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ON ∇-STATISTICAL CONVERGENCE IN RANDOM 2-NORMED SPACE

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Abstract. Recently in [19], Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. In this paper, we define and study the notion of ∇-statistical convergence and ∇-statistical Cauchy sequences using by λ-sequences in random 2-normed spaces and prove some theorems.

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1. INTRODUCTION

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [30] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy [6], Šalát [29], Çakalli [2], Maio and Kocinac [16], Miller [18], Maddox [15], Leindler [14], Mursaleen and Alotaibi [22], Mursaleen and Edely [24], Mursaleen and Edely [26], and many others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability, (see [3]).

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \quad \text{exists.}$$

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Definition 1. *A sequence $x = (x_k)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$*

$$\delta\big(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \big) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell\big(S\big)$ and S denotes the set of all statistically convergent sequences.

The probabilistic metric space was introduced by Menger [17] which is an interesting and important generalization of the notion of a metric space. Karakus [11] studied the concept of statistical convergence in probabilistic normed spaces. The theory of probabilistic normed spaces was initiated and developed in [1], [31], [32], [33], [35] and further it was extended to random/probabilistic 2-normed spaces by Goleţ [8] using the concept of 2-norm which is defined by Gähler [7], and Gürdal and Pehlivan [10] studied statistical convergence in 2-Banach spaces. Recently, Savas [36] defined and studied generalized statistical convergence in random 2-normed space.

Let $\lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to infinity such that

$$\lambda_{n+1} \leq \lambda_n + 1,\ \ \lambda_1 = 1.$$

The generalized de la Vallee-Pousin mean is defined by

$$t_n\left(x\right) = \tfrac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = \left[n - \lambda_n + 1, n\right]$. The collection of such sequence $\lambda = (\lambda_n)$ will be denoted by ∇. Let $K \subseteq \mathbb{N}$ be a set of positive integers. Then

$$\delta_\lambda\left(K\right) = \lim_n \tfrac{1}{\lambda_n} \big| \{n - \lambda_n + 1 \leq k \leq n :\ k \in K\} \big|$$

is said to be λ-density of K. In case $\lambda_n = n$, the λ-density reduces to natural density.

[20] Mursaleen introduced the λ-statistical convergence as follows: A sequence $x = (x_k)$ is said to be λ-*statistically convergent* or S_λ-convergent to ℓ if for every $\varepsilon > 0$

$$\lim_n \tfrac{1}{\lambda_n} \big| \{k \in I_n :\ \ |x_k - \ell| \geq \varepsilon\} \big|.$$

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [34] and intutionistic fuzzy normed spaces [12], [27], [28] and [13]. Further details on generalization of statistical convergence can be found in [23], [24], [25] and [26].

2. PRELIMINARIES

Definition 2. *A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is a non-decreasing and left continuous with $\inf_{t \in \mathbb{R}} f(t) = 0$ and $\sup_{t \in \mathbb{R}} f(t) =$*

1. *By* D^+, *we denote the set of all distribution functions such that* $f(0) = 0$. *If* $a \in \mathbb{R}_0^+$, *then* $H_a \in D^+$, *where*

$$H_a(t) = \begin{cases} 1, & \text{if } t > a; \\ 0, & \text{if } t \leq a \end{cases}$$

It is obvious that $H_0 \geq f$ *for all* $f \in D^+$.

A *t-norm* is a continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c \in [0,1]$. A triangle function τ is a binary operation on D^+, which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

In [7], Gähler introduced the following concept of 2-normed space.

