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ANALYTIC AND EMPIRICAL STUDY OF THE RATE OF CONVERGENCE OF SOME ITERATIVE METHODS

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Dedicated to prof. I. Păvăloiu on the occasion of his 75th anniversary

Abstract. We study analytically and empirically the rate of convergence of two k-step fixed point iterative methods in the family of methods

$$x_{n+1} = T(x_{i_0+n-k+1}, x_{i_1+n-k+1}, \dots, x_{i_{k-1}+n-k+1}),\ n \geq k-1, \tag{1}$$

where $T\colon X^k \to X$ is a mapping satisfying some Prešić type contraction conditions and $(i_0, i_1, \dots, i_{k-1})$ is a permutation of $(0, 1, \dots, k-1)$.

We also consider the Picard iteration associated to the fixed point problem $x = T(x, \dots, x)$ and compare analytically and empirically the rate and speed of convergence of three iterative methods. Our approach opens a new perspective on the study of the rate of convergence / speed of convergence of fixed point iterative methods and also illustrates the essential difference between them by means of some concrete numerical experiments.

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1. INTRODUCTION

In the book [23] (see also [28]), I. Păvăloiu studied some multistep iterative methods for solving the scalar equation

$$x = \varphi(x) \tag{2}$$

where $\varphi\colon I \to I$ is a function and $I \subset \mathbb{R}$ is an interval. In order to solve (2), he considers a function $g\colon I^s \to I$, where $s \geq 1$ is an integer, and the restriction of g to the diagonal of I^s coincides with φ, that is,

$$g(x, x, \dots, x) = \varphi(x), \qquad \forall x \in I. \tag{3}$$

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Then, by choosing $x_0, x_1, \dots, x_{s-1} \in I$, one constructs the s-point iterative sequence $\{x_n\}$ defined by

$$x_{s+p} = g(x_p, x_{p+1}, \dots, x_{p+s-1}), \qquad p = 0, 1, \dots \tag{4}$$

The convergence of the iterative method (4) is established in Theorem 4.2.1 in [23] (Theorem 5.3.1 in [28]), which essentially states that, if φ and g are defined as above and there exist constants $\alpha_i \in (0, 1)$, $i = 1, 2, \dots, s$ satisfying

$$\alpha_1 + \alpha_2 + \cdots + \alpha_s < 1 \tag{5}$$

such that

$$|g(u_0, \dots, u_{s-1}) - g(u_1, \dots, u_s)| \le \alpha_1 |u_0 - u_1| + \cdots + \alpha_s |u_{s-1} - u_s|, \tag{6}$$

for all $u_0, u_1, \dots, u_{k-1} \in I$, then the sequence $\{x_n\}$ given by (4) converges to $\overline{x} \in I$, the unique solution of equation (2), for any initial values $x_0, x_1, \dots, x_{s-1} \in I$.

Subsequently, by considering the family of $s!$ iterative methods

$$x_{n+1} = g(x_{i_0+n-s+1}, x_{i_1+n-s+1}, \dots, x_{i_{s-1}+n-s+1}), \qquad n \ge s - 1, \tag{7}$$

where $(i_0, i_1, \dots, i_{s-1})$ is a permutation of $(0, 1, \dots, s-1)$, the authors in [23] and [28] search for a certain iterative method in that family for which the best error estimate is obtained (by means of Theorem 4.2.1 [23]).

The conclusion (see Theorem 5.3.3 in [28]) is that the optimal method in this respect corresponds to the particular method obtained from (7) in case of the permutation $(i_0, i_1, \dots, i_{s-1})$ of $(0, 1, \dots, s-1)$ for which one has

$$\alpha_{i_0} \ge \alpha_{i_1} \ge \cdots \ge \alpha_{i_{s-1}}.$$

Starting from the fact that, in [23] and [28], no direct proof is given of the fact that the methods in (7) are also convergent, our aim in this paper is quadruple:

- First, to give a different proof of Theorem 4.2.1 in [23] (Theorem 5.3.1 in [28]) in the more general case of mappings defined on a complete metric space X;
- Second, to consider the one-point iterative method

$$y_{n+1} = g(y_n, y_n, \dots, y_n), \qquad n \ge 0, \tag{8}$$

  and prove that it converges to $\overline{x}$, for any initial value y_0;
- Third, to show analytically that all the three iterative methods mentioned above have linear rate of convergence;
- Fourth, to define a suitable concept of speed of convergence and to show empirically that the rate of convergence and the speed of convergence are distinct concepts and, additionally, to present some examples that show that two methods having the same rate of convergence may exhibit a different speed of convergence.

