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EXPANDING THE APPLICABILITY OF NEWTON-TIKHONOV METHOD FOR ILL-POSED EQUATIONS

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Abstract. We present a new semilocal convergence analysis of Newton-Tikhonov methods for solving ill-posed operator equations in a Hilbert space setting. Using more precise majorizing sequences and under the same computational cost as in earlier studies such as [13]–[20], we provide: weaker sufficient convergence criteria; tighter error estimates on the distances involved and an at least as precise information on the location of the solution. Applications include Hammertein nonlinear integral equations.

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution of non-linear ill-posed equation

$$A(x) = y \tag{1.1}$$

where A is a nonlinear operator defined on a subset $D = D(A)$ of a Hilbert space X, and with range $R(A)$ in a Hilbert space Y. Equation (1.1) is ill-posed in the sense that the solution of (1.1) does not depend continuously on the data y. Many regularization methods such as Tikhonov regularization [9, 10, 23, 25, 27, 32], Gauss-Newton method [6] and other methods [21], [22] have been used to approximate solution of equation (1.1).

Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modelling [1], [5], [7], [8], [29], [30]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial

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point, to give conditions ensuring the convergence of the iterative procedure, while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

George and collaborators (see [13]–[20]) solved equation (1.1) for the special case when A is a Hammerstein-type operator. A Hammerstein-type operator is of the form $A = MF$, where $F : D(F) \subset X \to Z$ is a nonlinear and $M : Z \to Y$ is a bounded linear operator with X, Y and Z are Hilbert spaces. Hence, (1.1) becomes

$$MF(x) = y. \tag{1.2}$$

In particular George and Kunhanandan [16] assumed that a solution $x^* \in D(F)$ of (1.2) satisfies

$$\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : MF(x) = y, x \in D(F)\}, \tag{1.3}$$

and $y^\delta \in Y$ are the available noisy data such that

$$\|y - y^\delta\| \leq \delta. \tag{1.4}$$

Then, for fixed $\alpha > 0, \delta > 0$, the Newton-Tikhonov (NT) method defined by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - F'(x^\delta_{n,\alpha})^{-1}(F(x^\delta_{n,\alpha}) - z^\delta_\alpha), \quad x^\delta_{0,\alpha} = x_0, \tag{1.5}$$

where z^δ_α is an approximation of the solution of the equation $M(z) = y^\delta$ (see Section 5) was used to generate a sequence $\{x^\delta_{n,\alpha}\}$ converging quadratically to a solution x^δ_α of the equation

$$F(x) = z^\delta_\alpha \tag{1.6}$$

provided that certain Kantrovich-type criteria are satisfied.

In the present paper we expand the applicability of (NT) by using more precise majorizing sequence for $\{x^\delta_{n,\alpha}\}$ than the ones given in [17]. This way we provide a semilocal convergence analysis for (NT) with the following advantages over the work in [16] under the same computational cost:

(a) Weaker sufficient convergence criteria;
(b) Tighter error estimates on the distances $\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\|$ and
(c) An at least as precise information on the location of the solution.

These advantages are obtained, since we use the more precise center-Lipschitz condition instead of the Lipschitz condition for the computation of the upper bounds on the norms $\|F'(x^\delta_{n,\alpha})^{-1}\|$ (see (C3) and (C4) in Section 3). We also study the semilocal convergence of the simplified Newton-Tikhonov method (SNT) defined by

$$x^\delta_{n+1,\alpha} = x^\delta_{n,\alpha} - F'(x^\delta_{0,\alpha})^{-1}(F(x^\delta_{n,\alpha}) - z^\delta_\alpha), \quad x^\delta_{0,\alpha} = x_0. \tag{1.7}$$

(SNT) method is used as predictor for (NT) method since the former converges under weaker sufficient convergence criteria than the latter.

The paper is organized as follows. Section 2 contains results on scalar sequences that are majorizing for (NT). Sections 3 and 4 contain respectively, the semilocal convergence of (NT) and (SNT). The applications are given in the concluding Section 5.

2. MAJORIZING SEQUENCES

We present auxiliary results on scalar sequences which shall be shown to be majorizing for $\{x^{\delta}_{n,\alpha}\}$ in Section 3.

DEFINITION 2.1. *Let* $L_0 > 0, L > 0, b > 0$ *and* $n > 0$. *Define scalar sequences* $\{r^{\delta}_{n,\alpha}\}, \{s^{\delta}_{n,\alpha}\}, \{t^{\delta}_{n,\alpha}\}$ *by*

$$r^{\delta}_{0,\alpha} = 0, \quad r^{\delta}_{1,\alpha} = r, \quad r^{\delta}_{2,\alpha} = r^{\delta}_{1,\alpha} + \frac{3bL_0(r^{\delta}_{1,\alpha} - r^{\delta}_{0,\alpha})^2}{2(1 - bL_0 r^{\delta}_{1,\alpha})},$$

$$r^{\delta}_{n+2,\alpha} = r^{\delta}_{n+1,\alpha} + \frac{3bL_0(r^{\delta}_{n+1,\alpha} - r^{\delta}_{n,\alpha})^2}{2(1 - bL_0 r^{\delta}_{n+1,\alpha})}, \quad \forall n = 1, 2, \ldots, \tag{2.8}$$

(2.9)

$$s^{\delta}_{0,\alpha} = 0, \ \ s^{\delta}_{1,\alpha} = r, \ \ s^{\delta}_{n+2,\alpha} = s^{\delta}_{n+1,\alpha} + \frac{3bL(s^{\delta}_{n+1,\alpha} - s^{\delta}_{n,\alpha})^2}{2(1 - bL_0 r^{\delta}_{n+1,\alpha})}, \quad \forall n = 0, 1, 2, \ldots,$$

and

(2.10)

$$t^{\delta}_{0,\alpha} = 0, \ \ t^{\delta}_{1,\alpha} = r, \ \ t^{\delta}_{n+2,\alpha} = t^{\delta}_{n+1,\alpha} + \frac{3bL(t^{\delta}_{n+1,\alpha} - t^{\delta}_{n,\alpha})^2}{2(1 - bLt^{\delta}_{n+1,\alpha})}, \quad \forall n = 0, 1, 2, \ldots.$$

Then, using a simple inductive argument we obtain the following result where we compare the three scalar sequences.

