JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 44 (2015) no. 1, pp. 42–61
ictp.acad.ro/jnaat

ON THE INTERPOLATION IN LINEAR NORMED SPACES USING MULTIPLE NODES

ADRIAN DIACONU*

Dedicated to prof. I. Păvăloiu on the occasion of his 75th anniversary

Abstract. In the papers [2], [3], [4], [6], [7] we indicated a method of extending the notion of interpolation polynomial to the case of a non-linear mapping $f : X \to Y$ where X and Y are linear spaces with special structures. This extension offers the possibility to establish, in this general and abstract case as well, the main properties known in the case of the interpolation of real functions.

To switch to the case using multiple nodes, case that compulsorily uses the notion of Fréchet differential of the first order as well as of higher orders, we will point out the definition and certain properties of these differentials. On this basis we can present the manner of building an abstract interpolation polynomial with multiple nodes.

MSC 2010. 41A05, 46B70.

Keywords. interpolation in linear normed spaces, multiple nodes.

1. INTRODUCTION

The topic of the interpolation of the functions defined between linear spaces or between linear normed spaces has been approached by Păvăloiu, I. in [9], [10], [11] Prenter, M. in [12], Argyros, I. K. [1], Makarov, V. L., Hlobistov, V. V. [8] and by myself in [2], [3], [4], [6], [7].

We will recall the elements of the construction of the abstract interpolation polynomial with simple nodes.

Let us consider X and Y two linear spaces and $f : X \to Y$ a non-linear mapping.

We note by $\mathcal{L}(X, Y)$ the set of the linear mappings from X to Y and by $(X, Y)^*$ the subset of $\mathcal{L}(X, Y)$ that contains linear and continuous mappings from X to Y.

For $n \geq 2$ we introduce the set $\mathcal{L}_n(X, Y) = \mathcal{L}(X, \mathcal{L}_{n-1}(X, Y))$ with $\mathcal{L}_1(X, Y) = \mathcal{L}(X, Y)$ and similarly $(X^n, Y)^* = \left(X, (X^{n-1}, Y)^*\right)^*$ with $(X^1, Y)^* = (X, Y)^*$.

**Babeş-Bolyai* University, Faculty of Mathematics and Computer Science, M. Kogălniceanu St, No. 1, 3400 Cluj-Napoca, Romania, e-mail: adiaconu@math.ubbcluj.ro.

For $n = 2$ the set $\mathcal{L}_2(X,Y)$ represents the set of the bi-linear mappings from $X \times X$ to Y.

Let θ_X and θ_Y be the null elements of the space X and Y respectively. We will note by Θ_n the null element of the space $\mathcal{L}_n(X,Y)$. For $n = 1$ we will use the notation Θ.

Let us consider now the bilinear mapping $B \in \mathcal{L}_2(Y,Y)$ that verifies $B(u,v) = B(v,u)$ for any $u, v \in Y$ and $B(B(u,v), w) = B(u, B(v,w))$ for any $u, v, w \in Y$.

We will now suppose the following properties:

1) there exists $u_0 \in Y$ the identity element of the semi-group (Y, B) and as well $Y_0 \subseteq Y$ with $u_0 \in Y_0$ so that (Y_0, B) form a group.
2) there exists $X_0 \subseteq X$ and the linear and bijective mapping $U_0 : X_0 \to Y_0$.
3) there exists the linear mapping $U : X \to Y$ so that $U|_{X_0} = U_0$.

Using the mappings $U \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}_2(Y,Y)$ we will introduce the sequence $(A_n)_{n\in\mathbb{N}}$ with $A_n : X^n \to Y$ where $A_n \in \mathcal{L}_n(X,Y)$ so that:

$$(1)\qquad \begin{aligned} &A_1(y) = U(y), && \text{for } y \in X;\\ &A_n(y_1,\ldots,y_n) = B(A_{n-1}(y_1,\ldots,y_{n-1}), U(y_n)), && \text{for } y_1,\ldots,y_n \in X. \end{aligned}$$

We consider now the points $x_0, x_1, \ldots, x_n \in X_0$ and the mapping:

$$(2)\qquad w_{0,n} : X \to Y, \quad w_{0,n}(x) = A_{n+1}(x - x_0, \ldots, x - x_n)$$

and for any $i \in \{0, 1, \ldots, n\}$ the mappings:

$$(3)\qquad \begin{gathered} w'_{0,n}(x_i) \in \mathcal{L}(X,Y),\\ w'_{0,n}(x_i)\, h = A_{n+1}(x_i - x_0, \ldots, x_i - x_{i-1}, x_i - x_{i+1}, \ldots, x_i - x_n, h), \end{gathered}$$

noting that $w'_{0,n}(x_i)$ represents the Fréchet differential of the mapping defined by (2), evidently in the case where X and Y are linear normed spaces.

A first result from [2], [7] shows that the restrictions to X_0 of the mappings (3) have values in Y_0 and are bijective, so we can speak of

$$w'_{0,n}(x_i)^{-1} : Y_0 \to X_0$$

and this mapping can be prolonged through linearity to $sp(Y_0)$ and in the case where Y has a topological structure (for example it is a linear normed space), the prolongation can be extended to $cl(sp(Y_0))$.

If we denote by Y_1 the maximal subspace from Y to which the introduced mappings can be extended and we suppose that for any $i \in \{0, 1, ..., n\}$ we have $f(x_i) \in Y_1$ and we can define the mappings:

$$(4)\qquad \begin{aligned} &\mathbf{L}(x_0, x_1, ..., x_n; f) : X \to Y;\\ &\mathbf{L}(x_0, x_1, ..., x_n; f)(x) =\\ &= \sum_{i=0}^{n} A_{n+1}(x - x_0, ..., x - x_{i-1}, x - x_{i+1}, ..., x - x_n; Z_{n,i}), \end{aligned}$$

where:

$$Z_{n,i} = w'_{0,n}(x_i)^{-1} f(x_i)$$

and these mappings verify, for any $i = \overline{0,n}$, the equalities:

$$\mathbf{L}(x_0, x_1, ..., x_n; f)(x_i) = f(x_i). \tag{5}$$

Because $A_{n+1} \in \mathcal{L}_{n+1}(X,Y)$, there exists $D_0 \in Y$ and for any $k \in \{1, 2, \ldots, n\}$ the mappings $D_k : X^k \to Y$ so that:

$$D_k x^k = D_k(\underbrace{x, \ldots, x}_{k \text{ times}}),$$

we will have:

$$\mathbf{L}(x_0, x_1, \ldots, x_n; f)(x) = D_n x^n + D_{n-1} x^{n-1} + \ldots + D_1 x + D_0. \tag{6}$$

The relation (6) is the expression of the character of an abstract polynomial of the mapping (4), so this relation together with the relation (5) prove the fact that this mapping can be denominated an abstract interpolation polynomial.