2. PRELIMINARIES

We first note that Theorem 4.2.1 in [23] (Theorem 5.3.1 in [28]) is a particular case of Prešić fixed point theorem, established in the general setting of a metric space [29].

Indeed, let (X, d) be a metric space and $T\colon X \to X$ a self mapping. Denote by $Fix\,(T) := \{x \in X : Tx = x\}$ the set of fixed points of T.

If (X, d) is complete and T is a contraction, *i.e.*, there exists a constant $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq a\, d(x, y), \qquad \text{for all } x, y \in X, \tag{9}$$

then, by the well known Banach contraction mapping principle (see [4], for example), we know that $Fix\,(T) = \{p\}$ and that, for any $x_0 \in X$, the Picard iteration, that is, the sequence defined by $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, converges to p, as $n \to \infty$.

The Banach contraction mapping principle has been extended by Prešić [29] (see also [42]), to mappings $f\colon X^k \to X$ satisfying a contractive condition that includes (9) in the particular case $k = 1$.

THEOREM 2.1 (S. Prešić [29], 1965). *Let (X, d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum_{i=1}^{k} \alpha_i = \alpha < 1$ and $f\colon X^k \to X$ a mapping satisfying*

$$d\,(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \alpha_1 d(x_0, x_1) + \cdots + \alpha_k d(x_{k-1}, x_k), \tag{10}$$

for all $x_0, \ldots, x_k \in X$.

Then:

1) *f has a unique fixed point $\overline{x}$, that is, there exists a unique $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$;*
2) *the sequence $\{x_n\}_{n \geq 0}$ defined by*

$$x_{n+1} = f(x_n, \ldots, x_{n-k+1}), \qquad n = k-1, k, k+1, \ldots \tag{11}$$

converges to $\overline{x}$, for any $x_0, \ldots, x_{k-1} \in X$.

It is easy to see that, subject to a change of notation, Theorem 4.2.1 in [23] is obtained from Theorem 2.1 for $X = I \subset \mathbb{R}$ and that, in the particular case $k = 1$, from Theorem 2.1, we get exactly the well-known Banach contraction mapping principle. In this case, the k-point iterative method (11) reduces to Picard iterations:

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, 3, \ldots, \tag{12}$$

Apart from applications in numerical analysis, Prešić fixed point theorem has other important applications in the study of global asymptotic stability of the equilibrium for nonlinear difference equations; see the paper [11] and the monograph [18].

On the other hand, some other Prešić type fixed point theorems have been obtained in [12], [18], [19], [17], [21], [30], and for more general contractive type conditions in [8], [9] and [11], with some applications to nonlinear cyclic systems of equations and difference equations.

Theorem 2.1 and other similar results, like the ones in [12], [19], [17], [30], have important applications in the iterative solution of nonlinear equations; see [23] and [28], [9], [10], as well as [31]-[41].

Another important generalization of Theorem 2.1 was obtained by I.A. Rus [30], for operators T fulfilling the more general condition

$$d(T(x_0,\dots,x_{k-1}),T(x_1,\dots,x_k)) \le \varphi(d(x_0,x_1),\dots,d(x_{k-1},x_k)), \tag{13}$$

for any $x_0,\dots,x_k \in X$, where $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfies certain appropriate conditions.

Another important generalization of Prešić's result was recently obtained by L. Cirić and S. Prešić in [12], where, instead of (10) and its generalization (13), the following contraction condition is considered:

$$d(T(x_0,\dots,x_{k-1}),T(x_1,\dots,x_k)) \le \lambda \max\{d(x_0,x_1),\dots,d(x_{k-1},x_k)\}, \tag{14}$$

for any $x_0,\dots,x_k \in X$, where $\lambda \in (0,1)$.

