\frac{1}{p}}\omega_r(f^{(n)},\xi)_p \left(\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \cdot \left[\frac{\left(\sum\limits_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}-1\right)|\nu|^{np-1}e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{p}}}{1+2\xi e^{-\frac{1}{\xi}}}\right] + \|f(x)\|_p\,|m_{\xi,P}-1| \tag{73}$$

holds. Additionally, as $\xi \to 0^+$, *we obtain that R.H.S. of* (73) *goes to zero.*

ii) *When* $p = 1$, *let* $f \in C^n(\mathbb{R}), f \in L_1(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, *and* $n \in \mathbb{N} - \{1\}$. *Then*

$$\|E_{n,P}(f,x)\|_1 \le \frac{\xi\omega_r(f^{(n)},\xi)_1}{(n-1)!(r+1)} \cdot \left[\frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1}-1\right)|\nu|^{n-1}e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right] + \|f(x)\|_1\,|m_{\xi,P}-1| \tag{74}$$

holds. Additionally, as $\xi \to 0^+$, *we obtain that R.H.S. of* (74) *goes to zero.*

iii) *When* $n = 0$, *let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, *and the rest as above in this section. Then*

$$\|E_{0,P}(f,x)\|_p \le \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{q}} \cdot \left[\frac{\left(\sum\limits_{\nu=-\infty}^{\infty}\left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}}\right)^{1/p}}{1+2\xi e^{-\frac{1}{\xi}}}\right] + \|f(x)\|_p\,|m_{\xi,P}-1| \tag{75}$$

holds. Hence, as $\xi \to 0^+$, *we obtain that R.H.S. of* (75) *goes to zero.*

iv) *When $n=0$ and $p=1$, the inequality*

$$\|E_{0,P}(f,x)\|_1 \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) \omega_r(f,\xi)_1 + \|f(x)\|_1 |m_{\xi,P}-1| \tag{76}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (76) *goes to zero.*

Next in [5], the authors gave quantitative results for $E_{n,W}(f,x)$

Theorem 11. **i**) *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $n \in \mathbb{N}$ such that $np \neq 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

(77)

$$\|E_{n,W}(f,x)\|_p \leq \frac{\xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p \left(\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \cdot$$
$$\cdot \left[\frac{\left(\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \|f(x)\|_p |m_{\xi,W}-1|$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (77) *goes to zero.*

ii) *For $p=1$, let $f \in C^n(\mathbb{R}), f \in L_1(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N}-\{1\}$. Then*

(78)

$$\|E_{n,W}(f,x)\|_1 \leq \frac{\xi \omega_r(f^{(n)},\xi)_1}{(n-1)!(r+1)} \cdot$$
$$\cdot \left[\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \|f(x)\|_1 |m_{\xi,W}-1|$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (78) *goes to zero.*

iii) *For $n=0$, let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\|E_{0,W}(f,x)\|_p \leq \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} \cdot \left[\frac{\left(\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] \tag{79}$$
$$+ \|f(x)\|_p |m_{\xi,W}-1|$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (79) *goes to zero.*

iv) *For $n = 0$ and $p = 1$, the inequality*

$$\|E_{0,W}(f,x)\|_1 \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi}\xi\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) \omega_r(f,\xi)_1 + \|f(x)\|_1 \left|m_{\xi,W} - 1\right| \tag{80}$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (80) *goes to zero.*

3. MAIN RESULTS

We start with global smoothness preservation properties of the operators $P^*_{r,\xi}$, $W^*_{r,\xi}$, and $\Theta^*_{r,\xi}$.