PROPOSITION 2.2. [1], [3]–[5] *Suppose that* $L_0 \le L$ *and*

$$t^{\delta}_{n+1,\alpha} < \tfrac{1}{bL}, \quad \forall n = 0, 1, 2, \ldots. \tag{2.11}$$

Then, the sequences $\{r^{\delta}_{n,\alpha}\}, \{s^{\delta}_{n,\alpha}\}$ *and* $\{t^{\delta}_{n,\alpha}\}$ *are well defined, increasing and converge to their unique least upper bounds* r^*, s^*, t^* *which satisfy for* $\gamma = \frac{6L}{3L+\sqrt{9L^2+24L_0L}}$,

$$r \le r^* \le s^* \le t^* \le \tfrac{1}{bL} \tag{2.12}$$

$r^* \le \bar{r}^* = r + \frac{bL_0 r^2}{2(1-\gamma)(1-bL_0 r)}, s^* \le \bar{s}^* = \frac{r}{1-\gamma}$ *and* $t^* \le \bar{t}^* \le 2r$. *Moreover, the following estimates hold for each* $n = 1, 2, \ldots$

$$r^{\delta}_{n,\alpha} \le s^{\delta}_{n,\alpha} \le t^{\delta}_{n,\alpha}, \tag{2.13}$$

and

$$r^{\delta}_{n+1,\alpha} - r^{\delta}_{n,\alpha} \le s^{\delta}_{n+1,\alpha} - s^{\delta}_{n,\alpha} \le t^{\delta}_{n+1,\alpha} - t^{\delta}_{n,\alpha}. \tag{2.14}$$

Further more strict inequality holds in (2.13) *and* (2.14) *if* $L_0 < L$.

REMARK 2.3. It follows from (2.12)–(2.14) that if $L_0 < L$ sequence $\{t^{\delta}_{n,\alpha}\}$ is the least tight. This sequence is a majorizing for $\{x^{\delta}_{n,\alpha}\}$ (cf. [17, Theorem 3.3]).

Next we present results respectively for $\{t^{\delta}_{n,\alpha}\}, \{s^{\delta}_{n,\alpha}\}, \{r^{\delta}_{n,\alpha}\}$.

LEMMA 2.4. [1, 4] *Suppose that*

$$h = 4Lbr \leq 1. \tag{2.15}$$

Then, sequence $\{t^{\delta}_{n,\alpha}\}$ is increasing convergent to t^*. The convergence is linear if $h = 1$ and quadratic if $h < 1$.

LEMMA 2.5. [3, Lemma 2.1] *Suppose that*

$$h_1 = \tfrac{b}{4}\Big(3L + 4L_0 + \sqrt{9L^2 + 24L_0L}\Big)r \leq 1. \tag{2.16}$$

Then, $\{s^{\delta}_{n,\alpha}\}$ is increasing convergent to s^*. The convergence is linear if $h_1 = 1$ and quadratic if $h_1 < 1$.

LEMMA 2.6. [4] *Suppose that*

$$h_2 = \tfrac{b}{4}\Big(4L_0 + \sqrt{3L_0L + 8L_0^2} + \sqrt{3L_0L}\Big)r \leq 1. \tag{2.17}$$

Then, $\{r^{\delta}_{n,\alpha}\}$ is increasing convergent to r^*. The convergence is linear if $h_2 = 1$ and quadratic if $h_2 < 1$.

We also have the following generalization of Lemma 2.6.

LEMMA 2.7. [4] *Suppose that there exists a minimum integer $N > 1$ such that $r^{\delta}_{i,\alpha} (i = 0, 1, \ldots, N-1)$ given by (2.14) are well-defined,*

$$r^{\delta}_{i,\alpha} < r^{\delta}_{i+1,\alpha} < \tfrac{1}{bL_0}, \quad i = 0, 1, \ldots, N-2. \tag{2.18}$$

Then, the following assertions hold

$$bL_0 r^{\delta}_{N,\alpha} < 1, \tag{2.19}$$

$$r^{\delta}_{N,\alpha} \leq \tfrac{1}{bL_0}(1 - (1 - bL_0)r^{\delta}_{N-1,\alpha}\gamma) \tag{2.20}$$

and

$$\gamma_{N-1} = \frac{3L(r^{\delta}_{N+1,\alpha} - r^{\delta}_{N,\alpha})}{2(1 - bL_0 r^{\delta}_{N+1,\alpha})} \leq \gamma \leq 1 - \frac{bL_0(r^{\delta}_{N+1,\alpha} - r^{\delta}_{N,\alpha})}{1 - bL_0 r^{\delta}_{N,\alpha}}. \tag{2.21}$$

REMARK 2.8. (a) Lemma 2.7 reduces to Lemma 2.6 if $N = 2$ (see also Remark 2.5 in [4]).

(b) It follows from (2.15), (2.16), (2.17) that

$$h \leq 1 \Longrightarrow h_1 \leq 1 \Longrightarrow h_2 \leq 1 \tag{2.22}$$

but not necessarily vice versa unless if $L_0 = L$. Moreover, we have that

$$\tfrac{h_4}{h} \to \tfrac{1}{4}, \tfrac{h_2}{h} \to 0 \ \textit{and} \ \tfrac{h_2}{h_1} \to 0 \ \textit{as} \ \tfrac{L_0}{L} \to 0. \tag{2.23}$$

Estimates (2.23) show by how many times the applicability of the method is expanded if weaker h_1 or h_2 are used instead of h.

(c) Numerical examples where $L_0 < L$ can be found in [1]–[5].

3. SEMILOCAL CONVERGENCE OF (NT)

We present the semi-local convergence of $\{x^\delta_{n,\alpha}\}$ in this section. Let $U(x,r)$ and $\overline{U(x,r)}$ stand, respectively, for the open and closed ball in X with center x and radius $r > 0$. Let $L(X,Y)$ stand for the space of bounded linear operators from X into Y. We assume throughout this section that the following (C) conditions hold:

(C1) $F : D(F) \subseteq X \to Y$ is Frèchet differentiable

(C2) There exists $x_0 \in D(F)$ such that $F'(x_0)^{-1} \in L(Y,X)$ and

$$\|F'(x_0)^{-1}\| \le b$$

(C3) There exists $L_0 > 0$ such that F' satisfies the center-Lipschitz conditions

$$\|F'(x) - F'(x_0)\| \le L_0\|x - x_0\|$$

holds for all $x \in D(F)$.

(C4) There exists $L > 0$ such that F' satisfies the Lipschitz condition

$$\|F'(x) - F'(y)\| \le L\|x - y\|$$

holds for all x and y in $D(F)$.