Other general Prešić type fixed point results have been very recently obtained by the third author in [18]-[21] based on alternative contractive conditions which are more general than (14), (13) and (10). For other related results, we refer to [31]-[41].

The following lemmas will be useful in proving our main results in this paper.

Lemma 1. ([29]) *Let $k \in \mathbb{N}, k \neq 0$ and $\alpha_1,\alpha_2,\dots,\alpha_k \in \mathbb{R}_+$ such that $\sum\limits_{i=1}^{k}\alpha_i = \alpha < 1$. If $\{\Delta_n\}_{n\ge 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \le \alpha_1\Delta_n + \alpha_2\Delta_{n+1} + \ldots + \alpha_k\Delta_{n+k-1}, \quad n \ge 1, \tag{15}$$

then there exist $L > 0$ and $\theta \in (0,1)$ such that

$$\Delta_n \le L\cdot\theta^n, \qquad \text{for all } n \ge 1. \tag{16}$$

The next Lemma is due to Ostrowski ([16]) and can also be found in an extended form in [4].

Lemma 2. *Let $\{a_n\}_{n\ge 0}$, $\{b_n\}_{n\ge 0}$ be two sequences of positive real numbers and $q \in (0,1)$ such that:*

i) $a_{n+1} \le q\, a_n + b_n, n \ge 0$;
ii) $b_n \to 0$ *as* $n \to \infty$.

Then:

$$\lim_{n\to\infty} a_n = 0.$$

A more general form of the previous lemma has been obtained in [2].

LEMMA 3. *Let* $\{a_n\}_{n\geq 0}$ *be a sequence of positive real numbers and let* $\{b_n\}_{n\geq 0}$ *be a sequence of non-negative real numbers for which there exist* $q \in (0,1)$ *and an integer* $k \geq 0$ *such that:*

i) $a_{n+1} \leq q\, a_{n-k} + b_n, \quad n \geq k;$

ii) $\sum_{n=1}^{\infty} b_n < +\infty.$

Then:

$$\sum_{n=1}^{\infty} a_n < +\infty.$$

Note that for $k = 0$, by Lemma 3, we actually get the conclusion of Lemma 2, *i.e.*,

$$\lim_{n\to\infty} a_n = 0.$$

3. MAIN RESULTS

Our first main result is an improved version of Prešić fixed point theorem in [29] (Theorem 2.1); see also [18].

THEOREM 3.1. *Let* (X,d) *be a complete metric space,* k *a positive integer and* $f : X^k \to X$ *a mapping for which there exist* $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}_+$, $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$ *such that*

$$d(f(x_0,\dots,x_{k-1}), f(x_1,\dots,x_k)) \leq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i), \tag{17}$$

for all $x_0, \dots, x_k \in X$.

Then:

1) f *has a unique fixed point* $\overline{x}$, *i.e.,* $f(\overline{x},\dots,\overline{x}) = \overline{x}$;
2) *the sequence* $\{y_n\}_{n\geq 0}$, *defined by*

$$y_{n+1} = f(y_n, y_n, \dots, y_n), \quad n \geq 0, \tag{18}$$

converges to $\overline{x}$, *for any* $y_0 \in X$;

3) *the sequence* $\{x_n\}_{n\geq 0}$ *defined by* $x_0, \dots, x_{k-1} \in X$ *and*

$$x_{n+1} = f(x_{n-k+1}, x_{n-k}, \dots, x_n), \quad n \geq k-1, \tag{19}$$

also converges to $\overline{x}$, *for all* $x_0, \dots, x_{k-1} \in X$.

4) *the sequence* $\{z_n\}_{n\geq 0}$ *defined by* $z_0, \dots, z_{k-1} \in X$ *and*

$$z_{n+1} = f(z_n, z_{n-1}, \dots, x_{n-k+1}), \quad n \geq k-1, \tag{20}$$

converges to $\overline{x}$, *for all* $z_0, \dots, z_{k-1} \in X$.