2. THE FRÉCHET DIFFERENTIAL OF A MAPPING. SPECIAL PROPERTIES

Let us consider the linear normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. We consider as well the nonlinear mapping $f : X \to Y$ with $D \subseteq X$. For $x \in D$ let be the set:

$$\mathcal{E}_{D,x} = \{h \in X : x + h \in D\}.$$

First we will have the following:

DEFINITION 1. *The mapping $f : D \to Y$ admits a Fréchet differential in the point $x \in D$, if there exists $T \in (X,Y)^*$ and $\omega : D \times (\mathcal{E}_{D,x} \setminus \{\theta_X\}) \to Y$ so that:*

$$f(x+h) - f(x) = T(h) + \|h\|_X \, \omega(x,h)$$

and:

$$\lim_{h \to \theta_X} \|\omega(x,h)\|_Y = 0.$$

For the mapping $T \in (X,Y)^*$ we have:

REMARK 2. We can easily prove that there exists at most a mapping $T \in (X,Y)^*$ that corresponds to the requirements of the Definition 1. □

In this way Definition 1 is completed with:

DEFINITION 3. *The mapping $T \in (X,Y)^*$ from Definition 1 that is attached to the function $f : D \to Y$ and to the point x, is called the Fréchet differential of this mapping at the point x, and is denoted through $f'(x)$.*

Now we have:

Remark 4. Usually there exists a subset $A \subseteq D$ so that for any $u \in A$, the function $f : D \to Y$ is Fréchet differentiable at every point u. In this case it is possible to define the function:

$$df : A \to (X, Y)^*, \quad df(u) = f'(u).$$

The definition of the previous function allows for the introduction of differentials with higher orders. □

Thus we have:

Definition 5. *Besides the data from Definition 1 let us consider a number* $p \in \mathbb{N}$.

If:

(i) *there exists* V *a neighborhood of the point* $x \in D$*, so that for any* $u \in V \cap D$ *there exists the differential of the order* $p-1$ *of the mapping* $f : D \to Y$ *at the point* u*, denoted by* $f^{(p-1)}(u) \in (X^{p-1}, Y)^*$*, so the function:*

$$d^{p-1}f : V \cap D \to \left(X^{p-1}, Y\right)^*; \quad \left(d^{p-1}f\right)(u) = f^{(p-1)}(u) \tag{7}$$

is defined;

(ii) *the function defined by* (7) *is a differential (of the first order) at the point* x; *then we can say that the mapping* $f : D \to Y$ *admits a differential with the order* p *at the point* x *and in this case:*

$$f^{(p)}(x) := \left(d^{p-1}f\right)'(x) = \left(f^{(p-1)}\right)'(x) \in \left(X, \left(X^{p-1}, Y\right)^*\right)^* = (X^p, Y)^*.$$

In the paper [5], we have established certain properties of the Fréchet differentials of higher orders, which are relevant for the statements below.

We will recall some of these properties.

I) Let us consider the bilinear and symmetrical mapping $B \in \mathcal{L}_2(Y, Y)$ together with the non-linear mappings $f, g : D \to Y$ with $D \subseteq X$.

With the aid of this mappings we consider:

$$F : D \to Y, \ \ F(x) = B(f(x), g(x)). \tag{8}$$

We have the following proposition:

Proposition 6. *If the non-linear mappings* $f, g : D \to Y$ *admit Fréchet differential up to the order* n*, included, at the point* $x \in D$*, then the mapping introduced by* (8) *admits a Fréchet differential up to the same order* n*, at the same point* x*, and for any* $h_1, \ldots, h_n \in X$ *we have:*

$$F^{(n)}(x)\, h_1...h_n = \sum_{k=0}^{n} \sum_{1 \le i_1 < ... < i_k \le n} B\big(f^{(k)}(x)\, h_{i_1}...h_{i_k}, g^{(n-k)}(x)\, h_{j_1}...h_{j_{n-k}}\big) \tag{9}$$

where, for a fixed $i_1, i_2, ..., i_k \in \mathbb{N}$ *with* $1 \le i_1 < ... < i_k \le n$*, we will choose* $\{j_1, ..., j_{n-k}\} \in \{1, ..., n\} \setminus \{i_1, ..., i_k\}$ *with* $j_1 < ... < j_{n-k}$.

For the case $h_1 = ... = h_n = h$ we have:

$$F^{(n)}(x)\,h^n = \sum_{k=0}^{n} \binom{n}{k} B\big(f^{(k)}(x)\,h^k, g^{(n-k)}(x)\,h^{n-k}\big) \tag{10}$$

We can notice that the equation (10) represents an extension of the well-known Leibnitz derivation formula.

II) It is necessary to generalize the property expressed by Proposition 6.

For this extension let us consider the sequence of mappings $(Q_m)_{m\in\mathbb{N},\, m\geq 2}$ with $Q_m \in \mathcal{L}_m(X,Y)$ and:

$$\begin{aligned} &Q_2(u_1,u_2) = B(u_1,u_2);\\ &Q_m(u_1,...,u_m) = B(Q_{m-1}(u_1,...,u_{m-1}),u_m), \quad m\in\mathbb{N},\ m\geq 2, \end{aligned} \tag{11}$$

where by $u_1,...,u_m$ we have denoted arbitrary elements of Y.

We will now consider the natural numbers p, s with $s \leq p$ and the set of distinct elements:

$$H = \{x_1, x_2, ..., x_p\}.$$

We introduce the set:

$$\mathcal{C}_{p,s}(H) = \{(x_{i_1},...,x_{i_s}) \in H^s : 1 \leq i_1 < i_2 < ... < i_s \leq p\}$$

and obviously:

$$|\mathcal{C}_{p,s}(H)| = \tfrac{p!}{s!(p-s)!},$$

where $|H|$ denotes the number of elements of the set H.

Let us now consider $m \in \mathbb{N}$ and $\alpha_1,...,\alpha_m \in \mathbb{N}$ so that $\alpha_1 + ... + \alpha_m = n$ and to start with we denote:

$$\begin{cases} H_1 = \{1,2,...,n\}; \quad p_1 = n;\\ G_1 = \left\{\left(i_1^{(1)},...,i_{\alpha_1}^{(1)}\right) \in H_1^{\alpha_1} : i_1^{(1)} < ... < i_{\alpha_1}^{(1)}\right\}. \end{cases}$$

For a fixed $\left(i_1^{(1)},...,i_{\alpha_1}^{(1)}\right) \in G_1$ we choose:

$$\begin{cases} H_2 = \{1,2,...,n\} \diagdown \left\{i_1^{(1)},...,i_{\alpha_1}^{(1)}\right\}; \quad p_2 = n - \alpha_1;\\ G_2 = \left\{\left(i_1^{(2)},...,i_{\alpha_2}^{(2)}\right) \in H_2^{\alpha_2} : i_1^{(2)} < ... < i_{\alpha_2}^{(2)}\right\}. \end{cases}$$

