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SOME PROPERTIES OF THE OPERATORS DEFINED BY LUPAŞ

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Abstract. In the present paper, we show that a subclass of the operators defined by Lupaş [12] preserve properties of the modulus of continuity function and Lipschitz constant and the order of a Lipschitz continuous function. We also concerned with the monotonicity of sequence of such operators for convex functions.

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1. INTRODUCTION

By means of the identity

$$\tfrac{1}{(1-a)^\alpha} = \sum_{k=0}^{\infty} \tfrac{(\alpha)_k}{k!} a^k, \quad |a| < 1,$$

where $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, $k \geq 1$, Lupaş [12] proposed the positive linear operators

$$T_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \tfrac{(nx)_k}{k!} f\left(\tfrac{k}{n}\right) a^k, \quad x \geq 0$$

for the functions $f : [0,\infty) \to \mathbb{R}$ and $n \in \mathbb{N}$. After that Agratini [3], by choosing $a = \frac{1}{2}$, for the operators

$$L_n(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \tfrac{(nx)_k}{2^k k!} f\left(\tfrac{k}{n}\right) \tag{1.1}$$

obtained some estimates to the order of approximation on a finite interval and proved a Voronovskaya type theorem. Furthermore, he again derived the positive linear operators L_n *via* a probabilistic approach and presented the Kantorovich and Durrmeyer variants of these operators. In [5], a better error estimation and statistical Korovkin type approximation properties of the

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operators L_n were examined by Dirik. Jain and Pethe [9], as a generalization of Szasz-Mirakjan operators, introduced the operators

$$M_{n,\alpha}(f;x) = (1+n\alpha)^{-\frac{x}{\alpha}} \sum_{\nu=0}^{\infty} \left(\alpha + \tfrac{1}{n}\right)^{-\nu} \tfrac{x^{(\nu,-\alpha)}}{\nu!} f\left(\tfrac{\nu}{n}\right), \quad x \geq 0$$

where $x^{(0,-\alpha)} = 1$, $x^{(\nu,-\alpha)} = x(x+\alpha)\cdots(x+(\nu-1)\alpha)$, $0 \leq n\alpha \leq 1$ and $n \in \mathbb{N}$. By setting $c = c_n = \frac{1}{n\alpha}$ such that $c \geq \beta$ for certain constant $\beta > 0$, Abel and Ivan [1] expressed these operators in the equivalent form

$$S_{n,c}(f;x) = \sum_{\nu=0}^{\infty} P_{n,\nu}^{[c]}(x) f\left(\tfrac{\nu}{n}\right), \quad x \geq 0$$

where $P_{n,\nu}^{[c]}(x) = \left(\frac{c}{1+c}\right)^{ncx} \binom{ncx+\nu-1}{\nu} (1+c)^{-\nu}$, and studied their local approximation properties and also obtained a complete asymptotic expansion formula. We remark that when $c = 1$ the operators $S_{n,c}$ reduce to the operators defined by (1.1). In [6], Erençin and Taşdelen introduced the following generalization of the operators L_n

$$L_n^*(f;x) = 2^{-a_n x} \sum_{k=0}^{\infty} \tfrac{(a_n x)_k}{2^k k!} f\left(\tfrac{k}{b_n}\right), \qquad x \geq 0$$

where (a_n), (b_n) are increasing and unbounded sequences of positive numbers such that

$$\tfrac{a_n}{b_n} = 1 + \mathcal{O}\left(\tfrac{1}{b_n}\right), \quad \lim_{n\to\infty} \tfrac{1}{b_n} = 0$$

and investigated their weighted approximation properties. Later, Erençin and Taşdelen [7] estimated the rate of convergence for the Kantorovich type version of the operators L_n^* by means of the modulus of continuity, elements of local Lipschitz class and Peetre's K-functional. Recently, A-statistical convergence properties of the operators L_n and their Kantorovich type modification were studied by Tarabie in [14].