5) *The following estimates hold:*

$$d(y_n, \overline{x}) \leq \tfrac{\alpha^n}{1-\alpha}\, d(y_1, y_0)\,, \quad n = 1, 2, \dots; \tag{21}$$

$$d(x_n, \overline{x}) \leq L\, \tfrac{\theta^n}{1-\theta}, \quad n = 1, 2, \dots; \tag{22}$$

where $L > 0$ and $\theta \in (0,1)$ are some constants.

(23) $$d(z_n, \overline{x}) \leq L_1 \frac{\theta_1^i}{1-\theta_1}, \quad n = 1, 2, \dots;$$

where $L_1 > 0$ and $\theta_1 \in (0,1)$ are some constants.

Proof. 1), 2) By considering the associate operator $F : X \to X, F(x) = f(x, \dots, x)$, for any $x \in X$ we have:

$$\begin{aligned} &d(F(x), F(y)) = d(f(x, x, \dots, x), f(y, y, \dots, y)) \leq \\ &\leq d(f(x, \dots, x), f(x, \dots, x, y)) + d(f(x, \dots, x, y), f(x, \dots, x, y, y)) + \\ &\quad + \dots + \\ &\quad + d(f(x, x, y, \dots, y), f(x, y, \dots, y)) + d(f(x, y, \dots, y), f(y, \dots, y)). \end{aligned}$$

By (17) we obtain:

$$\begin{aligned} d(F(x), F(y)) \quad &\leq \quad [\alpha_1 d(x,x) + \alpha_2 d(x,x) + \dots + \alpha_{k-1} d(x,x) + \alpha_k d(x,y)] + \\ &+ \quad [\alpha_1 d(x,x) + \alpha_2 d(x,x) + \dots + \alpha_{k-1} d(x,y) + \alpha_k d(y,y)] + \\ &+ \quad \dots + \\ &+ \quad [\alpha_1 d(x,y) + \alpha_2 d(y,y) + \dots + \alpha_{k-1} d(y,y) + \alpha_k d(y,y)], \end{aligned}$$

so

$$d(F(x), F(y)) \leq \sum_{i=1}^{k} \alpha_i d(x,y) = \alpha d(x,y),$$

for any $x, y \in X$, which shows that F is a Banach contraction with constant $\alpha \in [0,1)$.

Consequently, by Banach contraction mapping principle, F has a unique fixed point $\overline{x} \in X$ that can be obtained by means of the Picard iterations corresponding to F starting from any $x_0 \in X$, which thus proves 1) and 2).

3) We prove now that the k-step iteration method $\{x_n\}_{n \geq 0}$, defined by (19) converges to the unique fixed point $\overline{x}$ of f. For $n \geq k$ we have:

$$d(x_n, x_{n+1}) = d(f(x_{n-k}, x_{n-k+1}, \dots, x_{n-1}), f(x_{n-k+1}, x_{n-k+2}, \dots, x_n)) \leq$$

(24) $$\leq \alpha_1 d(x_{n-k}, x_{n-k+1}) + \alpha_2 d(x_{n-k+1}, x_{n-k+2}) + \dots + \alpha_k d(x_{n-1}, x_n).$$

If

$$\Delta_n = d(x_{n-1}, x_n), \quad n \geq 1,$$

then, by (24), we obtain that the sequence $\{\Delta_n\}_{n \geq 1}$ satisfies:

$$\Delta_{n+1} \leq \alpha_1 \Delta_{n-k+1} + \alpha_2 \Delta_{n-k+2} + \dots + \alpha_k \Delta_n, \quad n \geq 1,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k > 0$ and $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$.