Theorem 12. *Let $h > 0$ and $0 < \xi \leq 1$.*

i) *Suppose $f \in C(\mathbb{R})$, and $P^*_{r,\xi}(f;x)$, $W^*_{r,\xi}(f;x)$, $Q^*_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, $\omega_m(f,h) < \infty$. Then*

$$\omega_m(P^*_{r,\xi}f,h) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h), \tag{81}$$

$$\omega_m(W^*_{r,\xi}f,h) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h), \tag{82}$$

and

$$\omega_m(Q^*_{r,\xi}f,h) \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h). \tag{83}$$

ii) *Suppose $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p \geq 1$. Then*

$$\omega_m(P^*_{r,\xi}f,h)_p \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h)_p, \tag{84}$$

$$\omega_m(W^*_{r,\xi}f,h)_p \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h)_p, \tag{85}$$

and

$$\omega_m(Q^*_{r,\xi}f,h)_p \leq \left(\sum_{j=0}^{r} |\alpha_j|\right)\omega_m(f,h)_p. \tag{86}$$

Proof. By Theorem 1. □

For $r = 1$, we get $\alpha_0 = 0$ and $\alpha_1 = 1$. Hence, we obtain

$$P^*_{1,\xi}(f;x) = P^*_{\xi}(f;x) = \frac{\sum\limits_{\nu=-\infty}^{\infty} f(x+\nu)e^{-\frac{|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}, \tag{87}$$

(88)
$$W_{1,\xi}^{*}\left(f;x\right)=W_{\xi}^{*}\left(f;x\right)=\frac{\sum\limits_{\nu=-\infty}^{\infty}f(x+\nu)e^{-\frac{\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{-\frac{\nu^{2}}{\xi}}},$$

and for $\beta>\frac{1}{\alpha}$, $\alpha\in\mathbb{N}$

(89)
$$Q_{1,\xi}^{*}\left(f;x\right)=Q_{\xi}^{*}\left(f;x\right)=\frac{\sum\limits_{\nu=-\infty}^{\infty}f(x+\nu)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}.$$

Therefore, by *Theorem* 12, we have

THEOREM 13. *Let $h>0$ and $0<\xi\leq1$.*
i) *Suppose $f\in C\left(\mathbb{R}\right)$, and $P_{\xi}^{*}\left(f;x\right), W_{\xi}^{*}\left(f;x\right), Q_{\xi}^{*}\left(f;x\right)\in\mathbb{R}$ for all $x\in\mathbb{R}$, $\omega_m(f,h)<\infty$. Then*

(90)
$$\omega_m(P_{\xi}^{*}f,h)\leq\omega_m(f,h),$$

(91)
$$\omega_m(W_{\xi}^{*}f,h)\leq\omega_m(f,h),$$

and

(92)
$$\omega_m(Q_{\xi}^{*}f,h)\leq\omega_m(f,h).$$

Inequalities (90), (91), *and* (92) *are sharp, that is attained by $f\left(x\right)=g\left(x\right)=x^m$.*
ii) *Suppose $f\in\left(C\left(\mathbb{R}\right)\cap L_p\left(\mathbb{R}\right)\right)$, $p\geq1$. Then*

(93)
$$\omega_m(P_{\xi}^{*}f,h)_p\leq\omega_m(f,h)_p,$$

(94)
$$\omega_m(W_{\xi}^{*}f,h)_p\leq\omega_m(f,h)_p,$$

and

(95)
$$\omega_m(Q_{\xi}^{*}f,h)_p\leq\omega_m(f,h)_p.$$

Proof. It suffices to show the attainability of the inequalities (90), (91), and (92). We notice that

(96)
$$\omega_m(g,h)=\omega_m(x^m,h)=m!h^m.$$

On the other hand, we have

$$\Delta_t^m (P_\xi^* g)(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} P_\xi^* g(x+jt) \tag{97}$$
$$= \frac{\sum\limits_{\nu=-\infty}^{\infty} \left[\sum\limits_{j=0}^{m} (-1)^{m-j} \binom{m}{j} ((x+\nu)+jt)^m \right] e^{-\frac{|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}$$
$$= \frac{\sum\limits_{\nu=-\infty}^{\infty} (\Delta_t^m (x+\nu)^m) e^{-\frac{|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}$$
$$= m! t^m.$$

Thus, we get

$$\omega_m(g,h) = \omega_m(P_\xi^* g, h). \tag{98}$$

Similarly, we obtain

$$\omega_m(g,h) = \omega_m(W_\xi^* g, h), \tag{99}$$

and

$$\omega_m(g,h) = \omega_m(Q_\xi^* g, h). \tag{100}$$

□

Next, we present the following theorem for the non-unitary operators $P_{r,\xi}$ and $W_{r,\xi}$