Definition 3. *Let* X *be a real vector space of dimension* $d > 1$ *(d may be infinite). A real-valued function* $\|.,.\|$ *from* X^2 *into* $\mathbb{R}$ *satisfying the following conditions:*

(1) $\|x_1, x_2\| = 0$ *if and only if* x_1, x_2 *are linearly dependent,*
(2) $\|x_1, x_2\|$ *is invariant under permutation,*
(3) $\|\alpha x_1, x_2\| = |\alpha| \cdot \|x_1, x_2\|$, *for any* $\alpha \in \mathbb{R}$,
(4) $\|x + \overline{x}, x_2\| \leq \|x, x_2\| + \|\overline{x}, x_2\|$

is called a 2*-norm on* X *and the pair* $(X, \|.,.\|)$ *is called a* 2*-normed space.*

A trivial example of an 2-normed space is $X = \mathbb{R}^2$, equipped with the Euclidean 2-norm $\|x_1, x_2\|_E =$ the volume of the parallellogram spanned by the vectors x_1, x_2 which may be given expicitly by the formula

$$\|x_1, x_2\|_E = |\det(x_{ij})| = \text{abs}\left(\det(\langle x_i, x_j \rangle)\right)$$

where $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2$ for each $i = 1, 2$.

Recently, Goleţ [8] used the idea of 2-normed space to define the random 2-normed space.

Definition 4. *Let* X *be a linear space of dimension* $d > 1$ *(d may be infinite),* τ *a triangle, and* $\mathcal{F} : X \times X \to D^+$. *Then* $\mathcal{F}$ *is called a probabilistic 2-norm and* $(X, \mathcal{F}, \tau)$ *a probabilistic 2-normed space if the following conditions are satisfied:*

$(P2N_1)$ $\mathcal{F}(x, y; t) = H_0(t)$ *if* x *and* y *are linearly dependent, where* $\mathcal{F}(x, y; t)$ *denotes the value of* $\mathcal{F}(x, y)$ *at* $t \in \mathbb{R}$,
$(P2N_2)$ $\mathcal{F}(x, y; t) \neq H_0(t)$ *if* x *and* y *are linearly independent,*
$(P2N_3)$ $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$, *for all* $x, y \in X$,
$(P2N_4)$ $\mathcal{F}(\alpha x, y; t) = \mathcal{F}(x, y; \frac{t}{|\alpha|})$, *for every* $t > 0, \alpha \neq 0$ *and* $x, y \in X$,
$(P2N_5)$ $\mathcal{F}(x + y, z; t) \geq \tau\left(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t)\right)$, *whenever* $x, y, z \in X$.

If $(P2N_5)$ *is replaced by*

$(P2N_6)$ $\mathcal{F}(x + y, z; t_1 + t_2) \geq \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$, *for all* $x, y, z \in X$ *and* $t_1, t_2 \in \mathbb{R}_0^+$;

then $(X, \mathcal{F}, *)$ *is called a random 2-normed space (for short, R2NS).*

Remark 5. Every 2-normed space $(X, \|.,.\|)$ can be made a random 2-normed space in a natural way, by setting

(i) $\mathcal{F}(x, y; t) = H_0(t - \|x, y\|)$, for every $x, y \in X, t > 0$ and $a * b = \min\{a, b\},\ a, b \in [0, 1]$;
(ii) $\mathcal{F}(x, y; t) = \frac{t}{t+\|x,y\|}$, for every $x, y \in X, t > 0$ and $a*b = ab, a, b \in [0, 1]$.

□

In [9], Gürdal and Pehlivan studied statistical convergence in 2-normed spaces and in 2-Banach spaces in [10]. In fact, Mursaleen [19] studied the concept of statistical convergence of sequences in random 2-normed space. Recently in [4], Esi and Özdemir introduced and studied the concept of generalized Δ^m-statistical convergence of sequences in probabilistic normed space.