By Lemma 1, there exist $L > 0$ and $\theta \in (0,1)$ such that $\Delta_n \leq L\theta^n, n \geq 1$, that is,

(25) $$d(x_{n-1}, x_n) \leq L\theta^n, \quad n \geq 1.$$

For $n \geq 1$ and $p \geq 1$, by (25) we obtain:

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \leq \\ &\leq L\theta^{n+1} + L\theta^{n+2} + \ldots + L\theta^{n+p} = \\ &= L\theta^{n+1}\left(1 + \theta + \theta^2 + \ldots + \theta^{p-1}\right), \end{aligned}$$

so

$$d(x_n, x_{n+p}) \leq L\theta^{n+1}\frac{1-\theta^p}{1-\theta}, \quad n \geq 1, p \geq 1. \tag{26}$$

Since $\theta \in (0,1)$, it follows that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence, which in the complete metric space (X, d) is convergent.

We prove that $\{x_n\}_{n\geq 0}$ in fact converges to $\overline{x}$, the unique fixed point of f. Indeed, for $n \geq 0$ we have:

$$\begin{aligned} &d(x_{n+1}, \overline{x}) \leq d(f(x_{n-k+1}, x_{n-k+2}, \ldots, x_n), f(\overline{x}, \overline{x}, \ldots, \overline{x})) \\ &\leq d(f(x_{n-k+1}, x_{n-k+2}, \ldots, x_n), f(x_{n-k+2}, x_{n-k+3}, \ldots, x_n, \overline{x})) + \\ &\quad + d(f(x_{n-k+2}, x_{n-k+3}, \ldots, x_n, \overline{x}), f(x_{n-k+3}, x_{n-k+4}, \ldots, x_n, \overline{x}, \overline{x})) + \\ &\quad + \ldots + d(f(x_n, \overline{x}, \ldots, \overline{x}), f(\overline{x}, \overline{x}, \ldots, \overline{x})), \end{aligned} \tag{27}$$

so by (10) we obtain:

$$\begin{aligned} d(x_{n+1}, \overline{x}) \leq {}& [\alpha_1 d(x_{n-k+1}, x_{n-k+2}) + \ldots + \alpha_{k-1} d(x_{n-1}, x_n) + \alpha_k d(x_n, \overline{x})] + \\ &+ [\alpha_1 d(x_{n-k+2}, x_{n-k+3}) + \ldots + \alpha_{k-1} d(x_n, \overline{x}) + \alpha_k d(\overline{x}, \overline{x})] + \\ &+ \ldots + \\ &+ [\alpha_1 d(x_n, \overline{x}) + \alpha_2 d(\overline{x}, \overline{x}) + \ldots + \alpha_k d(\overline{x}, \overline{x})]. \end{aligned}$$

Now using (25) it follows that:

$$\begin{aligned} d(x_{n+1}, \overline{x}) \leq {}& [\alpha_1 L\theta^{n-k+2} + \alpha_2 L\theta^{n-k+3} + \ldots + \alpha_{k-1} L\theta^n + \alpha_k d(x_n, \overline{x})] + \\ &+ [\alpha_1 L\theta^{n-k+3} + \alpha_2 L\theta^{n-k+4} + \ldots + \alpha_{k-2} L\theta^n + \alpha_{k-1} d(x_n, \overline{x}) + \alpha_k \cdot 0] + \\ &+ \ldots \\ &+ [\alpha_1 d(x_n, \overline{x}) + 0] = \\ = {}& \alpha_1 L\theta^{n-k+2} + (\alpha_1 + \alpha_2) L\theta^{n-k+3} + \ldots + (\alpha_1 + \alpha_2 + \ldots + \alpha_{k-1}) L\theta^n + \\ &+ (\alpha_1 + \alpha_2 + \ldots + \alpha_k) d(x_n, \overline{x}). \end{aligned}$$

Finally we obtain that:

$$\begin{aligned} d(x_{n+1}, \overline{x}) \leq \alpha d(x_n, \overline{x}) + L\theta^n [&\alpha_1 \theta^{2-k} + (\alpha_1 + \alpha_2)\theta^{3-k} + \ldots + \\ &+ (\alpha_1 + \alpha_2 + \ldots + \alpha_{k-2})\theta + (\alpha_1 + \alpha_2 + \ldots + \alpha_{k-1})], \quad n \geq 1, \end{aligned}$$

where $\theta \in (0,1)$. This inequality shall lead to estimate

$$d(x_{n+1}, \overline{x}) \leq \alpha d(x_n, \overline{x}) + M \cdot \theta^n, n \geq 0, \tag{28}$$