For $k \in \mathbb{N},\ k \leq m$ and a fixed

$$\left(i_1^{(1)},...,i_{\alpha_1}^{(1)}\right) \in G_1, ..., \left(i_1^{(k-1)},...,i_{\alpha_{k-1}}^{(k-1)}\right) \in G_{k-1}$$

we choose:

$$\begin{cases} H_k = \{1,2,...,n\} \diagdown \left\{i_1^{(1)},...,i_{\alpha_1}^{(1)},...,i_1^{(k-1)},...,i_{\alpha_{k-1}}^{(k-1)}\right\};\\ p_k = n - (\alpha_1 + ... + \alpha_{k-1});\\ G_k = \left\{\left(i_1^{(k)},...,i_{\alpha_k}^{(k)}\right) \in H_k^{\alpha_k} : i_1^{(k)} < ... < i_{\alpha_k}^{(k)}\right\}. \end{cases}$$

Finally, for a fixed $(i_1^{(1)},...,i_{\alpha_1}^{(1)}) \in G_1,...,(i_1^{(m-1)},...,i_{\alpha_{m-1}}^{(m-1)}) \in G_{m-1}$ we choose:

$$\begin{cases} H_m = \{1,2,...,n\} \setminus \{i_1^{(1)},...,i_{\alpha_1}^{(1)},...,i_1^{(m-1)},...,i_{\alpha_{m-1}}^{(m-1)}\}; \\ p_m = n - (\alpha_1 + ... + \alpha_{m-1}); \\ G_m = \left\{ (i_1^{(m)},...,i_{\alpha_m}^{(m)}) \in H_m^{\alpha_m} : i_1^{(m)} < ... < i_{\alpha_m}^{(m)} \right\}. \end{cases}$$

It is clear that for any $k \in \mathbb{N}$ we have $G_k \in \mathcal{C}_{p_k,\alpha_k}(H_k)$ and consequently:

$$|G_k| = \frac{p_k!}{\alpha_k!(p_k-\alpha_k)!} = \frac{[n-(\alpha_1+...+\alpha_{k-1})]!}{\alpha_k![n-(\alpha_1+...+\alpha_{k-1}+\alpha_k)]!}.$$

Let us denote by $\mathcal{A}_n^{[\alpha_1,...,\alpha_m]}$ the set of all systems $(G_1,\ldots,G_m)$, where for any $i \in \{1,2,\ldots,m\}$ the system of indexes G_i obtained in the aforementioned manner.

It is obvious that:

$$\begin{aligned} \left|\mathcal{A}_n^{[\alpha_1,...,\alpha_m]}\right| &= |G_1| \cdot ... \cdot |G_m| = \prod_{k=1}^{m} \frac{p_k!}{\alpha_k!(p_k-\alpha_k)!} \\ &= \frac{1}{\alpha_1!...\alpha_m!} \prod_{k=1}^{m} \frac{[n-(\alpha_1+...+\alpha_{k-1})]!}{[n-(\alpha_1+...+\alpha_{k-1}+\alpha_k)]!} \\ &= \frac{1}{\alpha_1!...\alpha_m!} \cdot \frac{n!}{(n-(\alpha_1+...+\alpha_m))!}. \end{aligned}$$

But $(n-(\alpha_1+\ldots+\alpha_m))! = (n-n)! = 0! = 1$, so:

$$\left|\mathcal{A}_n^{[\alpha_1,...,\alpha_m]}\right| = \frac{n!}{\alpha_1!...\alpha_m!}.$$

Let us consider now the non-linear mappings $f_i : A \to Y$, for $i = \overline{1,m}$ where $A \subseteq X$. Using these mappings and the n-linear mapping $Q_m : Y^m \to Y$ introduced by (11) consider the mapping:

$$F : A \to Y, \quad F(x) = Q_m(f_1(x),\ldots,f_m(x)), \tag{12}$$

which represents a more general case of the mappings (8).

In this way we have the following extension of the Proposition 6.

Proposition 7. *If the mappings $f_i : A \to Y$; $i = \overline{1,m}$ and $A \subseteq X$ admit the Fréchet differentials of the n order at the point $x \in A$, then the mapping $F : A \to Y$ defined by (12) admits as well the Fréchet differential of the n order at the same point x and for any $h_1,\ldots,h_n \in X$ we have the equality:*

$$F^{(n)}(x)\, h_1...h_n = \sum_{\alpha_1+...+\alpha_m=n} \; \sum_{(G_1,...,G_m)\in\mathcal{A}_n^{[\alpha_1,...,\alpha_m]}} T_{G_1,...,G_m}, \tag{13}$$

where for $T_{G_1,...,G_m}$ we have denoted the expression:

$$\sum_{\left(i_1^{(k)},...,i_{\alpha_k}^{(k)}\right)\in G_k;\; k=\overline{1,m}} Q_m\big(f_1^{(\alpha_1)}(x)\, h_{i_1^{(1)}}...h_{i_{\alpha_1}^{(1)}},\ldots,f_m^{(\alpha_m)}(x)\, h_{i_1^{(m)}}...h_{i_{\alpha_m}^{(m)}}\big). \tag{14}$$

For the case $h_1 = ... = h_m = h$ *we have:*

$$F^{(n)}(x)\,h^n = \sum_{\alpha_1+...+\alpha_m=n} \tfrac{n!}{\alpha_1!...\alpha_m!} Q_m\big(f_1^{(\alpha_1)}(x)\,h^{\alpha_1},\dots,f_m^{(\alpha_m)}(x)\,h^{\alpha_m}\big). \tag{15}$$

III) Let us consider now for a fixed $a \in X$, the mapping:

$$T_m : X \to Y, \quad T_m(x) = A_m(\underbrace{x-a,\dots,x-a}_{m \text{ times}}) \overset{\text{not}}{=} A_m(x-a)^m, \tag{16}$$

where $A_m \in \mathcal{L}_m(X,Y)$ is introduced by (1) and the mappings $B \in \mathcal{L}_2(Y,Y)$ and $U \in \mathcal{L}(X,Y)$ verify the specified conditions.

For these mappings we have:

Proposition 8. *The mappings defined by* (16) *admit Fréchet differentials of any order* $n \in \mathbb{N}$ *and for any* $h_1,...,h_n \in X$ *we have:*

$$T_m^{(n)} h_1...h_n = \begin{cases} \theta_Y, & \text{for } n > m \\ A_m(h_1,\dots,h_m), & \text{for } m = n \\ \frac{m!}{(m-n)!} A_n (x-a)^{m-n} h_1...h_n, & \text{for } m < n. \end{cases} \tag{17}$$

IV) Taking into account the mappings $(A_m)_{m\in\mathbb{N}}$, with $A_m \in \mathcal{L}_m(X,Y)$ introduced by (1), and if the numbers $r_1,\dots,r_m \in \mathbb{N}$ we can consider the mapping:

$$F : X \to Y, \quad F(x) = A_{r_1+...+r_m}\left((x-x_1)^{r_1},\dots,(x-x_m)^{r_m}\right), \tag{18}$$

where $x_1,\dots,x_m \in X$ are arbitrary.