Note that from Lemma 1 in [3] we have

$$L_n(1;x) = 1,$$
$$L_n(t;x) = x.$$

In this paper, for the operators L_n defined by (1.1), we firstly show that when f is a general function of modulus of continuity, then $L_n(f;x) := L_n(f)$ is also a function of modulus of continuity with the help of the same technique of Li [11]. Later, we also show that the operators L_n preserve the Lipschitz constant and the order of a Lipschitz continuous function. Furthermore, we discuss the monotonicity of the operators L_n for n under the convexity of f. We note that in the literature there are a number of papers containing preservation properties of positive linear operators. Some of them are [2], [4], [8], [10] and [15].

2. SOME PROPERTIES OF THE OPERATORS L_n

In order to give some properties of the operators defined by (1.1) let us recall some definitions.

Let f be a real valued continuous function defined on $[0,\infty)$. Then f is said to be Lipschitz continuous of order γ $(0<\gamma\le 1)$ on $[0,\infty)$, if there exists $M>0$ such that

$$|f(x)-f(y)|\le M\,|x-y|^{\gamma}$$

for all $x,y\in[0,\infty)$. The set of Lipschitz continuous functions of order γ with Lipschitz constant M is denoted by $\mathrm{Lip}_M(\gamma)$.

A real valued continuous function f is said to be convex on $[0,\infty)$, if

$$f\Big(\sum_{i=1}^{n}\alpha_i t_i\Big)\le\sum_{i=1}^{n}\alpha_i f(t_i)$$

for all $t_1,t_2,\cdots,t_n\in[0,\infty)$ and for all non-negative numbers $\alpha_1,\alpha_2,\cdots,\alpha_n$ such that $\alpha_1+\alpha_2+\cdots+\alpha_n=1$.

Also, a continuous and non-negative function ω defined on $[0,\infty)$ is called the modulus of continuity function, if each of the following conditions is satisfied:

a) $\omega(u+v)\le\omega(u)+\omega(v)$ for $u,v\in[0,\infty)$, i.e., ω is subadditive,
b) $\omega(u)\ge\omega(v)$ for $u\ge v$, i.e., ω is non-decreasing,
c) $\lim_{u\to 0^+}\omega(u)=\omega(0)=0$

(see p. 106 in [13]).

Theorem 1. *If ω is a modulus of continuity function, then $L_n(\omega)$ is also a modulus of continuity function.*

Proof. Let $x,y\in[0,\infty)$ and $x\le y$. Then we have

$$L_n(f;y)=2^{-ny}\sum_{k=0}^{\infty}\tfrac{(ny)_k}{2^k k!}f\left(\tfrac{k}{n}\right)=2^{-ny}\sum_{k=0}^{\infty}\tfrac{(n(x+(y-x)))_k}{2^k k!}f\left(\tfrac{k}{n}\right).$$

Since

$$(n(x+(y-x)))_k=\sum_{i=0}^{k}\tbinom{k}{i}(nx)_i(n(y-x))_{k-i}$$

one may write

$$L_n(f;y)=2^{-ny}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\tfrac{1}{2^k k!}\tbinom{k}{i}(nx)_i(n(y-x))_{k-i}f\left(\tfrac{k}{n}\right).$$

Changing the order of the above summations and then taking $k-i=j$, we reach to

$$L_n(f;y)=2^{-ny}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\tfrac{1}{2^k k!}\tbinom{k}{i}(nx)_i(n(y-x))_{k-i}f\left(\tfrac{k}{n}\right)= \tag{2.1}$$

$$= 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} f\left(\frac{i+j}{n}\right).$$

On the other hand using the identity $2^{n(y-x)} = \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!}$, we get

$$\begin{aligned} L_n(f;x) =& 2^{-nx} \sum_{i=0}^{\infty} \frac{(nx)_i}{2^i i!} f\left(\frac{i}{n}\right) \\ =& 2^{-ny} 2^{n(y-x)} \sum_{i=0}^{\infty} \frac{(nx)_i}{2^i i!} f\left(\frac{i}{n}\right) \\ =& 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} f\left(\frac{i}{n}\right). \end{aligned} \tag{2.2}$$