where

$$M = L[\alpha_1\theta^{2-k} + (\alpha_1+\alpha_2)\theta^{3-k} + \ldots + (\alpha_1+\alpha_2+\ldots+\alpha_{k-2})\theta + (\alpha_1+\alpha_2+\ldots+\alpha_{k-1})]$$

is a fixed positive number (since k is fixed). Considering

$$a_n = d(x_n, \overline{x}),$$
$$q = \alpha \in [0,1)$$
$$b_n = M\theta^n, \quad n \geq 1,$$

the conditions of Lemma 2 are fulfilled, so by its conclusion

$$d(x_n, \overline{x}) \to 0, \quad n \to \infty.$$

Consequently, the sequence $\{x_n\}_{n\geq 0}$ converges to $\overline{x}$, the unique fixed point of f.

4) First, we observe that for

$$\Delta_n = d(z_n, z_{n-1}), \quad n \geq 1,$$

proceeding as in the previous case, we obtain by Lemma 1, that there exist $L_1 > 0$ and $\theta_1 \in (0,1)$ such that $\Delta_n \leq L_1\theta_1^n,\ n \geq 1$, that is,

$$d(z_{n-1}, z_n) \leq L_1\theta_1^n, \quad n \geq 1. \tag{29}$$

Next, in a similar way to the case of $\{x_n\}_{n\geq 0}$ but by following slightly different computations, we find that

$$d(z_{n+1}, \overline{x}) \leq \alpha d(z_{n-k+1}, \overline{x}) + M_1\,\theta_1^n,$$

for a certain constant $M_1 > 0$.

Now, simply use Lemma 3 to get the conclusion that the sequence $\{z_n\}_{n\geq 0}$ converges to $\overline{x}$, too.

5) The error estimate (21) follows by the Banach contraction mapping principle in the form given in [4], while the estimates (22) and (23) are obtained by (26) and its version for $\{z_n\}_{n\geq 0}$, respectively, by letting $p \to \infty$. □

4. RATE OF CONVERGENCE VERSUS SPEED OF CONVERGENCE

As before, let $\{x_n\}_{n\geq 0}$ be a convergent sequence with limit $\overline{x}$. If, for some r, we have

$$\lim_{n\to\infty} \frac{d(\bar{x}_{n+1}, \bar{x}^*)}{[d(\bar{x}_n, \bar{x}^*)]^r} = \lambda < +\infty,$$

then r is called the *rate of convergence* of $\{x_n\}_{n\geq 0}$, while λ is termed as its *asymptotic error*; see [15] for more details.

If $r = 1$, we say that the convergence of $\{x_n\}_{n\geq 0}$ is *linear*, if $r = 2$, we say that the convergence is *quadratic*, while, for $1 < r < 2$, we say that the convergence is *superlinear*.

Now, let $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be two convergent sequences with the same limit $\overline{x}$. If

$$\lim_{n\to\infty} \frac{d(\bar{x}_n, \bar{x}^*)}{d(y_n, \bar{x}^*)} = \beta < +\infty,$$

exists and $\beta = 0$, then we say that $\{x_n\}_{n\geq 0}$ converges *faster* than $\{y_n\}_{n\geq 0}$ to $\overline{x}$, and if $\beta \neq 0$, we say that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ have the same speed of convergence.

Clearly, if $\beta = \infty$, then $\{y_n\}_{n\geq 0}$ converges faster than $\{x_n\}_{n\geq 0}$ to $\overline{x}$ (for more details see [1], [3], [4]).

This concept of convergence can be defined in a more general context, when $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are convergent sequences with different limits, $\overline{x}$ and $\overline{y}$, respectively; see [1] and [4].