We have the following result:

Proposition 9. *The mappings defined by* (18) *admit Fréchet differentials up to the order* n *included, where* $n \le r_1 + r_2 + \ldots + r_m$ *at any point* $x \in X$ *and:*

(19)

$$F^{(n)}(x)\,h_1...h_n =$$
$$= n! \sum_{\alpha_1+...+\alpha_m=n} \prod_{i=1}^{m} \binom{r_i}{\alpha_i} A_{r_1+...+r_m}\Big((x-x_1)^{r_1-\alpha_1},\dots,(x-x_m)^{r_m-\alpha_m}, h_1,\dots,h_n\Big)$$

V) We will also consider another extension of Leibnitz' formula concerning the derivative with the n order.

In this way let us consider X, Y, Z linear normed spaces and the mappings $f : X \to (Y,Z)^*$ and $g : X \to Y$.

Using the considered functions we consider the function:

$$F : X \to Y, \quad F(x) = [f(x)]\,g(x) \tag{20}$$

and for this function we have:

Proposition 10. *If the mappings* $f : X \to (Y, Z)^*$ *and* $g : X \to Y$ *admit Fréchet differentials of the order* n, *at the point* $x \in X$, *then the mapping defined by* (20) *also admits the Fréchet differential of the same order at the same point* x *and:*

$$F^{(n)}(x)\, h_1 ... h_n =$$

$$= \sum_{k=0}^{n} \sum_{1 \le i_1 < ... < i_k \le n} [f^{(k)}(x)\, h_{i_1} ... h_{i_k}]\, g^{(n-k)}(x)\, h_{j_1} ... h_{j_{n-k}} \tag{21}$$

where $\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ *with* $j_1 < j_2 < ... < j_{n-k}$.

For the case where $h_1 = ... h_n = h$ *we have:*

$$F^{(n)}(x)\, h^n = \sum_{k=0}^{n} \binom{n}{k} \left[f^{(k)}(x)\, h^k \right] g^{(n-k)}(x)\, h^{n-k}. \tag{22}$$

VI) Let us consider now the mapping $f : X \to (X, Y)^*$ supposing that for every $x \in X$ the linear mapping $f(x) : X \to Y$ has an inverse mapping $[f(x)]^{-1} : Y \to X$.

Therefore we can consider the mapping:

$$g : X \to (Y, X)^*, \quad g(x) = [f(x)]^{-1}. \tag{23}$$

We obtain the following result:

Proposition 11. *If the non-linear mapping* $f : X \to (X, Y)^*$ *has a Fréchet differential at the point* x, *then the mapping* $g : X \to (Y, X)^*$ *introduced by* (23) *also has a Fréchet differential at the same point* x, *and:*

$$g'(x)\, h = -[f(x)]^{-1}\, f'(x)\, h\, [f(x)]^{-1}, \tag{24}$$

for every $h \in Y$.

3. THE FRÉCHET DIFFERENTIAL OF CERTAIN ESSENTIAL MAPPINGS THAT APPEAR IN THE INTERPOLATION WITH MULTIPLE NODES

We will consider the sequence of mappings $(A_n)_{n \in \mathbb{N}}$ where $A_n : X^n \to Y$ are given by (1) and $B \in (Y^2, Y)^*$, $U \in (X, Y)^*$ verify the assumptions specified in the first paragraph of the present paper.

For $x_1, \ldots, x_m \in X$ and $p_1, \ldots, p_m \in \mathbb{N}$ we consider the mapping:

$$\omega \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x \Big) \in (X, Y)^*,$$

$$\omega \left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x \right) h = A_{p_1 + ... + p_m + 1}\left((x - x_1)^{p_1}, \ldots, (x - x_m)^{p_m}, h \right). \tag{25}$$

It is clear that if $x - x_i \in X_0$ for any $i = \overline{1, m}$, from the imposed hypotheses it results that the mappings defined by (25) are bijections from X_0 to Y_0. Thus we can consider the inverse mapping defined on Y_0, which can be extended first by linearity to $sp(Y_0)$ and then by continuity to $cl(sp(Y_0))$.

We can thus consider the mapping:

$$g : X \to (cl\,(sp\,(Y_0))\,, X)^*\,, \quad g\,(x) = \left[\omega \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\right]^{-1}. \tag{26}$$

Concerning this mapping we have:

Proposition 12. *If for any $k = \overline{1, m}$ we have $x - x_k \in X_0$, then for any $n \in \mathbb{N}$, the mapping defined by* (26) *admits a Fréchet differential of the order n. For any $h_1, ..., h_n \in X$ and $t \in cl\,(sp\,(Y_0))$ we have:*

(27)

$$\left[g^{(n)}\,(x)\,h_1...h_n\right] t =$$

$$= (-1)^n\, n! \sum_{\alpha_1 + ... + \alpha_m = n} \prod_{i=1}^{m} \binom{p_i + \alpha_i - 1}{\alpha_i} A_{n+1}\left(\left[\omega\left(\begin{smallmatrix} x_1 \\ p_1+\alpha_1 \end{smallmatrix}, ..., \begin{smallmatrix} x_m \\ p_m+\alpha_m \end{smallmatrix}; x\right)\right]^{-1} t, h_1, ..., h_n\right)$$

Proof. Based on Proposition 11 we deduce that:

$$g'\,(x)\,h = -\left[P\,(x)\right]^{-1} P'\,(x)\,h\left[P\,(x)\right]^{-1}, \tag{28}$$

where:

$$P\,(x) = \omega \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Bigr).$$

Taking into account Propositions 7 and 9 we deduce that:

$$P'\,(x)\,hu = \omega'\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right) hu = \sum_{k=1}^{m} p_k C^{(k)}_{p_1,...,p_m}\,(x, h, u)\,, \tag{29}$$

for any $h, u \in X$, where we denote by $C^{(k)}_{p_1,...,p_m}\,(x, h, u)$ the value of the mapping $A_{p_1+...+p_m+1}$ on the arguments:

$$(x - x_1)^{p_1}, ..., (x - x_{k-1})^{p_{k-1}}, (x - x_k)^{p_k - 1}, (x - x_{k+1})^{p_{k+1}}, ..., (x - x_m)^{p_m}, h, u.$$

If we choose $t \in cl\,(sp\,(Y_0))$ and:

$$u = \left[\omega \left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t \in X,$$

after that for $k \in \{1, 2, ..., m\}$ it has been proved that:

$$C^{(k)}_{p_1,...,p_m}\,(x, h, u) = B\left(E^{(k)}_{p_1,...,p_{k-1},p_k-1,p_{k+1},...,p_m}\,(x, h)\,, U\,(u)\right),$$

where $E^{(k)}_{p_1,...,p_{k-1},p_k-1,p_{k+1},...,p_m}\,(x, h)$ is the value of $A_{p_1+...+p_m}$ at the arguments:

$$(x - x_1)^{p_1}, ..., (x - x_{k-1})^{p_{k-1}}, (x - x_k)^{p_k - 1}, (x - x_{k+1})^{p_{k+1}}, ..., (x - x_m)^{p_m}, h;$$

so:

$$E^{(k)}_{p_1,...,p_{k-1},p_k-1,p_{k+1},...,p_m}\,(x, h) = \omega \left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k - 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right) h.$$