Definition 6. [19] *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be statistical-convergent or S^{R2N}-convergent to some $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\theta \in (0, 1)$ and for non zero $z \in X$ such that*

$$\delta\big(\{k \in \mathbb{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) \le 1 - \theta\}\big) = 0,$$

In other words we can write the sequence (x_k) statistical converges to ℓ in random 2-normed space $(X, \mathcal{F}, *)$ if

$$\lim_{m\to\infty} \tfrac{1}{m}\big|\{k \le m : \mathcal{F}(x_k - \ell, z; \varepsilon) \le 1 - \theta\}\big| = 0.$$

or equivalently

$$\delta\big(\{k \in \mathbb{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}\big) = 1,$$

i.e.

$$S - \lim_{k\to\infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1.$$

In this case we write $S^{R2N} - \lim x = \ell$ and ℓ is called the $S^{R2N} -$ limit of x. Let $S^{R2N}(X)$ denotes the set of all statistical convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

In this paper we define and study ∇-statistical convergence in random 2-normed space using by λ sequences which is quite a new and interesting idea to work with. We show that some properties of ∇-statistical convergence of real numbers also hold for sequences in random 2-normed spaces. We find some relations related to λ-statistical convergent sequences in random 2-normed spaces. Also we find out the relation between ∇-statistical convergent and ∇-statistical Cauchy sequences in this spaces.

3. ∇-STATISTICAL CONVERGENCE IN RANDOM 2-NORMED SPACE

In this section we define ∇-statistical convergent sequence in random 2-normed $(X, \mathcal{F}, *)$. Also we obtained some basic properties of this notion in random 2-normed space.

Definition 7. *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇-convergent to $\ell \in X$ with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\beta \in (0,1)$ there exists an positive integer n_0 such that $\mathcal{F}\Big(\frac{1}{\lambda_n}\sum\limits_{k\in I_n} x_k - \ell, z; \varepsilon\Big) > 1-\beta$, whenever $k \geq n_0$ and for non-zero $z \in X$. In this case we write $\mathcal{F} - \lim_k x_k = \ell$, and ℓ is called the $\mathcal{F}_\nabla$-limit of $x = (x_k)$.*

Definition 8. *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇-Cauchy with respect to $\mathcal{F}$ if for each $\varepsilon > 0$, $\beta \in (0,1)$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $\mathcal{F}(\frac{1}{\lambda_n}\sum\limits_{k\in I_n}(x_k - x_s), z; \varepsilon) < 1-\theta$, whenever $k, s \geq n_0$ and for non-zero $z \in X$.*

Definition 9. *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇-satistically convergent or S_∇-convergent to $\ell \in X$ with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non zero $z \in X$ such that*

$$\delta_\nabla\left(\left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) \leq 1-\beta\right\}\right) = 0.$$

In other ways we can write

$$\left|\left\{k \in I_n : \mathcal{F}\left(\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) \leq 1-\beta\right\}\right| = 0,$$

or, equivalently,

$$\delta_\nabla\left(\left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) > 1-\beta\right\}\right) = 1,$$

i.e.,

$$S_\nabla - \lim_{n\to\infty}\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) = 1.$$

In this case we write $S_\nabla^{R2N} - \lim x = \ell$ or $x_k \to \ell(S_\nabla^{R2N})$ and

$$S_\nabla^{R2N}(X) = \big\{x = (x_k) : \exists\ \ell \in \mathbb{R}, S_\nabla^{R2N} - \lim x = \ell\big\}.$$

In this case we write $S_\nabla^{R2N} - \lim x = \ell$ and ℓ is called the $S_\nabla^{R2N} -$ limit of x. Let $S_\nabla^{R2N}(X)$ denotes the set of all statistical convergent sequences in random 2-normed space $(X, \mathcal{F}, *)$.

Definition 10. *A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be ∇-statistical Cauchy with respect to $\mathcal{F}$ if for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non-zero $z \in X$ there exists a positive integer $n = n(\varepsilon)$ such that for all $k, s \geq n$*

$$\delta_\nabla\left(\left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}(x_k - x_s), z; \varepsilon\right) \leq 1-\beta\right\}\right) = 0,$$

or, equivalently,

$$\delta_{\nabla}\left(\left\{k \in I_n : \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n}(x_k - x_s), z; \varepsilon\right) > 1-\beta\right\}\right) = 1.$$

Definition 3.3, immediately implies the following Lemma.