(C5) $\|F'(x_0)^{-1}(F(x_0) - z^\delta_\alpha)\| \le r$

(C6) $h_2 = bL_2 r \le 1$, where $L_2 = \frac{1}{4}(4L_0 + \sqrt{3L_0L + 8L_0^2} + \sqrt{3L_0L})$ and

(C7) $\overline{U(x,r^*)} \subseteq D(F)$, where r^* is defined in (2.12).

We present the following semi-local convergence result for $\{x^\delta_{n,\alpha}\}$.

THEOREM 3.1. *Suppose that the conditions* (C1)–(C7) *hold. Then, the sequence* $\{x^\delta_{n,\alpha}\}$ *generated by* (NT) *is well defined, remains in* $\overline{U(x_0,r^*)}$ *for all* $n \ge 0$ *and converges to some* $x^\delta_\alpha \in \overline{U(x_0,r^*)}$ *such that* $F(x^\delta_\alpha) = z^\delta_\alpha$. *Moreover, the following estimates hold for each* $n = 0,1,2\ldots$

$$\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\| \le r^\delta_{n+1,\alpha} - r^\delta_{n,\alpha} \tag{3.24}$$

and

$$\|x^\delta_{n+1,\alpha} - x^\delta_\alpha\| \le r^* - r^\delta_{n,\alpha} \tag{3.25}$$

Proof. We use mathematical induction to prove that

$$\|x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}\| \le r^\delta_{k+1,\alpha} - r^\delta_{k,\alpha} \tag{3.26}$$

and

$$\overline{U(x^\delta_{k+1,\alpha}, r^* - r^\delta_{k+1,\alpha})} \subseteq \overline{U(x^\delta_{k\,\alpha}, r^* - r^\delta_{k+1,\alpha})} \tag{3.27}$$

for each $k = 0, 1, 2, \ldots$. Let $v \in \overline{U(x^\delta_{1,\alpha}, r^* - r^\delta_{1,\alpha})}$. Then, we obtain

$$\begin{aligned} \|v - x^\delta_{0,\alpha}\| &\leq \|v - x^\delta_{1,\alpha}\| + \|x^\delta_{1,\alpha} - x^\delta_{0,\alpha}\| \\ &\leq r^* - r^\delta_{1,\alpha} + r^\delta_{1,\alpha} - r^\delta_{0,\alpha} \\ &= r^* - r^\delta_{0,\alpha} \end{aligned}$$

which implies $v \in \overline{U(x^\delta_{0,\alpha}, r^* - r^\delta_{0,\alpha})}$. Note also that

$$\begin{aligned} \|x^\delta_{1,\alpha} - x^\delta_{0,\alpha}\| &= \|F'(x^\delta_{0,\alpha})^{-1}(F(x^\delta_{0,\alpha}) - z^\delta_\alpha)\| \\ &\leq r = r^\delta_{1,\alpha} - r^\delta_{0,\alpha}. \end{aligned}$$

Hence, estimates (3.26) and (3.27) hold for $k = 0$. Suppose that these estimates hold for $n \leq k$. Then, we have that

$$\begin{aligned} \|x^\delta_{k+1,\alpha} - x^\delta_{0,\alpha}\| &\leq \sum_{i=1}^{k+1} \|x^\delta_{i,\alpha} - x^\delta_{i-1,\alpha}\| \\ &\leq \sum_{i=1}^{k+1} (r^\delta_{i,\alpha} - r^\delta_{i-1,\alpha}) \\ &\leq r^\delta_{i+1,\alpha} \leq r^* \end{aligned}$$

and

$$\|x^\delta_{k,\alpha} + \theta(x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}) - x^\delta_{0,\alpha}\| \leq r^\delta_{k,\alpha} + \theta(r^\delta_{k+1,\alpha} - r^\delta_{k,\alpha}) - r^\delta_{0,\alpha} \leq r^*$$

for each $\theta \in [0, 1]$.

Using (C3), Lemmas 2.5 (see also Lemma 2.1 in [3]) and the induction hypotheses we get

$$\begin{aligned} \|F'(x^\delta_{n+1,\alpha}) - F'(x^\delta_{0,\alpha})\| &\leq L_0\|x^\delta_{k+1,\alpha} - x^\delta_{0,\alpha}\| \\ &\leq L_0(r^\delta_{k+1,\alpha} - r^\delta_{0,\alpha}) \\ &\leq L_0 r^\delta_{k+1,\alpha} < \tfrac{1}{b}. \end{aligned} \tag{3.28}$$

It follows from (3.28) and the Banach Lemma on invertible operators [1], [5] that $F'(x^\delta_{k+1,\alpha})^{-1} \in L(Y, X)$ and

$$\begin{aligned} ||F'(x^\delta_{k+1,\alpha})^{-1}\| &\leq \frac{b}{1 - bL_0\|x^\delta_{k+1,\alpha} - x^\delta_{0,\alpha}\|} \\ &\leq \frac{b}{1 - bL_0(r^\delta_{k+1,\alpha} - r^\delta_{0,\alpha})}. \end{aligned} \tag{3.29}$$

Using (NT) we obtain the approximation

$$\begin{aligned} F(x^\delta_{k+1,\alpha}) - z^\delta_\alpha &= F'(x^\delta_{k+1,\alpha})(x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}) + F(x^\delta_{k+1,\alpha}) - F(x^\delta_{k,\alpha}) \\ &\quad + [F'(x^\delta_{k,\alpha}) - F'(x^\delta_{k+1,\alpha})]??(x^\delta_{k+1,\alpha} - x^\delta_{k,\alpha}). \end{aligned} \tag{3.30}$$

In view of (C4), (2.8), (3.30), (C3) for $k=0$ and the induction hypotheses we get

$$\begin{aligned}\|F(x^\delta_{k+1,\alpha})-z^\delta_\alpha\| &\leq \tfrac{L}{2}\|x^\delta_{k+1,\alpha}-x^\delta_{k,\alpha}\|^2+L\|x^\delta_{k+1,\alpha}-x^\delta_{k,\alpha}\|^2\\ &= \tfrac{3L}{2}\|x^\delta_{k+1,\alpha}-x^\delta_{k,\alpha}\|^2\\ &\leq \tfrac{3L}{2}(r^\delta_{k+1,\alpha}-r^\delta_{k,\alpha})^2. \qquad (3.31)\end{aligned}$$