Thus:

$$\omega'\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big) h \left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t =$$

$$(30) \qquad = \sum_{k=1}^{m} p_k B\left(\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k - 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right) h, U(u)\right)$$

We now show the equality:

$$U(u) = U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t$$

$$(31) \qquad = A_2\left(x - x_k, \left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t\right).$$

Because the extension from $t \in Y_0$ to $t \in cl\,(sp\,(Y_0))$ is evident it is enough to suppose that $t \in Y_{0.}$

From the bijectivity of the mappings (25) for $t \in Y_0$ we deduce the existence with a unique determination of the elements $h, u \in X_0$ so that:

$$(32) \quad t = \omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right) h = \omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right) u.$$

The first member of this equality can be written as:

$$B\left(A_{p_1+...+p_m}\left((x - x_1)^{p_1}, ..., (x - x_m)^{p_m}\right), U(h)\right)$$

and the second is the value of the mapping $A_{p_1+...+p_m+2}$ at the arguments:

$$(x - x_1)^{p_1}, ..., (x - x_{k-1})^{p_{k-1}}, (x - x_k)^{p_k+1}, (x - x_{k+1})^{p_{k+1}}, ..., (x - x_m)^{p_m}, u$$

and this can be written as:

$$B\left(A_{p_1+...+p_m}\left((x - x_1)^{p_1}, ..., (x - x_m)^{p_m}\right), A_2(x - x_k, u)\right),$$

consequently the equality (32) becomes:

(33)

$$B\Big(A_{p_1+...+p_m}\left((x - x_1)^{p_1}, ..., (x - x_m)^{p_m}\right), U(h) - A_2(x - x_k, u)\Big) = \theta_Y.$$

Because for every $k \in \{1, 2, ..., m\}$ we have $x - x_k \in X_0$ we deduce that:

$$A_{p_1+...+p_m}\left((x - x_1)^{p_1}, ..., (x - x_m)^{p_m}\right) \in Y_0,$$

consequently the equality (33) will be possible only if:

$$(34) \qquad U(h) = A_2(x - x_k, u).$$

But from the same relation (32) it is clear that:

$$u = \left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t$$

and:

$$h = \left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t,$$

after that replacing these values in (34), we obtain the relation (31).

Because of the relation (30) we have the equality:

$$\omega'\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big) h \left[\omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big)\right]^{-1} t =$$

$$= \sum_{k=1}^{m} p_k B(E^{(k)}_{p_1,...,p_{k-1},p_k-1,p_{k+1},...,p_m}(x,h), A_2\left(x-x_k, q_k^{-1}t\right)) \tag{35}$$

where:

$$q_k = \omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_{k-1} \\ p_{k-1} \end{pmatrix}, \begin{pmatrix} x_k \\ p_k+1 \end{pmatrix}, \begin{pmatrix} x_{k+1} \\ p_{k+1} \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big).$$

So:

$$B(E^{(k)}_{p_1,...,p_{k-1},p_k-1,p_{k+1},...,p_m}(x,h), A_2\left(x-x_k, q_k^{-1}t\right)) =$$
$$= A_{p_1+...+p_m+2}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}, q_k^{-1}t, h\right)$$
$$= B\left(A_{p_1+...+p_m+1}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}, q_k^{-1}t\right), U(h)\right)$$
$$= B\left(\omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big) q_k^{-1}t, U(h)\right),$$

consequently:

$$\omega'\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big) h \left[\omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big)\right]^{-1} t =$$

$$= \sum_{k=1}^{m} p_k B\left(\omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\Big) q_k^{-1}t, U(h)\right) \tag{36}$$

We remark that for any $a \in Y_0$ it exists a $\overline{y} \in Y_0$ so that for any $s \in Y$ we have:

$$B(s, B(a, \overline{y})) = s, \tag{37}$$

this fact being evident, the element $\overline{y} \in Y_0$ is the symmetrical element of the element a in the group (Y_0, B).

Because the fact that $x - x_k \in X_0$ for any $k = \overline{1,m}$ we can deduce that:

$$a = A_{p_1+...+p_m}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}\right) \in Y_0$$

from where, using the relation (37), we deduce that for any $s \in Y$ we have:

$$B\left(s, B\left(A_{p_1+...+p_m}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}\right), \overline{y}\right)\right) = s,$$

from where through the properties of the bilinear mapping $B \in \left(Y^2, Y\right)^*$ we have:

$$s = B\left(A_{p_1+...+p_m}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}\right), B(s, \overline{y})\right)$$
$$= B\left(A_{p_1+...+p_m}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}\right), U\left(U^{-1}B(s, \overline{y})\right)\right)$$
$$= A_{p_1+...+p_m+1}\left((x-x_1)^{p_1}, ..., (x-x_m)^{p_m}, U^{-1}B(s, \overline{y})\right);$$

consequently:

$$\omega\begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, ..., \begin{pmatrix} x_m \\ p_m \end{pmatrix}; x\end{pmatrix} U^{-1} B\left(s, \overline{y}\right) = s,$$

from where:

$$B\left(s, \overline{y}\right) = U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} s.$$

For the beginning let be $s = B\left(u, v\right) \in Y$ with $u, v \in Y$ in the previous relation, so we have:

$$\begin{aligned} U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} B\left(u, v\right) &= B\left(B\left(u, v\right), \overline{y}\right) = B\left(B\left(\overline{y}u\right), v\right) \\ &= B\left(U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} u, v\right). \end{aligned}$$

Then in the previous relation we consider, with $t \in Y$, the elements:

$$u = \omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t$$

and:

$$v = U\left(h\right),$$

consequently:

$$\begin{aligned} &U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} B\left(u, U\left(h\right)\right) = \\ &= B\left(U\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t, h\right) \\ &= A_2\left(\left[\omega\left(\begin{matrix} x_1 \\ p_1 \end{matrix}, ..., \begin{matrix} x_{k-1} \\ p_{k-1} \end{matrix}, \begin{matrix} x_k \\ p_k + 1 \end{matrix}, \begin{matrix} x_{k+1} \\ p_{k+1} \end{matrix}, ..., \begin{matrix} x_m \\ p_m \end{matrix}; x\right)\right]^{-1} t, h\right), \end{aligned}$$

thus:

$$\begin{aligned} &\left[g'\left(x\right)h\right]t = \\ &= -\sum_{k=1}^{m} p_k U^{-1} A_2\left(\left[\omega\left(\begin{smallmatrix} x_1 \\ p_1 \end{smallmatrix}, ..., \begin{smallmatrix} x_{k-1} \\ p_{k-1} \end{smallmatrix}, \begin{smallmatrix} x_k \\ p_k+1 \end{smallmatrix}, \begin{smallmatrix} x_{k+1} \\ p_{k+1} \end{smallmatrix}, ..., \begin{smallmatrix} x_m \\ p_m \end{smallmatrix}; x\right)\right]^{-1} t, h\right) \\ &= -\sum_{\alpha_1+...+\alpha_m=1} \prod_{i=1}^{m} \binom{p_i+\alpha_i-1}{\alpha_i} U^{-1} A_2\left(\left[\omega\left(\begin{smallmatrix} x_1 \\ p_1+\alpha_1 \end{smallmatrix}, ..., \begin{smallmatrix} x_m \\ p_m+\alpha_m \end{smallmatrix}; x\right)\right]^{-1} t, h\right) \quad (38) \end{aligned}$$

so the relation (27) is true for $n = 1$.