LEMMA 11. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ is a sequence in X, then for every $\varepsilon > 0$, $\beta \in (0,1)$ and for non zero $z \in X$, then the following statements are equivalent.*

(i) $S_{\nabla} - \lim_{k\to\infty} x_k = \ell$.

(ii) $\delta_{\nabla}\left(\left\{k \in I_n : \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) \le 1-\beta\right\}\right) = 0$.

(iii) $\delta_{\nabla}\left(\left\{k \in I_n : \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) > 1-\beta\right\}\right) = 1$.

(iv) $S_{\nabla} - \lim_{k\to\infty} \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \varepsilon\right) = 1$.

THEOREM 12. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ is a sequence in X such that $S_{\nabla}^{R2N} - \lim x_k = \ell$ exists, then it is unique.*

Proof. Suppose that there exist elements ℓ_1, ℓ_2 $(\ell_1 \neq \ell_2)$ in X such that

$$S_{\nabla}^{R2N} - \lim_{k\to\infty} x_k = \ell_1; \qquad S_{\nabla}^{R2N} - \lim_{k\to\infty} x_k = \ell_2.$$

Let $\varepsilon > 0$ be given. Choose $s > 0$ such that

$$(1-s) * (1-s) > 1-\varepsilon. \tag{3.1}$$

Then, for any $t > 0$ and for non zero $z \in X$ we define

$$K_1(s,t) = \left\{k \in I_n : \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell_1, z; \tfrac{t}{2}\right) \le 1-s\right\};$$

$$K_2(s,t) = \left\{k \in I_n : \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell_2, z; \tfrac{t}{2}\right) \le 1-s\right\}.$$

Since

$$S_{\nabla}^{R2N} - \lim_{k\to\infty} x_k = \ell_1 \text{ and } S_{\nabla}^{R2N} - \lim_{k\to\infty} x_k = \ell_2,$$

we have

$$\delta_{\nabla}(K_1(s,t)) = 0 \text{ and } \delta_{\nabla}(K_2(s,t)) = 0 \text{ for all } t > 0.$$

Now let $K(s,t) = K_1(s,t) \cup K_2(s,t)$, then it is easy to observe that $\delta_{\nabla}(K(s,t)) = 0$. But we have $\delta_{\nabla}(K^c(s,t)) = 1$.

Now if $k \in K^c(s,t)$ then we have

$$\mathcal{F}(\ell_1 - \ell_2, z; t) \ge$$

$$\ge \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell_1, z; \tfrac{t}{2}\right) * \mathcal{F}\left(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell_2, z; \tfrac{t}{2}\right) > (1-s) * (1-s).$$

It follows by (3.1) that

$$\mathcal{F}(\ell_1 - \ell_2, z; t) > (1 - \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $\mathcal{F}(\ell_1 - \ell_2, z; t) = 1$ for all $t > 0$ and non zero $z \in X$. Hence $\ell_1 = \ell_2$.

Next theorem gives the algebraic characterization of λ-statistical convergence on random 2-normed spaces.

Theorem 13. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space, and $x = (x_k)$ and $y = (y_k)$ be two sequences in X.*

(a) *If $S_{\nabla}^{R2N} - \lim x_k = \ell$ and $c(\neq 0) \in \mathbb{R}$, then $S_{\nabla}^{R2N} - \lim cx_k = c\ell$.*

(b) *If $S_{\nabla} - \lim x_k = \ell_1$ and $S_{\nabla}^{R2N} - \lim y_k = \ell_2$, then $S_{\nabla}^{R2N} - \lim(x_k + y_k) = \ell_1 + \ell_2$.*

The proof of this theorem is straightforward, and thus will be omitted.