THEOREM 14. *Let $h > 0$ and $0 < \xi \leq 1$.*

i) *Suppose that $f \in C(\mathbb{R})$, and $P_{r,\xi}^*(f;x)$, $W_{r,\xi}^*(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, $\omega_m(f,h) < \infty$. Then*

$$\omega_m(P_{r,\xi} f, h) \leq \left(\frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}} \right) \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h), \tag{101}$$

$$\omega_m(W_{r,\xi} f, h) \leq \left(1 + \frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h). \tag{102}$$

ii) *Suppose $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p \geq 1$. Then*

$$\omega_m(P_{r,\xi} f, h)_p \leq \left(\frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}} \right) \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h)_p, \tag{103}$$

$$\omega_m(W_{r,\xi} f, h)_p \leq \left(1 + \frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) \left(\sum_{j=0}^{r} |\alpha_j| \right) \omega_m(f,h)_p. \tag{104}$$

Proof. By (7), (27) , (29) and Theorem 12, we have

$$\omega_m(P_{r,\xi}f,h) \leq \lambda_1(\xi)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h), \tag{105}$$

$$\omega_m(P_{r,\xi}f,h)_p \leq \lambda_1(\xi)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_p, \tag{106}$$

and

$$\omega_m(W_{r,\xi}f,h) \leq \lambda_2(\xi)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h), \tag{107}$$

$$\omega_m(W_{r,\xi}f,h)_p \leq \lambda_2(\xi)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f,h)_p. \tag{108}$$

Additionally, in [3], it was shown that

$$\lambda_1(\xi) \leq \frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}}, \tag{109}$$

and

$$\lambda_2(\xi) \leq 1+\frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{110}$$

Thus, by (105)–(110), we obtain the inequalities (101)–(104). □

Now, we give our results for the derivatives of the unitary operators $P_{r,\xi}^{*}(f;x)$, $W_{r,\xi}^{*}(f;x)$, and $Q_{r,\xi}^{*}(f;x)$ mentioned above. First, we get

Theorem 15. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n, r \in \mathbb{N}$, $0 < \xi \leq 1$. Additionally, suppose that for each $x \in \mathbb{R}$ the function $f^{(i)}(x+j\nu) \in L_1(\mathbb{R},\mu_\xi)$ as a function of ν, for all $i = 0, 1, \ldots, n-1$; $j = 1, \ldots, r$. Assume that there exist $g_{i,j} \geq 0$, $i = 1, \ldots, n$; $j = 1, \ldots, r$, with $g_{i,j} \in L_1(\mathbb{R},\mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$|f^{(i)}(x+j\nu)| \leq g_{i,j}(\nu), \tag{111}$$

for μ_ξ-almost all $\nu \in \mathbb{R}$, all $i = 1, \ldots, n$; $j = 1, 2, \ldots, r$. Then, $f^{(i)}(x+j\nu)$ defines a μ_ξ-integrable function with respect to ν for each $x \in \mathbb{R}$, all $i = 1, \ldots, n$; $j = 1, \ldots, r$.

i) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}$$

we get

$$\left(P_{r,\xi}^{*}(f;x)\right)^{(i)} = P_{r,\xi}^{*}\left(f^{(i)};x\right), \tag{112}$$

for all $x \in \mathbb{R}$, and for all $i = 1, \ldots, n$.

ii) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}$$

we have

$$\left(W^*_{r,\xi}\left(f;x\right)\right)^{(i)} = W^*_{r,\xi}\left(f^{(i)};x\right), \tag{113}$$

for all $x \in \mathbb{R}$, *and for all* $i = 1, \ldots, n$.

iii) *Let* $\alpha \in \mathbb{N}$, *and* $\beta > \frac{1}{\alpha}$. *When*

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},$$

we obtain

$$\left(Q^*_{r,\xi}\left(f;x\right)\right)^{(i)} = Q^*_{r,\xi}\left(f^{(i)};x\right), \tag{114}$$

for all $x \in \mathbb{R}$, *and for all* $i = 1, \ldots, n$.