Let us suppose now that the relation (27) is true for $n = k$, namely:

(39)

$$\left[g^{(k)}(x)\, h_2...h_{k+1}\right]t = (-1)^k\, k!\times$$
$$\times \sum_{\alpha_1+..+\alpha_m=k} \prod_{i=1}^{m} \tbinom{p_i+\alpha_i-1}{\alpha_i} U^{-1}A_{k+1}\left(\left[\omega\left(\tfrac{x_1}{p_1+\alpha_1},...,\tfrac{x_m}{p_m+\alpha_m};x\right)\right]^{-1}t,h_2,...,h_{k+1}\right)$$

and for $h_1 \in X$ as well:

(40)

$$\left[g^{(k)}(x+h_1)\, h_2...h_{k+1}\right]t = (-1)^k\, k!\times$$
$$\times \sum_{\alpha_1+..+\alpha_m=k} \prod_{i=1}^{m} \tbinom{p_i+\alpha_i-1}{\alpha_i} U^{-1}A_{k+1}\left(\left[\omega\left(\tfrac{x_1}{p_1+\alpha_1},...,\tfrac{x_m}{p_m+\alpha_m};x+h_1\right)\right]^{-1}t,h_2,...,h_{k+1}\right)$$

From the relations (39) and (40) through substraction, member by member, we obtain:

$$\left[\left(g^{(k)}(x+h_1)-g^{(k)}(x)\right)h_2...h_{k+1}\right]t =$$
$$= (-1)^k\, k! \sum_{\alpha_1+...+\alpha_m=k} \prod_{i=1}^{m} \tbinom{p_i+\alpha_i-1}{\alpha_i} U^{-1}A_{k+1}\left(Z(x,h_1)\,t,h_2,...,h_{k+1}\right), \tag{41}$$

where:

$$Z(x,h_1) = \left[\omega\left(\frac{x_1}{p_1+\alpha_1},...,\frac{x_m}{p_m+\alpha_m};x+h_1\right)\right]^{-1} -$$
$$-\left[\omega\left(\frac{x_1}{p_1+\alpha_1},...,\frac{x_m}{p_m+\alpha_m};x\right)\right]^{-1} \in (Y,X)^*.$$

At the same time:

$$Z(x,h_1) = \left\{\left[\omega\left(\frac{x_1}{p_1+\alpha_1},...,\frac{x_m}{p_m+\alpha_m};x\right)\right]^{-1}\right\}' h_1 + \|h_1\|_X\, R(x,h_1), \tag{42}$$

where $R(x,h_1) \in (Y,X)^*$ and $\lim\limits_{h_1\to\theta_X} \|R(x,h_1)\| = 0$.

Because $A_{k+1} : X^{k+1} \to Y$ is a $k+1$ linear mapping, from the already proved relation (38) we obtain:

$$\left[\left(g^{(k)}(x+h_1)-g^{(k)}(x)\right)h_2...h_{k+1}\right]t =$$
$$= (-1)^{k+1}\, k! \sum_{\alpha_1+...+\alpha_m=k} \prod_{i=1}^{m} \tbinom{p_i+\alpha_i-1}{\alpha_i} \sum_{\beta_1+...+\beta_m=1} \prod_{i=1}^{m} \tbinom{p_i+\alpha_i+\beta_i-1}{\beta_i}\times$$
$$\times U^{-1}A_{k+1}\left(U^{-1}A_2\left(\Omega_{\alpha_1+\beta_1,...,\alpha_m+\beta_m}(x)\,t,h_1\right),h_2,...,h_m\right)+$$
$$+\|h_1\|_X\cdot \mathcal{R}(x;h_1,h_2,...,h_{k+1}) \tag{43}$$

where:

$$\Omega_{\alpha_1+\beta_1,\ldots,\alpha_m+\beta_m}(x) = \left[\omega\begin{pmatrix} x_1 & , \ldots, & x_m & ; x \\ p_1+\alpha_1+\beta_1 & & p_m+\alpha_m+\beta_m & \end{pmatrix}\right]^{-1} \in (Y,X)^*$$

and:

$$\mathcal{R}(x;h_1,h_2,\ldots,h_{k+1}) = $$
$$= (-1)^k k! \sum_{\alpha_1+\ldots+\alpha_m=k} \prod_{i=1}^{m} \binom{p_i+\alpha_i-1}{\alpha_i} U^{-1}A_{k+1}\left(R(x,h_1)t, h_2,\ldots,h_m\right).$$

Therefore:

$$\|\mathcal{R}(x;h_1,h_2,\ldots,h_{k+1})\|_Y \le k! \left\|U^{-1}\right\| \cdot \|A_{k+1}\| \cdot \|t\|_Y \cdot \|h_2\|_X \ldots \|h_m\|_X \times$$
$$\times \sum_{\alpha_1+\ldots+\alpha_m=k} \prod_{i=1}^{m} \binom{p_i+\alpha_i-1}{\alpha_i} \|R(x,h)\|_X,$$

from where:

$$\lim_{h_1\to\theta_X} \|\mathcal{R}(x;h_1,h_2,\ldots,h_{k+1})\|_Y = 0.$$

If we denote $\gamma_i = \alpha_i + \beta_i$ for $i = \overline{1,m}$ it evidently results that:

$$\gamma_1 + \ldots + \gamma_m = (\alpha_1 + \ldots + \alpha_m) + (\beta_1 + \ldots + \beta_m) = k+1$$

and:

$$\binom{p_i+\alpha_i-1}{\alpha_i} \cdot \binom{p_i+\alpha_i+\beta_i-1}{\beta_i} = \binom{\gamma_i}{\alpha_i} \cdot \binom{p_i+\gamma_i-1}{\gamma_i},$$

so if we denote:

$$T_{\alpha_1+\beta_1,\ldots,\alpha_m+\beta_m} = U^{-1}A_{k+1}\left(U^{-1}A_2\left(\Omega_{\alpha_1+\beta_1,\ldots,\alpha_m+\beta_m}(x)t, h_1\right), h_2,\ldots,h_m\right),$$

we have:

$$\sum_{\alpha_1+\ldots+\alpha_m=k} \prod_{i=1}^{m} \binom{p_i+\alpha_i-1}{\alpha_i} \sum_{\beta_1+\ldots+\beta_m=1} \prod_{i=1}^{m} \binom{p_i+\alpha_i+\beta_i-1}{\beta_i} T_{\alpha_1+\beta_1,\ldots,\alpha_m+\beta_m} =$$
$$= \sum_{\gamma_1+\ldots+\gamma_m=k+1} \prod_{i=1}^{m} \binom{p_i+\gamma_i-1}{\gamma_i} T_{\gamma_1,\ldots,\gamma_m} \cdot \sum_{\alpha_1+\ldots+\alpha_m=k} \prod_{i=1}^{m} \binom{\gamma_i}{\alpha_i}$$
$$= (k+1) \sum_{\gamma_1+\ldots+\gamma_m=k+1} \prod_{i=1}^{m} \binom{p_i+\gamma_i-1}{\gamma_i} T_{\gamma_1,\ldots,\gamma_m}.$$