Theorem 14. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ be a sequence in X such that $\mathcal{F}_{\nabla} - \lim x_k = \ell$ then $S_{\nabla}^{R2N} - \lim x_k = \ell$.*

Proof. Let $\mathcal{F}_{\nabla} - \lim x_k = \ell$. Then for every $\varepsilon > 0, t > 0$ and non zero $z \in X$, there is a positive integer n_0 such that

$$\mathcal{F}\left(\frac{1}{\lambda_n} \sum_{k \in I_n} x_k - \ell, z; t\right) > 1 - \varepsilon$$

for all $k \geq n_0$. Since the set

$$K(\varepsilon, t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n} \sum_{k \in I_n} x_k - \ell, z; t\right) \leq 1 - \varepsilon\right\}$$

has at most finitely many terms. Also, since every finite subset of $\mathbb{N}$ has δ_{∇}-density zero, and consequently we have $\delta_{\nabla}(K(\varepsilon, t)) = 0$. This shows that $S_{\nabla}^{R2N} - \lim x_k = \ell$.

Remark 15. The converse of the above theorem is not true in general. It follows from the following example. □

Example 16. Let $X = \mathbb{R}^2$, with the 2-norm $\|x, z\| = |x_1 z_2 - x_2 z_1|$, $x = (x_1, x_2), z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0, 1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t + \|x, z\|}$, for all $x, z \in X, z_2 \neq 0$, and $t > 0$. Now we define a sequence $x = (x_k)$ by

$$\frac{1}{\lambda_n} \sum_{k \in I_n} x_k = \begin{cases} (k, 0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n;\ n \in \mathbb{N} \\ (0, 0), & \text{otherwise.} \end{cases}$$

Now for every $0 < \varepsilon < 1$ and $t > 0$, write

$$K(\varepsilon, t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n} \sum_{k \in I_n} x_k - \ell, z; t\right) \leq 1 - \varepsilon\right\},$$

where $\ell = (0,0)$. Then

$$\begin{aligned} K(\varepsilon,t) &= \left\{ k \in I_n : \tfrac{t}{t+|\frac{1}{\lambda_n}\sum_{k\in I_n} x_k|} \leq 1-\varepsilon \right\} \\ &= \left\{ k \in I_n : |\tfrac{1}{\lambda_n} \sum_{k\in I_n} x_k| \geq \tfrac{t\varepsilon}{1-\varepsilon} > 0 \right\} \\ &= \{k \in I_n : x_k = (k,0)\} \\ &= \{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n; n \in \mathbb{N}\}, \end{aligned}$$

so we get

$$\tfrac{1}{\lambda_n}|K(\varepsilon,t)| \ \leq \tfrac{1}{\lambda_n}|\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \leq k \leq n; n \in \mathbb{N}\}| \ \leq \tfrac{[\sqrt{\lambda_n}]}{\lambda_n}.$$

Taking limit n approaches to ∞, we get

$$\delta_\theta(K(\varepsilon,t)) = \lim_{n\to\infty} \tfrac{1}{\lambda_n}|K(\varepsilon,t)| \leq \lim_{n\to\infty} \tfrac{[\sqrt{\lambda_n}]}{\lambda_n} = 0.$$

This shows that $x_k \to 0$ $(S_{\nabla}^{R2N}(X))$.

On the other hand the sequence is not $\mathcal{F}_{\nabla}$-convergent to zero as

$$\begin{aligned} \mathcal{F}\Big(\tfrac{1}{\lambda_n} \sum_{k\in I_n} x_k - \ell, z; t\Big) &= \frac{t}{t+\left|\tfrac{1}{\lambda_n} \sum_{k\in I_n} x_k\right|} \\ &= \begin{cases} \frac{t}{t+kz_2}, & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n; n \in \mathbb{N} \\ 1, & \text{otherwise.} \end{cases} \\ &\leq 1. \quad \square \end{aligned}$$