Proof. By Theorem 2. □

Next, we present our results for the derivatives of non-unitary operators $P_{r,\xi}\left(f;x\right)$ and $W_{r,\xi}\left(f;x\right)$.

PROPOSITION 16. *Let the assumptions of the Theorem* 15 *be valid.*

i) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{\frac{-1}{\xi}}}$$

we get

$$\left(P_{r,\xi}\left(f;x\right)\right)^{(i)} = P_{r,\xi}\left(f^{(i)};x\right), \tag{115}$$

for all $x \in \mathbb{R}$, *and for all* $i = 1, \ldots, n$.

ii) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}$$

we have

$$\left(W_{r,\xi}\left(f;x\right)\right)^{(i)} = W_{r,\xi}\left(f^{(i)};x\right), \tag{116}$$

for all $x \in \mathbb{R}$, *and for all* $i = 1, \ldots, n$.

Proof. By (27) and (29) we have

$$\left(P_{r,\xi}\left(f;x\right)\right)^{(i)} = \lambda_1\left(\xi\right)\left(P^*_{r,\xi}\left(f;x\right)\right)^{(i)} \tag{117}$$

and

$$\left(W_{r,\xi}\left(f;x\right)\right)^{(i)} = \lambda_2\left(\xi\right)\left(W^*_{r,\xi}\left(f;x\right)\right)^{(i)}. \tag{118}$$

Thus by *Theorem* 15, we get

$$(119)\qquad \left(P_{r,\xi}\left(f;x\right)\right)^{(i)} = \lambda_1\left(\xi\right)P^*_{r,\xi}(f^{(i)};x) = P_{r,\xi}(f^{(i)};x)$$

and

$$(120)\qquad \left(W_{r,\xi}\left(f;x\right)\right)^{(i)} = \lambda_2\left(\xi\right)W^*_{r,\xi}(f^{(i)};x) = W_{r,\xi}(f^{(i)};x).$$

□

We have the following application of the *Theorem* 15 for the case of $r = 1$.

PROPOSITION 17. *Let $f \in C^{n-1}(\mathbb{R})$, such that $f^{(n)}$ exists, $n \in \mathbb{N}$, $0 < \xi \leq 1$. Additionally, suppose that for each $x \in \mathbb{R}$ the function $f^{(i)}(x+\nu) \in L_1(\mathbb{R}, \mu_\xi)$ as a function of ν, for all $i = 0, 1, \ldots, n-1$. Assume that there exist $g_i \geq 0$, $i = 1, \ldots, n$ with $g_i \in L_1(\mathbb{R}, \mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$(121)\qquad |f^{(i)}(x+\nu)| \leq g_i(\nu),$$

for μ_ξ-almost all $\nu \in \mathbb{R}$, all $i = 1, \ldots, n$. Then, $f^{(i)}(x+\nu)$ defines a μ_ξ-integrable function with respect to ν for each $x \in \mathbb{R}$, and for all $i = 1, \ldots, n$.

i) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}$$

we get

$$(122)\qquad \left(P^*_\xi\left(f;x\right)\right)^{(i)} = P^*_\xi(f^{(i)};x),$$

for all $x \in \mathbb{R}$, and for all $i = 1, \ldots, n$.

ii) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}$$

we have

$$(123)\qquad \left(W^*_\xi\left(f;x\right)\right)^{(i)} = W^*_\xi(f^{(i)};x),$$

for all $x \in \mathbb{R}$, and for all $i = 1, \ldots, n$.

iii) *Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When*

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}},$$

we obtain

$$(124)\qquad \left(Q^*_\xi\left(f;x\right)\right)^{(i)} = Q^*_\xi(f^{(i)};x),$$

for all $x \in \mathbb{R}$, and for all $i = 1, \ldots, n$.