Moreover, by (NT), (2.8), (3.29) and (3.31) we get that

$$\begin{aligned}\|x^\delta_{k+2,\alpha}-x^\delta_{k+1,\alpha}\| &\leq \|F'(x^\delta_{k+1,\alpha})^{-1}\|\|F(x^\delta_{k+1,\alpha})-z^\delta_\alpha\|\\ &\leq \tfrac{b}{1-bL_0(r^\delta_{k+1,\alpha}-r^\delta_{0,\alpha})}\tfrac{3L}{2}(r^\delta_{k+1,\alpha}-r^\delta_{k,\alpha})^2\\ &= r^\delta_{k+2,\alpha}-r^\delta_{k,\alpha}\end{aligned}$$

which completes the induction for (3.26). Furthermore, let $w\in\overline{U(x^\delta_{k+2,\alpha},r^*-r^\delta_{k+2,\alpha})}$. Then, we have that

$$\begin{aligned}\|w-x^\delta_{k+1,\alpha}\| &\leq \|w-x^\delta_{k+2,\alpha}\|+\|x^\delta_{k+2,\alpha}-x^\delta_{k+1,\alpha}\|\\ &\leq r^*-r^\delta_{k+2,\alpha}+r^\delta_{k+21,\alpha}-r^\delta_{k+1,\alpha}\\ &= r^*-r^\delta_{k+1,\alpha}.\end{aligned}$$

That is $w\in\overline{U(x^\delta_{k+1,\alpha},r^*-r^\delta_{k+1,\alpha})}$. Lemma 2.6 implies that $\{r^\delta_{k,\alpha}\}$ is a complete sequence. It then follows from (3.26) and (3.27) that $\{x^\delta_{k,\alpha}\}$ is also complete sequence in the Hilbert space X and as such it converges to some $x^\delta_\alpha\in\overline{U(x_0,r^*)}$ (since $\overline{U(x_0,r^*)}$ is a closed set). By letting $k\to\infty$ in (3.31) we obtain $F(x^\delta_\alpha)=z^\delta_\alpha$. Estimate (3.25) is obtained from (3.24) by using standard majorization techniques [1], [5], [15]. The proof of the Theorem is complete. □

Remark 3.2. (a) If $L_0=L, r^\delta_{n,\alpha}=t^\delta_{n,\alpha}$ then Theorem 3.1 reduces to Theorem 3.3 in [17] (with corresponding changes). Otherwise, i.e., if $L_0<L$, according to Section 2 it constitutes an improvement with advantages as stated in the introduction of this study.

(b) Upper bounds on r^* and s^* in terms of L_0, L, b and n have been given in [1], [3]–[5] (see also (2.12)). these bounds are given in closed form and can certainly replace r^* in (C7).

(c) Note that $L_0\leq L$ holds in general and $\frac{L}{L_0}$ can be arbitrarily large [1], [3]–[5].

The rest of the results in [16] can be improved along the same lines by simply using, respectively, $\bar{r}^*, L_0$ instead of $\bar{t}^*, L$. In order for us to make the paper is self contained as possible we present the proof of one of them and for the proof of the rest we refer the reader to [16].

PROPOSITION 3.3. *Suppose that* (2.17), (C3) *and* (C4) *hold. Moreover, suppose that*

$$\|x_0 - x^*\| \le \bar{r}^* < r < \tfrac{1}{bL_0} = \lambda_0 \tag{3.32}$$

and

$$\overline{U(x_0,\lambda_0)} \subseteq D(F). \tag{3.33}$$

Then, the following assertion holds:

$$\|x^* - x_\alpha^\delta\| \le \tfrac{b}{1-bL_0 r}\|F(x^*) - z_\alpha^\delta\|. \tag{3.34}$$

Proof. Using (C3) (instead of (C4)) used in [16], we get that

$$\begin{aligned}
\|x^* - x_\alpha^\delta\| &= \|x^* - x_\alpha^\delta + F'(x_0)^{-1}[F(x_\alpha^\delta) - F(x^*) + F(x^*) - z_\alpha^\delta]\| \\
&\le \|F'(x_0)^{-1}[F'(x_0)(x^* - x_\alpha^\delta) - (F(x^*) - F(x_\alpha^\delta))]\| \\
&\quad + \|F'(x_0)^{-1}[F(x^*) - z_\alpha^\delta]\| \\
&\le bL_0 r\|x^* - x_\alpha^\delta\| + b\|F(x^*) - z_\alpha^\delta\|.
\end{aligned}$$

which shows (3.34). The proof of the proposition is complete. □

The following is a consequence of Theorem 3.1 and Proposition 3.3.

COROLLARY 3.4. *Suppose that hypotheses of Theorem* 3.1, $\bar{r}^* < r$ *and* $bL_0 r < 1$ *hold. Then, the following assertion holds*

$$\|x^* - x_{n,\alpha}^\delta\| \le \tfrac{b}{1-bL_0 r}\|F(x^*) - z_\alpha^\delta\| + r^* - r_{n,\alpha}^\delta$$

for each $n = 0, 1, 2, \ldots$.

REMARK 3.5. if $L_0 = L$ Proposition 3.3 and Corollory 3.4 reduce to the corresponding ones in [16]. Otherwise, i.e., $L_0 < L$ our results constitute an improvement.

4. SEMILOCAL CONVERGENCE OF (SNT)

We present the semilocal convergence of (SNT).

THEOREM 4.1. *Suppose that* (C1)–(C3), (C5) *hold. If, in addition,*

(C8) $h_0 = \tau b L_0 r < 1$
and
(C9) $\overline{U(x_0,\rho)} \subseteq D(F)$, *where* $\rho = \frac{1-\sqrt{1-h_0}}{bL_0}$,

then, sequence $\{x_{n,\alpha}^\delta\}$ *generated by* (SNT) *is well defined, remain in* $\overline{U(x_0,\rho)}$ *for all* $n \ge 0$ *and converges to some* $x_\alpha^\delta \in \overline{U(x_0,\rho)}$ *such that* $F(x_\alpha^\delta) = z_\alpha^\delta$. *Moreover the following estimate hold for each* $n = 0, 1, 2, \ldots$

$$\|x_{n+2,\alpha}^\delta - x_{n+1,\alpha}^\delta\| \le q\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\|$$

and

$$\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \le \tfrac{q^n r}{1-q},$$

where $q = 1 - \sqrt{1-h_0}$.