We have used the identity:

$$\sum_{\alpha_1+\ldots+\alpha_m=k} \prod_{i=1}^{m} \binom{\gamma_i}{\alpha_i} = \binom{\gamma_1+\ldots+\gamma_m}{\alpha_1+\ldots+\alpha_m} = \binom{k+1}{k} = k+1.$$

In this way, because $k!\,(k+1) = (k+1)!$, we are able to write the relation:

$$\left[\left(g^{(k)}\left(x+h_1\right)-g^{(k)}\left(x\right)\right)h_2...h_{k+1}\right]t =$$
$$= \left(-1\right)^{k+1}\left(k+1\right)! \sum_{\gamma_1+...+\gamma_m=k+1} \prod_{i=1}^{m} \binom{p_i+\gamma_i-1}{\gamma_i} T_{\gamma_1,...,\gamma_m}$$
$$+ \left\|h_1\right\|_X \cdot \mathcal{R}\left(x; h_1, h_2, ..., h_{k+1}\right);$$

whence we deduce that there exists the mapping $g^{(k+1)}(x) \in \left(Y^{k+1}, X\right)^*$.

It is clear that:

$$\begin{aligned} T_{\gamma_1,...,\gamma_m} &= U^{-1}A_{k+1}\left(U^{-1}A_2\left(\left[\omega\left(\begin{smallmatrix} x_1 \\ p_1+\gamma_1 \end{smallmatrix}, \ldots, \begin{smallmatrix} x_m \\ p_m+\gamma_m \end{smallmatrix}; x\right)\right]^{-1} t, h_1\right), h_2, \ldots, h_{k+1}\right) \\ &= U^{-1}A_{k+1}\left(U^{-1}A_2\left(W_{\gamma_1,...\gamma_m}\left(t\right), h_1\right), h_2, ..., h_{k+1}\right) \\ &= U^{-1}B\left(UU^{-1}A_2\left(W_{\gamma_1,...\gamma_m}\left(t\right), h_1\right), A_k\left(h_2, ..., h_{k+1}\right)\right) \\ &= U^{-1}A_{k+1}\left(W_{\gamma_1,...\gamma_m}\left(t\right), h_1, h_2, ..., h_{k+1}\right), \end{aligned}$$

where we have denoted:

$$W_{\gamma_1,...\gamma_m} = \left[\omega\left(\begin{matrix} x_1 \\ p_1+\gamma_1 \end{matrix}, ..., \begin{matrix} x_m \\ p_m+\gamma_m \end{matrix}; x\right)\right]^{-1} \in \left(Y, X\right)^*.$$

Therefore:

$$\left[g^{(k+1)}\left(x\right)h_1...h_{k+1}\right]\left(t\right) = \left(-1\right)^{k+1}\left(k+1\right)! \times$$
$$\times \sum_{\gamma_1+...+\gamma_m=k+1} \prod_{i=1}^{m} \binom{p_i+\gamma_i-1}{\gamma_i} U^{-1}A_{k+1}\left(W_{\gamma_1,...\gamma_m}\left(t\right), h_1, h_2, ..., h_{k+1}\right),$$

and so the equality (27) is true for $n = k+1$.

Based on the principle of mathematical induction the equality (27) is true for every $n \in \mathbb{N}$.

Proposition 12 is proved. □

4. THE CONSTRUCTION OF AN ABSTRACT INTERPOLATION POLYNOMIAL WITH MULTIPLE NODES

Let us consider the linear normed spaces X and Y, the set $D \subseteq X$ and the function $f : D \to Y$.

The general interpolation problem has the following setting.

Being given the distinct elements $x_0, x_1, ..., x_m \in D$, the numbers $r_0, r_1, ..., r_m \in N$ and the values $f^{(j)}(x_i) \in \left(X^j, Y\right)^*$ with $j = \overline{0, r_i - 1}$; $i = \overline{0, m}$, determine a $(U - B)$ polynomial $P : X \to Z$ with the minimum degree so that for any $i = \overline{0, m}$ and $j = \overline{0, r_i - 1}$ we have $P^{(j)}(x_i) = f^{(j)}(x_i)$, the equality being understood as one between the elements of $\left(X^j, Y\right)^*$.

To provide an answer to this problem let us suppose that the hypotheses from the preliminaries are fulfilled. Then let us consider the mappings that we have introduced by (26) .

To simplify the writing we will introduce:

$$\begin{aligned} X_i &= (x_1, ..., x_{i-1}, x_{i+1}, ..., x_m) \in X^{m-1}, \\ R_i &= (r_1, ..., r_{i-1}, r_{i+1}, ..., r_m) \in \mathbb{N}^{m-1}, \\ J_i &= (\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_m) \in \mathbb{N}^{m-1}, \\ |J_i| &= \alpha_1 + ... + \alpha_{i-1} + \alpha_{i+1} + ... + \alpha_m \in \mathbb{N}. \end{aligned}$$

together with the mapping from $(X, Y)^*$:

(44)

$$\mathcal{W}_{m,i}^{(X_i,R_i,J_i)} = \omega \left(\begin{matrix} x_1 \\ r_1 + \alpha_1 \end{matrix}, ..., \begin{matrix} x_{i-1} \\ r_{i-1} + \alpha_{i-1} \end{matrix}, \begin{matrix} x_{i+1} \\ r_{i+1} + \alpha_{i+1} \end{matrix}, ..., \begin{matrix} x_m \\ r_m + \alpha_m \end{matrix}; x_i \right).$$

If $x_i - x_j \in X_0$ for any $i, j \in \{1, 2, ..., m\}$ then, based on what we have established in the previous paragraph, the mapping (44) is invertible on Y_0, and it is possible to prolong the inverted mapping to $cl\,(sp\,(Y_0))$.

We denote this inverted mapping as $\left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} \in (Y, X)^*$.