We obtain

Theorem 18. *Let $h>0$ and the assumptions of the Theorem 15 be valid.*
i) *Suppose that $\omega_m(f^{(i)},h)<\infty$, for all $i=0,1,...,n$. Then*

$$\omega_m((P^*_{r,\xi}f)^{(i)},h)\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h), \tag{125}$$

$$\omega_m((W^*_{r,\xi}f)^{(i)},h)\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h), \tag{126}$$

and

$$\omega_m((Q^*_{r,\xi}f)^{(i)},h)\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h). \tag{127}$$

ii) *Assume $f^{(i)}\in(C(\mathbb{R})\cap L_p(\mathbb{R}))$, $i=0,1,...,n$, $p\geq1$. Then*

$$\omega_m((P^*_{r,\xi}f)^{(i)},h)_p\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h)_p, \tag{128}$$

$$\omega_m((W^*_{r,\xi}f)^{(i)},h)_p\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h)_p, \tag{129}$$

and

$$\omega_m((Q^*_{r,\xi}f)^{(i)},h)_p\leq\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h)_p. \tag{130}$$

Proof. By Theorems 12, 15. □

Next, we state our results for the non-unitary operators

Theorem 19. *Let $h>0$ and the assumptions of the Theorem 15 be valid.*
i) *Suppose that $\omega_m(f^{(i)},h)<\infty$, for all $i=0,1,...,n$. Then*

$$\omega_m((P_{r,\xi}f)^{(i)},h)\leq\left(\frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}}\right)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h), \tag{131}$$

and

$$\omega_m((W_{r,\xi}f)^{(i)},h)\leq\left(1+\frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}\right)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h). \tag{132}$$

ii) *Assume $f^{(i)}\in(C(\mathbb{R})\cap L_p(\mathbb{R}))$, $i=0,1,...,n$, $p\geq1$. Then*

$$\omega_m((P_{r,\xi}f)^{(i)},h)_p\leq\left(\frac{1+2e^{\frac{-1}{\xi}}(\xi+1)}{1+2\xi e^{\frac{-1}{\xi}}}\right)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h)_p, \tag{133}$$

and

$$\omega_m((W_{r,\xi}f)^{(i)},h)_p\leq\left(1+\frac{2e^{\frac{-1}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}\right)\left(\sum_{j=0}^{r}|\alpha_j|\right)\omega_m(f^{(i)},h)_p. \tag{134}$$

Proof. By (27) , (29) , (109) , (110) , and Theorem 18. □

For the case of $r = 1$ we have

PROPOSITION 20. *Let $h > 0$ and the assumptions of the Proposition 17 be valid.*

i) *Assume that $\omega_m(f^{(i)}, h) < \infty$, for all $i = 0, 1, ..., n$. Then*

$$\omega_m((P_\xi^* f)^{(i)}, h) \leq \omega_m(f^{(i)}, h), \tag{135}$$

$$\omega_m((W_\xi^* f)^{(i)}, h) \leq \omega_m(f^{(i)}, h), \tag{136}$$

and

$$\omega_m((Q_\xi^* f)^{(i)}, h) \leq \omega_m(f^{(i)}, h). \tag{137}$$

ii) *Suppose that $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $i = 0, 1, ..., n$, $p \geq 1$. Then*

$$\omega_m((P_\xi^* f)^{(i)}, h)_p \leq \omega_m(f^{(i)}, h)_p, \tag{138}$$

$$\omega_m((W_\xi^* f)^{(i)}, h)_p \leq \omega_m(f^{(i)}, h)_p, \tag{139}$$

and

$$\omega_m((Q_\xi^* f)^{(i)}, h)_p \leq \omega_m(f^{(i)}, h)_p. \tag{140}$$

Proof. By Theorem 13 and Proposition 17. □

Now, we demonstrate our simultaneous results for the operators $P_{r,\xi}^*$, $W_{r,\xi}^*$, and $Q_{r,\xi}^*$. We start with

THEOREM 21. *Let $f \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}$, $\rho \in \mathbb{Z}^+$ and $f^{(n+i)} \in C_u(\mathbb{R})$, $i = 0, 1, \ldots, \rho$, and $0 < \xi \leq 1$. We consider the assumptions of Theorem 15 valid for $n = \rho$ there.*

$$\left\| \left(P_{r,\xi}^*(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_{\infty,x} \leq \tag{141}$$

$$\leq \frac{\omega_r\left(f^{(n+i)}, \xi\right)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \right),$$

$$\left\| \left(W_{r,\xi}^*(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_{\infty,x} \leq \tag{142}$$

$$\leq \frac{\omega_r\left(f^{(n+i)}, \xi\right)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \right),$$

and

$$\left\| \left(Q^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!}\delta_k q^*_{k,\xi} \right\|_{\infty,x} \leq \tag{143}$$

$$\leq \frac{\omega_r\left(f^{(n+i)},\xi\right)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}} \right).$$

where $\beta > \frac{n+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$.