Hence from (2.1) and (2.2) it follows that

$$L_n(f;y) - L_n(f;x) = 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} \left[f\left(\frac{i+j}{n}\right) - f\left(\frac{i}{n}\right) \right]. \tag{2.3}$$

Thus by means of the equality (2.3) and the subadditivity of ω, we can write

$$\begin{aligned} L_n(\omega;y) - L_n(\omega;x) =& 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} \left[\omega\left(\frac{i+j}{n}\right) - \omega\left(\frac{i}{n}\right) \right] \\ \leq& 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} \omega\left(\frac{j}{n}\right) \\ =& 2^{-nx} \sum_{i=0}^{\infty} \frac{(nx)_i}{2^i i!} 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \omega\left(\frac{j}{n}\right) \\ =& L_n(1;x) 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \omega\left(\frac{j}{n}\right) \\ =& 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \omega\left(\frac{j}{n}\right) \\ =& L_n(\omega; y-x). \end{aligned}$$

This shows the subadditivity of $L_n(\omega)$. We also infer from (2.3) that $L_n(\omega;y) \geq L_n(\omega;x)$ for $y \geq x$ which means that $L_n(\omega)$ is non-decreasing. Finally the property $L_n(\omega;0) = \omega(0) = 0$ is clear. Thus we may conclude that $L_n(\omega)$ is a modulus of continuity function. □

Now we introduce the second result of this section with the following theorem.

THEOREM 2. *If* $f \in \mathrm{Lip}_M(\gamma)$, *then* $L_n(f) \in \mathrm{Lip}_M(\gamma)$.

Proof. Suppose that $x \leq y$. By using the facts $f \in \mathrm{Lip}_M(\gamma)$ and $L_n(1,x) = 1$ from (2.3) we can write

$$\begin{aligned} |L_n(f;y) - L_n(f;x)| &\leq M 2^{-ny} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(nx)_i}{2^i i!} \frac{(n(y-x))_j}{2^j j!} \left(\frac{j}{n}\right)^{\gamma} \\ &= M 2^{-nx} \sum_{i=0}^{\infty} \frac{(nx)_i}{2^i i!} 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \left(\frac{j}{n}\right)^{\gamma} \\ &= M L_n(1,x) 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \left(\frac{j}{n}\right)^{\gamma} \\ &= M 2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \left(\frac{j}{n}\right)^{\gamma}. \end{aligned}$$

Now applying Hölder's inequality one gets

$$|L_n(f;y) - L_n(f;x)| \leq M \left(2^{-n(y-x)} \sum_{j=0}^{\infty} \frac{(n(y-x))_j}{2^j j!} \frac{j}{n} \right)^{\gamma} = M \left(L_n(t; y-x) \right)^{\gamma}.$$

Since $L_n(t;x) = x$ the above inequality implies that

$$|L_n(f;y) - L_n(f;x)| \leq M(y-x)^{\gamma}.$$

Similarly, it can be shown that when $x > y$ our claim is valid. □

Now, we will study the monotonicity of the sequence of the operators $L_n(f;x)$ defined by (1.1) when the function f is convex.

THEOREM 3. *If f is a convex function defined on $[0,\infty)$, then $L_n(f;x)$ is strictly monotonically decreasing, unless f is the linear function (in which case $L_n(f;x) = L_{n+1}(f;x)$ for all n).*

Proof. We have

$$\begin{aligned} &L_n(f;x) - L_{n+1}(f;x) = \\ &= 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) - 2^{-(n+1)x} \sum_{k=0}^{\infty} \frac{((n+1)x)_k}{2^k k!} f\left(\frac{k}{n+1}\right) \\ &= 2^{-(n+1)x} \left\{ 2^x \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) - \sum_{k=0}^{\infty} \frac{((n+1)x)_k}{2^k k!} f\left(\frac{k}{n+1}\right) \right\}. \end{aligned}$$