EXAMPLE 4.1. If we consider the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{c_n\}_{n\geq 0}$ given by

$$a_n = \tfrac{1}{n+1}, \quad b_n = \tfrac{1}{2^n}, \quad c_n = 2^{-2^n},$$

then, obviously, $a_n \to 0$, $b_n \to 0$ and $c_n \to 0$, as $n \to \infty$, and since

$$a)\ \lim_{n\to\infty} \tfrac{a_{n+1}}{a_n} = 1, \quad b)\ \lim_{n\to\infty} \tfrac{b_{n+1}}{b_n} = \tfrac{1}{2}, \quad c)\ \lim_{n\to\infty} \tfrac{b_n}{a_n} = 0,$$

it follows that $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ have the same rate of convergence (linear). However, $\{b_n\}_{n\geq 0}$ converges faster than $\{a_n\}_{n\geq 0}$ to 0.

Moreover, since

$$\lim_{n\to\infty} \tfrac{c_{n+1}}{(c_n)^2} = 1,$$

it follows that the sequence $\{c_n\}_{n\geq 0}$ has quadratic rate of convergence and, as an immediate consequence, converges faster than both $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$.

If we now use the proof of Theorem 3.1 and the complete form of the Banach contraction principle – see for example [4] – then we obtain for the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ the following error estimates:

$$d(y_{n+1}, \overline{x}) \leq \alpha d(y_n, \overline{x}), \quad n \geq 0; \tag{30}$$

$$d(x_{n+1}, \overline{x}) \leq \alpha d(x_n, \overline{x}) + M \cdot \theta^n, \quad n \geq 0, \tag{31}$$

where $M > 0$ and $\theta \in (0, 1)$ are constant, and also

$$d(z_{n+1}, \overline{x}) \leq \alpha d(z_n, \overline{x}) + M_1 \cdot \theta_1^n, \quad n \geq 0. \tag{32}$$

Thus, the estimates (30)-(32) show that the sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ have all linear rate of convergence, while the estimates (21)-(23) offer information on the speed of convergence of these sequences.

In the proof of Lemma 1 in [29], we note that θ in (22) is the unique positive root of the polynomial equation

$$t^k - \alpha_1\, t^{k-1} - \cdots - \alpha_{k-1}\, t - \alpha_k = 0,$$

while θ_1 in (23) is the unique positive root of the polynomial equation

$$t^k - \alpha_k\, t^{k-1} - \cdots - \alpha_2\, t - \alpha_1 = 0.$$

Therefore, in view of the estimates (22)-(23), to compare the iterative methods $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$, it suffices to establish the order relation between the constants

$$\alpha, \theta, \theta_1 \in (0, 1).$$

5. EXAMPLES AND CONCLUDING REMARKS

The following example illustrates Theorem 3.1.

EXAMPLE 5.1. Let $X = \mathbb{R}$ with the usual metric and $f\colon X^2 \to X$ be defined by

$$f(x,y) = \tfrac{x+2y}{4}, \quad \forall (x,y) \in X^2.$$

It is easy to check that f satisfies condition (10) (with $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$), condition (13) (with $\varphi(t_1,t_2) = \frac{1}{4}t_1 + \frac{1}{2}t_2$), as well as condition (14) (with $\lambda = \frac{3}{4}$).

Consider the sequence $\{x_n\}$

$$x_{n+1} = \tfrac{x_n + 2x_{n-1}}{4}, \quad n \geq 1,$$

corresponding to the identity permutation $(0,1)$. Then we have

$$x_n = c_1\left(\tfrac{1-\sqrt{33}}{8}\right)^n + c_2\left(\tfrac{1+\sqrt{33}}{8}\right)^n, \quad n \geq 1,$$

where c_1 and c_2 are some constants. Consider now the sequence $\{z_n\}$

$$z_{n+1} = \tfrac{2z_n + z_{n-1}}{4}, \quad n \geq 1,$$

corresponding to the permutation $(1,0)$ of $(0,1)$. Similarly, we obtain

$$z_n = a_1\left(\tfrac{1-\sqrt{5}}{4}\right)^n + a_2\left(\tfrac{1+\sqrt{5}}{4}\right)^n, \quad n \geq 1,$$

where a_1 and a_2 are some constants. Now, considering the sequence $\{y_n\}$ given by $y_0 \in X$ and