Proof. By Theorems 3, 4. □

Next we have

THEOREM 22. *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then*

i)

$$\left\| \left(P^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!}\delta_k c^*_{k,\xi} \right\|_{p,x} \leq \tag{144}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(M^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r\left(f^{(n+i)},\xi\right)_p,$$

ii)

$$\left\| \left(W^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!}\delta_k p^*_{k,\xi} \right\|_{p,x} \leq \tag{145}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(N^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r\left(f^{(n+i)},\xi\right)_p,$$

iii) *for* $\beta > \frac{p(n+r)+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!}\delta_k q^*_{k,\xi} \right\|_{p,x} \leq \tag{146}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(S^*_{p,\xi}\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r\left(f^{(n+i)},\xi\right)_p.$$

Proof. By Theorems 7-9. □

Now, we give our results for the special case of $n = 0$.

PROPOSITION 23. *Let* $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *We consider the assumptions of Theorem 15 as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$, *we have*

i)

$$\left\| \left(P^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) \right\|_{p,x} \leq \left(\bar{M}^*_{p,\xi}\right)^{\frac{1}{p}} \omega_r(f^{(i)}, \xi)_p, \tag{147}$$

ii)

$$\left\| \left(W^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) \right\|_{p,x} \leq \left(\bar{N}^*_{p,\xi}\right)^{\frac{1}{p}} \omega_r(f^{(i)}, \xi)_p, \tag{148}$$

iii) *for* $\beta > \frac{p(r+2)+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) \right\|_{p,x} \leq \left(\bar{S}^*_{p,\xi}\right)^{\frac{1}{p}} \omega_r(f^{(i)}, \xi)_p. \tag{149}$$

Proof. By Theorems 7-9. □

For the special case of $p = 1$, we obtain

Theorem 24. *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_1(\mathbb{R})$, $n \in \mathbb{N} - \{1\}$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem 15 as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$, *we have*

i)

$$\left\| \left(P^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k c^*_{k,\xi} \right\|_{1,x} \leq \tag{150}$$
$$\leq \frac{1}{(n-1)!(r+1)} M^*_{1,\xi} \xi \omega_r(f^{(n+i)}, \xi)_1,$$

ii)

$$\left\| \left(W^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k p^*_{k,\xi} \right\|_{1,x} \leq \tag{151}$$
$$\leq \frac{1}{(n-1)!(r+1)} N^*_{1,\xi} \xi \omega_r(f^{(n+i)}, \xi)_1,$$

iii) *for* $\beta > \frac{n+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) - \sum_{k=1}^{n} \frac{f^{(i+k)}(x)}{k!} \delta_k q^*_{k,\xi} \right\|_{1,x} \leq \tag{152}$$
$$\leq \frac{1}{(n-1)!(r+1)} S^*_{1,\xi} \xi \omega_r(f^{(n+i)}, \xi)_1.$$

Proof. By Theorems 7–9. □

For $p = 1$ and $n = 0$, we give

Proposition 25. *Let* $f^{(i)} \in \left(C(\mathbb{R}) \cap L_1(\mathbb{R})\right)$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$, *we have*

i)

$$\left\| \left(P^*_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x) \right\|_{1,x} \leq \bar{M}^*_{1,\xi} \omega_r(f^{(i)}, \xi)_1, \tag{153}$$

ii)

$$\left\| \left(W^{*}_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x)\right\|_{1,x} \le \bar{N}^{*}_{1,\xi}\omega_r(f^{(i)},\xi)_1, \tag{154}$$

iii) *for* $\beta > \frac{r+3}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q^{*}_{r,\xi}(f;x)\right)^{(i)} - f^{(i)}(x)\right\|_{1,x} \le \bar{S}^{*}_{1,\xi}\omega_r(f^{(i)},\xi)_1. \tag{155}$$