THEOREM 17. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. If $x = (x_k)$ be a sequence in X, then $S_{\nabla}^{R2N} - \lim x_k = \ell$ if and only if there exists a subset $K \subseteq \mathbb{N}$ such that $\delta_{\nabla}(K) = 1$ and $\mathcal{F}_{\nabla} - \lim x_k = \ell$.*

Proof. Suppose first that $S_{\nabla}^{R2N} - \lim x_k = \ell$. Then for any $t > 0$, $s = 1, 2, 3, \ldots$ and non zero $z \in X$, let

$$A(s,t) = \left\{ k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n} \sum_{k\in I_n} x_k - \ell, z; t\Big) > 1 - \tfrac{1}{s} \right\}$$

and

$$K(s,t) = \left\{ k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n} \sum_{k\in I_n} x_k - \ell, z; t\Big) \leq 1 - \tfrac{1}{s} \right\}.$$

Since $S_{\nabla}^{R2N} - \lim x_k = \ell$ it follows that

$$\delta_{\nabla}(K(s,t)) = 0.$$

Now for $t > 0$ and $s = 1, 2, 3, \ldots$, we observe that

$$A(s,t) \supset A(s+1,t)$$

and

$$\delta_\theta(A(s,t)) = 1. \tag{3.2}$$

Now we have to show that, for $k \in A(s,t), \mathcal{F}_{\nabla} - \lim x_k = \ell$. Suppose that for $k \in A(s,t), x = (x_k)$ not convergent to ℓ with respect to $\mathcal{F}_{\nabla}$. Then there exists some $u > 0$ such that

$$\Big\{k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; t\Big) \le 1 - u\Big\}$$

for infinitely many terms x_k. Let

$$A(u,t) = \Big\{k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; t\Big) > 1 - u\Big\}$$

and

$$u > \tfrac{1}{s}, \qquad s = 1, 2, 3, \ldots$$

Then we have

$$\delta_\lambda(A(u,t)) = 0.$$

Furthermore, $A(s,t) \subset A(u,t)$ implies that $\delta_{\nabla}(A(s,t)) = 0$, which contradicts (3.2) as $\delta_{\nabla}(A(s,t)) = 1$. Hence $\mathcal{F}_{\nabla} - \lim x_k = \ell$.

Conversely, suppose that there exists a subset $K \subseteq \mathbb{N}$ such that $\delta_{\nabla}(K) = 1$ and $\mathcal{F}_{\nabla} - \lim x_k = \ell$.

Then for every $\varepsilon > 0$, $t > 0$ and non zero $z \in X$, we can find out a positive integer n such that

$$\mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; t\Big) > 1 - \varepsilon$$

for all $k \ge n$. If we take

$$K(\varepsilon,t) = \Big\{k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; t\Big) \le 1 - \varepsilon\Big\}$$

then it is easy to see that

$$K(\varepsilon,t) \subset \mathbb{N} - \{n_{k+1}, n_{k+2}, \ldots\}$$

and consequently

$$\delta_{\nabla}(K(\varepsilon,t)) \le 1 - 1.$$

Hence $S_{\nabla}^{R2N} - \lim x_k = \ell$.

Finally, we establish the Cauchy convergence criteria in random 2-normed spaces.

Theorem 18. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space. Then a sequence $x = (x_k)$ in X is ∇-statistically convergent if and only if it is ∇-statistically Cauchy.*

Proof. Let $x = (x_k)$ be a ∇-statistically convergent sequence in X. We assume that $S_{\nabla}^{R2N} - \lim x_k = \ell$. Let $\varepsilon > 0$ be given. Choose $s > 0$ such that (3.1) is satisfied. For $t > 0$ and for non zero $z \in X$ define

$$A(s,t) = \Big\{k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \tfrac{t}{2}\Big) \le 1 - s\Big\}$$

and

$$A^c(s,t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \frac{t}{2}\right) > 1 - s\right\}.$$