Proof. Let us define operator T on $\overline{U(x_0,\rho)}$ by

$$T(x) = x - F'(x_0)^{-1}(F(x) - z_\alpha^\delta).$$

Then, we shall show T is a contraction on $\overline{U(x_0,\rho)}$ and maps $\overline{U(x_0,\rho)}$ into itself. Indeed, we have that for $x, y \in \overline{U(x_0,\rho)}$

$$\begin{aligned} T(x) - T(y) &= x - y - F'(x_0)^{-1}(F(x) - z_\alpha^\delta) + F'(x_0)^{-1}(F(y) - z_\alpha^\delta) \\ &= -F'(x_0)^{-1}(F(x) - F(y) - F'(x_0)(x-y)) \end{aligned}$$

and

$$\begin{aligned} \|T(x) - T(y)\| &= \|F'(x_0)^{-1}(F(x) - F(y) - F'(x_0)(x-y))\| \\ &\le bL_0\rho\|x-y\|. \end{aligned}$$

But, we have $bL_0\rho < 1$. Hence, T is a contraction operator. Let $x \in \overline{U(x_0,\rho)}$. Then, we have that

$$T(x) - x_0 = T(x) - T(x_0) + T(x_0) - x_0$$

and

$$\begin{aligned} \|T(x) - x_0\| &\le \|T(x) - T(x_0)\| + \|T(x_0) - x_0\| \\ &= \frac{bL_0\rho^2}{2+r} = \rho \end{aligned}$$

by the choice of ρ. The proof of Theorem is complete. □

Remark 4.2. (a) More subtle arguments show that $h_0 < 1$ can be replaced by $h_0 \le 1$ (see [1], [5]).

(b) We have that $h^0 = 2bLr \le 1 \Rightarrow h_0 \le 1$ but not vice versa even if $L_0 = L$. Moreover, we have that $\frac{h_0}{h^0} \to 0$ as $\frac{L_0}{L} \to 0$. Note that the h^0 condition was given in [17]. Therefore method (SNT) can be used as a predictor until a certain finite iterate N such that $h \le 1$ holds for $x_{N,\alpha}^\delta$, being the initial point of method (NT). Such an approach has been used by us in [2] for modified Newton and Newton's method.

5. APPLICATIONS

Let us consider the nonlinear Hammerstein operator equation (c.f. [28])

$$(MFx)(t) = \int_0^1 m(s,t)p(s,x(s))x(s)ds$$

where m is continuous and p is differentiable with respect to the second variable. Define $F : D(F) = H^1(]0,1[) \to L^2(]0,1[)$ by

$$F(x)(s) = p(s,x(s)), \quad s \in [0,1]$$

and $M : L^2(]0,1[) \to L^2(]0,1[)$ by

$$Mu(t) = \int_0^1 m(s,t)u(s)ds, \quad t \in [0,1].$$

Then, F is Frèchet differentiable and we have that

$$[F'(x)]u(t) = \partial_2 p(t, x(t))u(t), \qquad t \in [0, 1].$$

If in addition $M_1 : H^1(]0,1[) \mapsto H^1(]0,1[)$ is defined by $(M_1 x)(t) := \partial_2 p(t, x(t))$ is locally Lipschitz continuous, one can compute the required constants L_0 and L. If we further assume the existence of a constant $\kappa_1 > 0$ such that $\partial_2 p(t, x(t)) \geq \kappa_1$ for all $t \in [0, 1]$ and $x(t) \in U(x_0, r^*)$, then $F'(x)^{-1}$ exists and is bounded operator.

Equation (1.2) is equivalent to

$$M[F(x) - F(x_0)] = y - MF(x_0) \tag{5.35}$$

where x_0, is an initial guess. Therefore the solution x of (1.2) is obtained by first solving

$$Mz = y - KF(x_0) \tag{5.36}$$

for z and then solving

$$F(x) = z + F(x_0) \tag{5.37}$$

for $x \in D(F)$. Let $\alpha > 0$, $\delta > 0$ be fixed. Then, we consider the regularized solution of (5.36) with y^δ in place of y as

$$z_\alpha^\delta = (M + \alpha I)^{-1}(y^\delta - MF(x_0)) + F(x_0) \tag{5.38}$$

if case M in (5.36) is positive self adjoint and $Z = Y$. Otherwise, we set

$$z_\alpha^\delta = (M^*M + \alpha I)^{-1}M^*(y^\delta - MF(x_0)) + F(x_0). \tag{5.39}$$

Note that (5.38) is the simplified or Lavrentiev regularization of equation (5.36) and (5.39) is the Tikhonov regularization of (5.36).

With these choices of operators the rest of the results in [16] involving (SNT) type methods (i.e., using L_0 instead of L) can be improved.

Proposition 5.1. *Suppose z_α^δ is given by* (5.39) *and*

$$\|F(x_0) - F(x^*)\| + \tfrac{\delta}{\sqrt{\alpha}} < \tfrac{r}{2b} < \tfrac{1}{2b^2 L_0}.$$

Then, the following assertion holds

$$\|x_0 - x^*\| \leq \bar{r}^* < r < \tfrac{1}{bL_0}.$$

Remark 5.2. (a) If $L_0 = L$ Proposition 5.1 reduces to Remark 3.4 in [16]. Otherwise, Proposition 5.1 improves Remark 3.4 in [16].

(b) The rest of the results in the literature (see, e.g. [13]–[20]) can be extended by simply using (C3) instead of (C4). Note also that there are examples where (C3) holds but not (C4).

Remark 5.3. Hereafter we consider z_α^δ as the Tikhonov regularization of (5.36) given in (5.39). All results in the forthcoming sections are valid for the simplified regularization of (5.36).

In view of the estimate in the Corollory 3.4, the next task is to find an estimate $\|F(x^*) - z_\alpha^\delta\|$. For this, let us introduce the notation;

$$z_\alpha := F(x_0) + (M^*M + \alpha I)^{-1} M^*(y - MF(x_0)).$$

We may observe that

$$\begin{aligned} \|F(x^*) - z_\alpha^\delta\| &\le \|F(x^*) - z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \\ &\le \|F(x^*) - z_\alpha\| + \tfrac{\delta}{\sqrt{\alpha}}, \end{aligned} \tag{5.40}$$

and

$$\begin{aligned} F(x^*) - z_\alpha &= F(x^*) - F(x_0) - (M^*M + \alpha I)^{-1} M^*M[F(x^*) - F(x_0)] \\ &= [I - (M^*M + \alpha I)^{-1} M^*M][F(x^*) - F(x_0)] \\ &= \alpha(M^*M + \alpha I)^{-1}[F(x^*) - F(x_0)]. \end{aligned} \tag{5.41}$$

Note that for $u \in R(M^*M)$ with $u = M^*Mz$ for some $z \in Z$,

$$\|\alpha(M^*M + \alpha I)^{-1} u\| = \|\alpha(M^*M + \alpha I)^{-1} M^*Mz\| \le \alpha\|z\| \to 0$$

as $\alpha \to 0$. Now since $\|\alpha(M^*M + \alpha I)^{-1}\| \le 1$ for all $\alpha > 0$, it follows that for every $u \in \overline{R(M^*M)}$, $\|\alpha(M^*M + \alpha I)^{-1} u\| \to 0$ as $\alpha \to 0$. Thus we have the following theorem.