Proof. By Theorems 7–9. □

Next, we state our simultaneous approximation results for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$. We obtain

Corollary 26. *Let* $f^{(i)} \in C_u(\mathbb{R})$, $i = 0,1,\ldots,\rho$, $\rho \in \mathbb{Z}^+$, *and* $0 < \xi \le 1$. *We consider the assumptions of Theorem 15 as valid for* $n = \rho$ *there. Then for all* $i = 0,1,\ldots,\rho$, *we have*

i)

$$\left|(E_{0,P}(f,x))^{(i)}\right| \le \left(\frac{\sum\limits_{\nu=-\infty}^{\infty}\omega_r(f^{(i)},|\nu|)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right) + \left|f^{(i)}(x)\right|\left|m_{\xi,P}-1\right|, \tag{156}$$

ii)

$$\left|(E_{0,W}(f,x))^{(i)}\right| \le \left(\frac{\sum\limits_{\nu=-\infty}^{\infty}\omega_r(f^{(i)},|\nu|)e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}\right) + \left|f^{(i)}(x)\right|\left|m_{\xi,W}-1\right|. \tag{157}$$

Proof. By (51), (52), also by $(E_{0,P}(f,x))^{(i)}=E_{0,P}(f^{(i)},x)$ and $(E_{0,W}(f,x))^{(i)} = E_{0,W}(f^{(i)},x)$. □

Theorem 27. *Let* $f \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}$, $\rho \in \mathbb{Z}^+$ *and* $f^{(n+i)} \in C_u(\mathbb{R})$, $i = 0,1,\ldots,\rho$, $0 < \xi \le 1$, *and* $\left\|f^{(i)}\right\|_{\infty,\mathbb{R}} < \infty$. *We consider the assumptions of Theorem 15 valid for* $n = \rho$. *Then for all* $i = 0,1,\ldots,\rho$, *we have*

i)

$$\left\|(E_{n,P}(f,x))^{(i)}\right\|_{\infty,x} \le \tag{158}$$

$$\le \frac{\omega_r(f^{(n+i)},\xi)}{n!}\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right) + \left\|f^{(i)}\right\|_{\infty,\mathbb{R}}\left|m_{\xi,P}-1\right|,$$

and

ii)

(159) $$\left\|\left(E_{n,W}(f,x)\right)^{(i)}\right\|_{\infty,x}\leq$$

$$\leq\frac{\omega_r(f^{(n+i)},\xi)}{n!}\left(\frac{\sum_{\nu=-\infty}^{\infty}|\nu|^n\left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}\right)+\left\|f^{(i)}\right\|_{\infty,\mathbb{R}}\left|m_{\xi,W}-1\right|.$$

Proof. By (53), (54), also by $(E_{n,P}(f,x))^{(i)}=E_{n,P}(f^{(i)},x)$ and $(E_{n,W}(f,x))^{(i)}$ $=E_{n,W}(f^{(i)},x)$. □

THEOREM 28. **i**) *Let* $f\in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)}\in L_p(\mathbb{R})$, $n\in\mathbb{N}$, $i=0,1,\dots,\rho\in\mathbb{Z}^+$. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $np\neq1$. *We consider the assumptions of Theorem* 15 *as valid for* $n=\rho$ *there. Then for all* $i=0,1,\dots,\rho$,

(160)

$$\left\|\left(E_{n,P}(f,x)\right)^{(i)}\right\|_p$$
$$\leq\frac{\xi^{\frac{1}{p}}\omega_r(f^{(n+i)},\xi)_p\left(\sum_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}\cdot\left[\frac{\left(\sum_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1}-1\right)|\nu|^{np-1}e^{\frac{-|\nu|}{\xi}}\right)^{\frac{1}{p}}}{1+2\xi e^{-\frac{1}{\xi}}}\right]$$
$$+\left\|f^{(i)}(x)\right\|_p\left|m_{\xi,P}-1\right|$$

holds.