Using the identity $2^x = \sum_{l=0}^{\infty} \frac{(x)_l}{2^l l!}$, one may write

$$\begin{aligned} &L_n(f;x) - L_{n+1}(f;x) = \\ &= 2^{-(n+1)x} \left\{ \sum_{l=0}^{\infty} \frac{(x)_l}{2^l l!} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right) - \sum_{k=0}^{\infty} \frac{((n+1)x)_k}{2^k k!} f\left(\frac{k}{n+1}\right) \right\} = \end{aligned} \tag{2.4}$$

$$= 2^{-(n+1)x} \left\{ \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \frac{(nx)_{k-l}(x)_l}{2^k l!(k-l)!} f\left(\frac{k-l}{n}\right) - \sum_{k=0}^{\infty} \frac{((n+1)x)_k}{2^k k!} f\left(\frac{k}{n+1}\right) \right\}$$

$$= 2^{-(n+1)x} \left\{ \sum_{k=0}^{\infty} \left[\sum_{l=0}^{k} \frac{(nx)_l(x)_{k-l}}{2^k l!(k-l)!} f\left(\frac{l}{n}\right) - \frac{((n+1)x)_k}{2^k k!} f\left(\frac{k}{n+1}\right) \right] \right\}.$$

Now we only need to show that for all $k = 0, 1, \ldots,$

$$f\left(\frac{k}{n+1}\right) \leq \frac{1}{((n+1)x)_k} \sum_{l=0}^{k} \binom{k}{l} (nx)_l (x)_{k-l} f\left(\frac{l}{n}\right), \tag{2.5}$$

which is a direct result of convexity. In fact, set

$$\alpha_l = \binom{k}{l} \frac{(nx)_l(x)_{k-l}}{((n+1)x)_k} \geq 0 \qquad \text{and} \qquad t_l = \frac{l}{n}.$$

Therefore with the help of the identity $((n+1)x)_k = \sum_{l=0}^{k} \binom{k}{l} (nx)_l (x)_{k-l}$, it is clear that

$$\sum_{l=0}^{k} \alpha_l = \frac{1}{((n+1)x)_k} \sum_{l=0}^{k} \binom{k}{l} (nx)_l (x)_{k-l} = 1.$$

On the other hand, we have

$$\begin{aligned} \sum_{l=0}^{k} \alpha_l t_l &= \frac{1}{((n+1)x)_k} \sum_{l=0}^{k} \binom{k}{l} (nx)_l (x)_{k-l} \frac{l}{n} \\ &= \frac{1}{n((n+1)x)_k} \sum_{l=1}^{k} \frac{k!}{(l-1)!(k-l)!} (nx)_l (x)_{k-l} \\ &= \frac{k}{n((n+1)x)_k} \sum_{l=0}^{k-1} \binom{k-1}{l} (nx)_{l+1} (x)_{k-l-1}. \end{aligned}$$

Since

$$(nx)_{l+1} = nx(nx+1)_l, \quad ((n+1)x)_k = (n+1)x((n+1)x+1)_{k-1}$$

and

$$((n+1)x+1)_{k-1} = \sum_{l=0}^{k-1} \binom{k-1}{l} (nx+1)_l (x)_{k-l-1}$$

one may write

$$\sum_{l=0}^{k} \alpha_l t_l = \frac{k}{(n+1)((n+1)x+1)_{k-1}} \sum_{l=0}^{k-1} \binom{k-1}{l} (nx+1)_l (x)_{k-l-1} = \frac{k}{n+1}$$

which, making use of the convexity of f, gives the inequality (2.5). Hence from (2.4) we arrive at the desired result. Clearly $L_n(f;x) = L_{n+1}(f;x)$ occurs only if

$$f\left(\frac{1}{((n+1)x)_k}\sum_{l=0}^{k}\binom{k}{l}(nx)_l(x)_{k-l}\frac{l}{n}\right) = f\left(\frac{k}{n+1}\right)$$

$$= \frac{1}{((n+1)x)_k}\sum_{l=0}^{k}\binom{k}{l}(nx)_l(x)_{k-l}f\left(\frac{l}{n}\right)$$

for all $k = 0, 1, \dots$ This implies that f is linear in $[0, \infty)$. Thus the proof is completed. □

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