Theorem 5.4. *If $F(x^*) - F(x_0) \in \overline{R(M^*M)}$, then $\|F(x^*) - z_\alpha\| \to 0$ as $\alpha \to 0$.*

5.1. Error bounds under source conditions. In view of the above theorem, we assume that

$$\|F(x^*) - z_\alpha\| \le \varphi(\alpha) \tag{5.42}$$

for some positive monotonic increasing function φ defined on $(0, \|M\|^2]$ such that

$$\lim_{\lambda \to 0} \varphi(\lambda) = 0.$$

Suppose φ is a source function in the sense that x^* satisfies a source condition of the form

$$F(x^*) - F(x_0) = \varphi(M^*M)w, \quad \|w\| \le 1,$$

such that

$$\sup_{0<\lambda<\|M\|^2} \tfrac{\alpha\varphi(\lambda)}{\lambda+\alpha} \le \varphi(\alpha), \tag{5.43}$$

then the assumption (5.42) is satisfied. Note that if $F(x^*) - F(x_0) \in R((M^*M)^\nu)$, for some ν with, $0 < \nu \le 1$, then by (5.41)

$$\begin{aligned} \|F(x^*) - z_\alpha\| &\le \|\alpha(M^*M + \alpha I)^{-1}(M^*M)^\nu \omega\| \\ &\le \sup_{0<\lambda\le\|M\|^2} \frac{\alpha\lambda^\nu}{\lambda + \alpha}\|\omega\| \le \alpha^\nu\|\omega\|. \end{aligned}$$

Thus in this case $\varphi(\lambda) = \frac{\lambda^\nu}{\|\omega\|}$ satisfies the assumption (5.42). Therefore by (5.40) and by the assumption (5.42), we have

$$\|F(x^*) - z_\alpha^\delta\| \leq \varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}. \tag{5.44}$$

So, we have the following theorem.

THEOREM 5.5. *Under the assumptions of Corollary* 3.4 *and* (5.44),

$$\|x^* - x_{n,\alpha}^\delta\| \leq \frac{b}{1-b\kappa_0 r}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}) + r^* - r_{n,\alpha}^\delta.$$

5.2. A priori choice of the parameter. Note that the estimate $\varphi(\alpha)+\frac{\delta}{\sqrt{\alpha}}$ in (5.43) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|M\|^2$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{5.45}$$

So the relation (5.44) leads to

$$\|F(x^*) - z_\alpha^\delta\| \leq 2\psi^{-1}(\delta).$$

Theorem 5.5 and the above observation leads to the following.

THEOREM 5.6. *Let* $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ *and the assumptions of Corollary* 3.4 *and* (5.42) *are satisfied. For* $\delta > 0$, *let* $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. *If*

$$n_\delta := \min\left\{n : (r^* - r_{n,\alpha}^\delta) < \frac{\delta}{\sqrt{\alpha_\delta}}\right\},$$

then

$$\|x^* - x_{\alpha_\delta,n_\delta}^\delta\| = O(\psi^{-1}(\delta)), \ as \ \delta \to 0.$$

5.3. An adaptive choice of the parameter. The error estimate in the above Theorem has optimal order with respect to δ. Unfortunately, an a priori parameter choice (5.45) cannot be used in practice since the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally unknown. There exist many parameter choice strategies in the literature, for example see [6], [11], [12], [18], [19], [31] and [33].

In [26], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter α_i are selected from some finite set $\{\alpha_i : 0 < \alpha_0 < \alpha_1 < \ldots < \alpha_N\}$ and the corresponding regularized solution, say $u_{\alpha_i}^\delta$ are studied on-line. Later George and Nair [20] considered the adaptive selection of the parameter for choosing the regularization parameter in Newton-Lavrentiev regularization method for solving Hammerstein-type operator equation. In this paper also, we consider the adaptive method for selecting the parameter α in $x_{\alpha,n}^\delta$. The rest of this section is essentially a reformulation of the adaptive method considered in [26] in a special context.

Let $i \in \{0,1,2,\ldots,N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta^2$. Let

$$l := \max\left\{i : \varphi(\alpha_i) \le \tfrac{\delta}{\sqrt{\alpha_i}}\right\} \text{ and} \tag{5.46}$$

$$k := \max\left\{i : \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \le \tfrac{4\delta}{\sqrt{\alpha_j}}, j = 0,1,2,\ldots,i\right\}. \tag{5.47}$$

The proof of the next theorem is analogous to the proof of Theorem 1.2 in [26], but for the sake of completeness, we supply its proof as well.

THEOREM 5.7. *Let l be as in* (5.46)*, k be as in* (5.47) *and $z^\delta_{\alpha_k}$ be as in* (5.39) *with $\alpha = \alpha_k$. Then $l \le k$ and*

$$\|F(x^*) - z^\delta_{\alpha_k}\| \le (2 + \tfrac{4\mu}{\mu-1})\mu\psi^{-1}(\delta).$$

Proof. Note that, to prove $l \le k$, it is enough to prove that, for $i = 1,\ldots,N$

$$\varphi(\alpha_i) \le \tfrac{\delta}{\sqrt{\alpha_i}} \Longrightarrow \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \le \tfrac{4\delta}{\sqrt{\alpha_j}}, \quad \forall j = 0,1,2,\ldots,i.$$

For $j \le i$,

$$\begin{aligned}
\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| &\le \|z^\delta_{\alpha_i} - F(\hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_j}\| \\
&\le \varphi(\alpha_i) + \tfrac{\delta}{\sqrt{\alpha_i}} + \varphi(\alpha_j) + \tfrac{\delta}{\sqrt{\alpha_j}} \\
&\le \tfrac{2\delta}{\sqrt{\alpha_i}} + \tfrac{2\delta}{\sqrt{\alpha_j}} \le \tfrac{4\delta}{\sqrt{\alpha_j}}.
\end{aligned}$$