ii) *Let* $f\in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)}\in L_1(\mathbb{R})$, $n\in\mathbb{N}-\{1\}$, $i=0,1,\dots,\rho\in\mathbb{Z}^+$. *We consider the assumptions of Theorem* 15 *as valid for* $n=\rho$ *there. Then for all* $i=0,1,\dots,\rho$,

(161) $$\left\|\left(E_{n,P}(f,x)\right)^{(i)}\right\|_1\leq\frac{\xi\omega_r(f^{(n+i)},\xi)_1}{(n-1)!(r+1)}\cdot\left[\frac{\sum_{\nu=-\infty}^{\infty}\left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1}-1\right)|\nu|^{n-1}e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}\right]$$
$$+\left\|f^{(i)}(x)\right\|_1\left|m_{\xi,P}-1\right|$$

holds.

iii) *Let* $f^{(i)}\in(C(\mathbb{R})\cap L_p(\mathbb{R}))$, $i=0,1,\dots,\rho\in\mathbb{Z}^+$; $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$. *We consider the assumptions of Theorem* 15 *as valid for* $n=\rho$

there. Then for all $i = 0, 1, \ldots, \rho$,

(162)

$$\left\| (E_{0,P}(f,x))^{(i)} \right\|_p \leq \omega_r(f^{(i)}, \xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}} \cdot \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \right)}{1+2\xi e^{-\frac{1}{\xi}}} ^{\frac{1}{p}} \right] + \left\| f^{(i)}(x) \right\|_p |m_{\xi,P} - 1|$$

holds.

iv) *Let* $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$,

(163)

$$\left\| (E_{0,P}(f,x))^{(i)} \right\|_1 \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) \omega_r(f^{(i)}, \xi)_1 + \left\| f^{(i)}(x) \right\|_1 |m_{\xi,P} - 1|$$

holds.

Proof. By Theorem 10 and by $(E_{n,P}(f,x))^{(i)} = E_{n,P}(f^{(i)}, x)$ for $n \in \mathbb{Z}^+$. $\square$

THEOREM 29. **i**) *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $np \neq 1$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$,

(164)

$$\left\| (E_{n,W}(f,x))^{(i)} \right\|_p \leq$$

$$\leq \frac{\xi^{\frac{1}{p}} \omega_r(f^{(n+i)}, \xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \cdot \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} ^{\frac{1}{p}} \right] + \left\| f^{(i)}(x) \right\|_p |m_{\xi,W} - 1|$$

holds.

ii) *Let* $f \in C^{n+\rho}(\mathbb{R})$, *with* $f^{(n+i)} \in L_1(\mathbb{R})$, $n \in \mathbb{N} - \{1\}$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$, *we have*

(165) $$\left\| \left(E_{n,W}(f,x)\right)^{(i)} \right\|_1 \leq \frac{\xi \omega_r(f^{(n+i)},\xi)_1}{(n-1)!(r+1)} \cdot \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \left\| f^{(i)}(x) \right\|_1 |m_{\xi,W} - 1|$$

holds.

iii) *Let* $f^{(i)} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$,

(166) $$\left\| \left(E_{0,W}(f,x)\right)^{(i)} \right\|_p \leq$$
$$\leq \omega_r(f^{(i)},\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} \cdot \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right] + \left\| f^{(i)}(x) \right\|_p |m_{\xi,W} - 1|$$

holds.

iv) *Let* $f^{(i)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $i = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem* 15 *as valid for* $n = \rho$ *there. Then for all* $i = 0, 1, \ldots, \rho$,

(167) $$\left\| \left(E_{0,W}(f,x)\right)^{(i)} \right\|_1 \leq$$
$$\leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{r} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) \omega_r(f^{(i)},\xi)_1 + \left\| f^{(i)}(x) \right\|_1 |m_{\xi,W} - 1|$$

holds.

Proof. By Theorem 11 and by $\left(E_{n,W}(f,x)\right)^{(i)} = E_{n,W}(f^{(i)},x)$ for $n \in \mathbb{Z}^+$. □

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