$$y_{n+1} = \tfrac{3}{4}y_n \geq 0,$$

we get

$$y_n = \left(\tfrac{3}{4}\right)^n y_0, \quad n \geq 0.$$

Since

$$\left|\tfrac{1-\sqrt{5}}{4}\right| < \left|\tfrac{1-\sqrt{33}}{8}\right| < \tfrac{3}{4} < \tfrac{1+\sqrt{5}}{4} < \tfrac{1+\sqrt{33}}{8},$$

we have

$$\lim_{n\to\infty} \tfrac{x_n}{y_n} = c_1 \cdot \lim_{n\to\infty}\left(\tfrac{1-\sqrt{33}}{6}\right)^n + c_2 \cdot \lim_{n\to\infty}\left(\tfrac{1+\sqrt{33}}{6}\right)^n = +\infty;$$

$$\lim_{n\to\infty} \tfrac{z_n}{y_n} = a_1 \cdot \lim_{n\to\infty}\left(\tfrac{1-\sqrt{5}}{3}\right)^n + a_2 \cdot \lim_{n\to\infty}\left(\tfrac{1+\sqrt{5}}{3}\right)^n = +\infty;$$

and

$$\lim_{n\to\infty} \tfrac{x_n}{z_n} = \lim_{n\to\infty} \frac{c_1\left(\frac{1-\sqrt{33}}{8}\right)^n + c_2\left(\frac{1+\sqrt{33}}{8}\right)^n}{a_1\left(\frac{1-\sqrt{5}}{4}\right)^n + a_2\left(\frac{1+\sqrt{5}}{4}\right)^n}$$

$$= \tfrac{c_2}{a_2} \cdot \lim_{n\to\infty}\left(\tfrac{1+\sqrt{33}}{2+2\sqrt{5}}\right)^n = +\infty,$$

as $c_1, c_2, a_1, a_2 \neq 0$.

These calculations prove that the sequence $\{y_n\}$ converges faster than $\{z_n\}$ to 0, $\{z_n\}$ converges faster than $\{x_n\}$, and $\{y_n\}$ converges faster than $\{z_n\}$, although all the three sequences have the same (linear) rate of convergence.

We thus can conclude that the above numerical tests confirm the theoretical results obtained in Păvăloiu [23]. Indeed, amongst the $k!$ iterative methods of the form

$$x_{n+1} = f(x_{i_0+n-k+1}, x_{i_1+n-k+1}, \dots, x_{i_{k-1}+n-k+1}), \quad n \geq k-1, \tag{33}$$

where $(i_0, i_1, \dots, i_{k-1})$ is a permutation of $(0, 1, \dots, k-1)$, the optimal method is $\{z_n\}$, which corresponds to the permutation $(i_0, i_1, \dots, i_{k-1})$ of $(0, 1, \dots, k-1)$ for which one has

$$\alpha_{i_0} \geq \alpha_{i_1} \geq \cdots \geq \alpha_{i_{k-1}}.$$

This also shows that, especially in the case of fixed point iteration procedures, which have generally linear rate a convergence, in order to decide about the fastest iterative method, we have to take into consideration the speed of convergence, usually deduced from the error estimates of the form (21)-(23).

An interesting conclusion that follows from the above example, is that, in this particular case, the one-point algorithm $\{y_n\}$ converges faster than the two two-step algorithms $\{x_n\}$ and $\{z_n\}$.

The problem is to study if this claim is valid in general. In view of (21)-(23), it would be sufficiently to show that

$$0 < \alpha < \theta_1 < \theta < 1,$$

which is a result similar to that given by Theorem 5.3.2 in [28].

We invite the reader to carry out all the calculations for the function f in the next example.

Example 5.2. Let $X = \mathbb{R}$ with the usual metric and $f\colon X^3 \to X$ be defined by

$$f(x, y, z) = \tfrac{x-2y+3z}{7}, \quad \forall (x, y, z) \in X^3,$$

which obviously satisfies condition (10) with $\alpha_1 = \frac{1}{7}$, $\alpha_2 = \frac{2}{7}$ and $\alpha_3 = \frac{3}{7}$.

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