This proves the relation $l \le k$. Now since $\sqrt{\alpha_{l+m}} = \mu^m\sqrt{\alpha_l}$, by using triangle inequality successively, we obtain

$$\begin{aligned}
\|F(\hat{x}) - z^\delta_{\alpha_k}\| &\le \|F(x^*) - z^\delta_{\alpha_l}\| + \sum_{j=l+1}^{k} \tfrac{4\delta}{\sqrt{\alpha_{j-1}}} \\
&\le \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \sum_{m=0}^{k-l-1} \tfrac{4\delta}{\sqrt{\alpha_l}\mu^m} \\
&\le \|F(\hat{x}) - z^\delta_{\alpha_l}\| + (\tfrac{\mu}{\mu-1})\tfrac{4\delta}{\sqrt{\alpha_l}}.
\end{aligned}$$

Therefore by (5.43) and (5.46) we have

$$\|F(x^*) - z^\delta_{\alpha_k}\| \le \varphi(\alpha_l) + \tfrac{\delta}{\sqrt{\alpha_l}} + (\tfrac{\mu}{\mu-1})\tfrac{4\delta}{\sqrt{\alpha_l}} \le (2 + \tfrac{4\mu}{\mu-1})\mu\psi^{-1}(\delta).$$

The last step follows from the inequality $\sqrt{\alpha_\delta} \le \sqrt{\alpha_{l+1}} \le \mu\sqrt{\alpha_l}$ and $\frac{\delta}{\sqrt{\alpha_\delta}} = \psi^{-1}(\delta)$. This completes the proof. □

5.4. **Stopping Rule.**

Note that

$$\begin{aligned}
e_0 = \|x^\delta_{1,\alpha} - x_0\| &= \|F'(x_0)^{-1}(M^*M + \alpha I)^{-1}M^*(y^\delta - MF(x_0))\| \\
&= \|F'(x_0)^{-1}(M^*M + \alpha I)^{-1}M^*(y^\delta - y + y - MF(x_0))\| \\
&\le b(\|(M^*M + \alpha I)^{-1}M^*(y^\delta - y)\| \\
&\quad + \|(M^*M + \alpha I)^{-1}M^*M(F(\hat{x}) - F(x_0))\|)
\end{aligned}$$

$$\leq \quad b(\omega + \tfrac{\delta}{\sqrt{\alpha}}),$$

so if

$$b(\omega + \tfrac{\delta}{\sqrt{\alpha}}) < \tfrac{1}{bL_2}, \tag{5.48}$$

and $r^* < r$. Then hypotheses of Theorem 3.1 hold. Again since $\alpha_j = \mu^{2j}\delta^2$, $\frac{\delta}{\sqrt{\alpha_k}} = \mu^{-k}$; the condition (5.48) with $\alpha = \alpha_k$ takes the form

$$b(\omega + \tfrac{1}{\mu^k}) < \tfrac{1}{bL_2}. \tag{5.49}$$

and $r^* < r$. Then, we have arrived at the algorithm which guarantees

$$\|x^* - x^\delta_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta)),\ as\ \delta \to 0$$

with $n_k = \min\{n : r^* - r^\delta_{n,\alpha} \leq \mu^{-j}, j = 1, 2, \ldots, i-1\}$ and improves the corresponding one in [16].

5.5. Algorithm: Note that for $i, j \in \{0, 1, 2, \ldots, n\}$

$$\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| = (\alpha_j - \alpha_i)(M^*M + \alpha_j I)^{-1}(M^*M + \alpha_i I)^{-1}M^*(y^\delta - MF(x_0)).$$

Therefore the adaptive algorithm associated with the choice of the parameter specified in the above theorem is as follows.

```
begin
    i=0
    repeat
        i=i+1
        Solve for w_i : (M*M + α_i I)w_i = M*(y^δ − MF(x_0))
        j=-1
        repeat
            j=j+1
            Solve for z_{i,j} : (M*M + α_j I)z_{i,j} = (α_j − α_i)w_i
        until (‖z_{i,j}‖ ≤ 4μ^{−j} AND j < i)
    until (‖z_{i,j}‖ ≤ 4μ^{−j})
    k=i-1.
    m=0
    repeat
        m=m+1
    until ((r* − r^δ_{m,α}) > 1/μ^k)
    n_k = m
    for l=1 to n_k
        Solve for u_{l−1} : F'(x^δ_{l−1,α_k})u_{l−1} = F(x^δ_{l−1,α_k}) − z^δ_{α_k}
        x^δ_{l,α_k} := x^δ_{l−1,α_k} − u_{l−1}
end
```

6. NUMERICAL EXAMPLE

In this section we consider a example for illustrating the algorithm mentioned in the above section.

EXAMPLE 6.1. In this example we consider the operator $KF : L^2(0,1) \longrightarrow L^2(0,1)$ where $K : L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$K(x)(t) = \int_0^1 k(t,s)x(s)ds$$

and $F : D(F) \subseteq H^1(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, 0 \le s \le t \le 1 \\ (1-s)t, 0 \le t \le s \le 1 \end{cases}.$$

The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds.$$

In our computation, we take $f(t) = \frac{1}{110}(\frac{t^{13}}{156} - \frac{t^3}{6} + \frac{25t}{156})$ and $f^\delta = f + \delta$ with $\delta = 0.01$. Then the exact solution

$$\hat{x}(t) = t^3.$$

We use

$$x_0(t) = t^3 + \tfrac{3}{56}(t - t^8)$$

as our initial guess. The results of the computation are presented in Table 1.

n	k	α_k	e_n	$\frac{e_n}{\psi^{-1}(\delta))}$
64	4	0.0011	0.5257	5.2541
128	4	0.0011	0.5234	5.2331
256	4	0.0011	0.5222	5.2216
512	4	0.0011	0.5216	5.2156
1024	4	0.0011	0.5211	5.2126
2048	4	0.0011	0.5211	5.2110
4096	4	0.0011	0.5210	5.2102

Table 1. Iterations and corresponding Error Estimates of Example 6.1

REMARK 6.2. We have considered an iterative regularization method, which is a combination of Newton iterative method with a Tikhonov regularization method, for obtaining approximate solution for a nonlinear Hammerstein-type operator equation $MF(x) = y$, with the available data y^δ in place of the exact data y. If the operator M is a positive self-adjoint bounded linear operator on a Hilbert space, then one may consider Newton Lavrentiev regularization method for obtaining an approximate solution for $MF(x) = y$. It is assumed that the Frèchet derivative $F'(x)$ of the non-linear operator F has a continuous inverse, in a neighborhood of some initial guess x_0 of the actual solution x^*. The procedure involves solving the equation

$$(M^*M + \alpha I)u_\alpha^\delta = M^*(y^\delta - MF(x_0))$$

and finding the fixed point of the function

$$G(x) = x - F'(x)^{-1}(F(x) - F(x_0) - u_\alpha^\delta)$$

in an iterative manner. For choosing the regularization parameter α and the stopping index for the iteration, we made use of the adaptive method suggested in [26]. □

REFERENCES

[1] I.K. Argyros, *Convergence and Application of Newton-type Iterations*, Springer, 2008.