Since $S_{\nabla}^{R2N} - \lim x_k = \ell$ it follows that $\delta_{\nabla}(A(s,t)) = 0$ and consequently $\delta_{\nabla}(A^c(s,t)) = 1$. Let $p \in A^c(s,t)$. Then

$$\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \frac{t}{2}\right) \leq 1 - s. \tag{3.3}$$

If we take

$$B(\varepsilon,t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; t\right) \leq 1 - \varepsilon\right\}$$

then to prove the result it is sufficient to prove that $B(\varepsilon,t) \subseteq A(s,t)$. Let $n \in B(\varepsilon,t)$, then for non zero $z \in X$

$$\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} (x_n - x_p), z; t\right) \leq 1 - \varepsilon. \tag{3.4}$$

If

$$\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} (x_n - x_p), z; t\right) \leq 1 - \varepsilon,$$

then we have

$$\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} x_n - \ell, z; \frac{t}{2}\right) \leq 1 - s$$

and therefore $n \in A(s,t)$. As otherwise i.e., if

$$\mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} x_n - \ell, z; \frac{t}{2}\right) > 1 - s$$

then by (3.1), (3.3) and (3.4) we get

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} (x_n - x_p), z; t\right) \\ &\geq \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} x_n - \ell, z; \frac{t}{2}\right) * \mathcal{F}\left(\frac{1}{\lambda_p}\sum_{p\in I_n} x_p - \ell, z; \frac{t}{2}\right) \\ &> (1 - s) * (1 - s) > (1 - \varepsilon) \end{aligned}$$

which is not possible. Thus $B(\varepsilon,t) \subset A(s,t)$. Since $\delta_{\nabla}(A(s,t)) = 0$, it follows that $\delta_{\nabla}(B(\varepsilon,t)) = 0$. This shows that (x_k) is ∇-statistically Cauchy.

Conversely, suppose (x_k) is ∇-statistically Cauchy but not ∇-statistically convergent. Then there exists positive integer p and for non zero $z \in X$ such that if we take

$$A(\varepsilon,t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{n\in I_n} (x_n - x_p), z; t\right) \leq 1 - \varepsilon\right\}$$

and

$$B(\varepsilon,t) = \left\{k \in I_n : \mathcal{F}\left(\frac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \frac{t}{2}\right) > 1 - \varepsilon\right\}$$

then

$$\delta_{\nabla}(A(\varepsilon,t)) = 0 = \delta_{\nabla}(B(\varepsilon,t))$$

and consequently

$$\delta_{\nabla}(A^c(\varepsilon,t)) = 1 = \delta_{\nabla}(B^c(\varepsilon,t)). \tag{3.5}$$

Since

$$\mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{n\in I_n}(x_n - x_p), z; t\Big) \geq 2\mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \tfrac{t}{2}\Big) > 1-\varepsilon,$$

if

$$\mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{k\in I_n} x_k - \ell, z; \tfrac{t}{2}\Big) > \tfrac{1-\varepsilon}{2}$$

then we have

$$\delta_{\nabla}\Big(\Big\{k \in I_n : \mathcal{F}\Big(\tfrac{1}{\lambda_n}\sum_{n\in I_n}(x_n - x_p), z; t\Big) > 1-\varepsilon\Big\}\Big) = 0$$

i.e. $\delta_{\nabla}(A^c(\varepsilon,t)) = 0$, which contradicts (3.5) as $\delta_{\nabla}(A^c(\varepsilon,t)) = 1$. Hence $x = (x_k)$ is ∇-statistically convergent.

Combining Theorem 3.7 and Theorem 3.8 we get the following corollary.

Corollary 19. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and and $x = (x_k)$ be a sequence in X. Then the following statements are equivalent:*

(a) *x is ∇-statistically convergent.*
(b) *x is ∇-statistically Cauchy.*
(c) *there exists a subset $K \subseteq \mathbb{N}$ such that $\delta_{\nabla}(K) = 1$ and $\mathcal{F}_{\nabla} - \lim x_k = \ell$.*

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