[2] I.K. Argyros, *Approximating solutions of equations using Newton's method with a modified Newton's method iterate as a starting point*, Rev. Anal. Numér. Théor. Approx., **36** (2007), pp. 123–138.

[3] I.K. Argyros, *A Semilocal convergence for directional Newton methods*, Math. Comp. **80**, (2011), pp. 327–343.

[4] I.K. Argyros and S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity, **28**, (2012), pp. 364–387.

[5] I.K. Argyros, Y.J. Cho and S. Hilout, *Numerical Methods for Equations and its Applications*, CRC Press, Taylor and Francis, New York, 2012.

[6] A. Bakushinsky and A. Smirnova, *On application of generalized discrepancy principle to iterative methods for nonlinear ill-posed problems*, Numer. Funct. Anal. Optim. **26** (2005), pp. 35–48.

[7] A. Binder, H.W. Engl, and S. Vessela, *Some inverse problems for a nonlinear parabolic equation connected with continuous casting of steel: stability estimate and regularization*, Numer. Funct. Anal. Optim., **11** (1990), pp. 643–671.

[8] H.W. Engl, M. Hanke and A. Neubauer, *Tikhonov regularization of nonlinear differential equations*, Inverse Methods in Action, P.C. Sabatier, ed., Springer-Verlag, New York 1990, pp. 92–105.

[9] H.W. Engl, K. Kunisch and A. Neubauer, *Convergence rates for Tikhonov regularization of nonlinear ill-posed problems*, Inverse Problems, **5** (1989), pp. 523–540.

[10] C.W. Groetsch, *Theory of Tikhonov Regularization for Fredholm Equation of the First Kind*, Pitmann Books, London, 1984.

[11] C.W. Groetsch and J.E. Guacaneme, *Arcangeli's method for Fredhom equations of the first kind*, Proc. Amer. Math. Soc. **99** (1987), pp. 256–260.

[12] J.E. Guacaneme, *Aparameter choice for simplified regularization*, Rostock. Math. Kolloq., **42** (1990), pp. 59–68.

[13] S. George, *Newton-Tikhonov regularization of ill-posed Hammerstein operator equation*, J. Inverse Ill-Posed Probl., **14** (2006), no. 2, pp. 135-146.

[14] S. George, *Newton-Lavrentiev regularization of ill-posed Hammerstein type operator equation*, J. Inverse Ill-Posed Probl., **14**, (2006), no. 6, pp. 573-582.

[15] S. George and A.I. Elmahdy, *A quadratic convergence yielding iterative method for nonlinear ill-posed operator equations*, Comput. Methods Appl. Math. **12**, no. 1 (2012), pp. 32–45.

[16] S. George and M. Kunhanandan, *An iterative regularization method for Ill-posed Hammerstein type operator equation*, J. Inverse Ill-Posed Probl., **17** (2009), pp. 831–844.

[17] S. George and M. Kunhanandan, *Iterative regularization methods for ill-posed Hammerstein type operator equation with monotone nonlinear part*, Int. J. Math. Anal., **4** (2010), no. 34, pp. 1673–1685.

[18] S. George and M.T. Nair, *An a posteriori parameter choice for simplified regularization of ill-posed problems*, Integral Equations Operator Theory, **16** (1993), pp. 392–399.

[19] S. George and M.T. Nair, *On a generalized Arcangeli's method for Tikhonov regularization with inexact data*, Numer. Funct. Anal. Optim., **19** (1998), (nos. 7–8), pp. 773–787.

[20] S. George and M.T. Nair, *A modified Newton-Lavrentiev regularization for nonlinear ill-posed Hammerstein-Type operator equation*, J. Complexity, **24** (2008), pp. 228–240.

[21] M. Hanke, A. Neubauer and O. Scherzer, *A convergence analysis of Landweber iteration of nonlinear ill-posed problems*, Numer. Math., **72** (1995), pp. 21–37.

[22] Jin Qi-nian and Hou Zong-yi, *Finite-dimensional approximations to the solutions of nonlinear ill-posed problems*, Appl. Anal., **62** (1996), pp. 253–261.

[23] Jin Qi-nian and Hou Zong-yi, *On an a posteriori parameter choice strategy for Tikhonov regularization of nonlinear ill-posed problems*, Numer. Math., **83** (1999), pp. 139–159.

[24] L.V. Kantorovich and G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1964.

[25] B.A. Mair, *Tikhonov regularization for finitely and infinitely smoothing operators*, SIAM J. Math. Anal., **25** (1994), pp. 135–147.

[26] S. Pereverzev and E. Schock, *On the adaptive selection of the parameter in regularization of ill-posed problems*, SIAM J. Numer. Anal., **43** (2005), no. 5, pp. 2060–2076.

[27] O. Scherzer, *A parameter choice for Tikhonov regularization for solving nonlinear inverse problems leading to optimal rates*, Appl. Math., **38** (1993), pp. 479–487.

[28] O. Scherzer, H.W. Engl and K. Kunisch, *Optimal a posteriori parameter choice for Tikhonov regularization for solving nonlinear ill-posed problems*, SIAM J. Numer. Anal., **30** (1993), no.6, pp. 1796–1838.

[29] O. Scherzer, *The use of Tikhonov regularization in the identification of electrical conductivities from overdetermined boundary data*, Results Mathematics, **22** (1992), pp. 599–618.

[30] O. Scherzer, H.W. Engl and R.S. Anderssen, *Parameter identification from boundary measurements in parabolic equation arising from geophysics*, Nonlinear Anal., **20** (1993), pp. 127-156.

[31] T. Raus, *On the discrepancy principle for the solution of ill-posed problems*, Uch. Zap. Tartu. Gos. Univ., **672** (1984), pp. 16–26 (In Russian).

[32] E. Schock, *On the asymptotic order of accuracy of Tikhonov regularization*, J. Optim. Theory Appl., **44** (1984), pp. 95-104.

[33] U. Tautenhahn, *On the method of Lavrentiev regularization for nonlinear ill-posed problems*, Inverse Problems, **18** (2002), pp. 191-207.

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