As an answer to the aforementioned interpolation problem, we have the following theorem:

Theorem 13. *If for every $i, j \in \{1, 2, ..., m\}$ with $i \neq j$ we have $x_i - x_j \in X_0$ and $f^{(j)}(x_i) \in (X, cl\,(sp\,(Y_0)))^*$, there exists a $(U - B)$ polynomial with the degree $n = r_1 + ... + r_m - 1$ that fulfills the conditions of the general interpolation problem. The expression of this $(U - B)$ polynomial is:*

$$\mathbf{H}_n \left(\begin{matrix} x_1 \\ r_1 \end{matrix}, ..., \begin{matrix} x_m \\ r_m \end{matrix}; f \right)(x) = \sum_{i=1}^{m} \sum_{j=0}^{r_i - 1} l_{ij}(f)(x), \tag{45}$$

where $l_{ij}(f) : X \to Y$ has the form:

$$\mathbf{l}_{ij}(f)(x) = \tfrac{1}{j!} \sum_{k=0}^{r_i - j - 1} (-1)^k \mathbf{Q}_{n,i,j,k}(f; x) \tag{46}$$

where $\mathbf{Q}_{n,i,j,k}(f; x)$ is the value of the mapping $A_{n-r_i+k+2} \in \left(X^{n-r_i+k+2}, Y\right)^*$ at the arguments:

$$(x - x_1)^{r_1}, ..., (x - x_{i-1})^{r_{i-1}}, (x - x_i)^k, (x - x_{i+1})^{r_{i+1}}, ..., (x - x_m)^{r_m};\ Z_{n,i,j,k}(x)$$

with:

$$Z_{n,i,j,k}(x) = \sum_{|J_i| = k} \prod_{j=1,\, j \neq i}^{m} \binom{r_j + \alpha_j - 1}{\alpha_j} \left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} f^{(j)}(x_i)(x - x_i)^j . \tag{47}$$

Proof. To simplify the writing we will use the notation:

$$\mathbf{H}_n\begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, ..., \begin{matrix} x_m \\ r_m \end{matrix}; f\Big) = \mathbf{H}_n : X \to Y.$$

The theorem is proved, if in the form (45) the element $l_{ij}(f)(x)$ is under the form (46) with the specification (47).

As $\mathbf{H}_n : X \to Y$ is an abstract interpolation polynomial it is necessary that:

$$[l_{ij}(f)]^{(t)}(x_s) = \delta_{is}\delta_{tj}f^{(j)}(x_i) \tag{48}$$

where δ_{pq} is Kronecker's symbol. In the equality (48) we have the values $i, s \in \{1, 2, ..., m\}$ and for a fixed i the indices $j, t \in \{0, 1, ..., r_{i-1}\}$.

In order to fulfill the conditions (48) we search the mappings $l_{ij}(f) : X \to Y$ under the form:

$$l_{ij}(f)(x) = B(g_i(x), S_{ij}(x)); \quad j = \overline{0, r_i - 1};\ i = \overline{1, m}; \tag{49}$$

where for any $i = \overline{1, m}$ we have:

$$g_i(x) = A_{n-r_i+1}\big((x - x_1)^{r_1}, ...\underset{i}{|}..., (x - x_m)^{r_m}\big) \tag{50}$$

here, $(x - x_1)^{r_1}, ...\underset{i}{|}..., (x - x_m)^{r_m}$ is an abbreviation for:

$$(x - x_1)^{r_1}, ..., (x - x_{i-1})^{r_{i-1}}, (x - x_{i+1})^{r_{i+1}}, ..., (x - x_m)^{r_m}$$

and:

$$S_{ij}(x) = B\big(A_j(x - x_i)^j, h_{ij}(x)\big);\ j = \overline{0, r_i - 1};\ i = \overline{1, m}; \tag{51}$$

where $h_{ij} : X \to Y$ is a mapping to be determined.

From the relations (49), (50) and (51) it is clear that:

$$l_{ij}^{(t)}(f)(x_s) = 0$$

for $s \neq i$, but also in the situation where $s = i$ and $t < j$.

We will determine the mappings $h_{ij} : X \to Y$ from (51) so that:

$$l_{ij}^{(j)}(f)(x_i) = f^{(j)}(x_i), \text{ and } l_{ij}^{(t)}(f)(x_i) = 0 \text{ for } t > j. \tag{52}$$

The relations (49)–(51) indicate that for the abstract interpolation polynomial to be determined, it is necessary to choose for h_{ij} a $(U - B)$ abstract polynomial with the degree $r_i - j - 1$.

We will now consider Taylor's formula for the case of non-linear mappings between linear normed spaces, formula which for a function $F : \Omega \to Y$, where Ω is an open and convex set of the linear normed space X, that admits Fréchet differentials up to the $n+1$ Fréchet differential, included, is expressed through:
(53)

$$\left\|F(x) - \sum_{k=0}^{n} \tfrac{F^{(k)}(x_0)}{k!}(x - x_0)^k\right\|_Y \leq \tfrac{\|x-x_0\|_X^{n+1}}{(n+1)!} \sup_{\theta\in[0,1]} \left\|F^{(n+1)}(x_0 + \theta(x - x_0))\right\|.$$

If $F:\Omega\to Y$ is a $(U-B)$ abstract polynomial with the degree $\leq n$ then using the proposition 8 we deduce that $F^{(n+1)}(y)=\Theta_{n+1}$, therefore from the inequality (53) we deduce that in this case we will have:

$$F(x)=\sum_{k=0}^{n}\tfrac{F^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{54}$$

Because the mapping $h_{ij}:X\to Y$ from the equality (51) is a $(U-B)$ abstract polynomial with the degree r_i-j-1, we have:

$$h_{ij}(x)=\sum_{k=0}^{r_i-j-1}\tfrac{1}{k!}h_{ij}^{(k)}(x_i)(x-x_i)^k. \tag{55}$$

From (49), (51) and (55) we deduce that:

$$l_{ij}(f)(x)=\sum_{k=0}^{r_i-j-1}\tfrac{1}{k!}B\left(g_i(x),B(A_{ij}(x-x_i)^j,h_{ij}^{(k)}(x_i)(x-x_i)^k)\right) \tag{56}$$

and this relation indicates that the problem is solved if the elements $h_{ij}^{(k)}(x_i)\in\left(X^k,Y\right)^*$ are determined and the equalities (52) are fulfilled.

Let us define for every $x\in X$ the mapping $\widetilde{g}_i(x)\in(Y,Y)^*$ by $\widetilde{g}_i(x)t=B(g_i(x),t)$ for $t\in Y$.

From this equality we deduce that:

$$B(A_j(x-x_i)^j,h_{ij}(x))=[\widetilde{g}_i(x)]^{-1}l_{ij}(f)(x). \tag{57}$$

Considering the equality of the Fréchet differentials of the order $j+k$ of the mappings from the first and the second member of the equality (57) and using the relations (10) and (22) we obtain:

$$\sum_{s=0}^{j+k}\tbinom{j+k}{s}B\left([A_j(x-x_i)^j]^{(j+k-s)}t^{j+k-s},h_{ij}^{(s)}(x)t^s\right)= \tag{58}$$
$$=\sum_{s=0}^{j+k}\tbinom{j+k}{s}\left\{([\widetilde{g}_i(x)]^{-1})^{(j+k-s)}t^{j+k-s}\right\}l_{ij}^{(s)}(x)t^s.$$

On the account of Proposition 8 and of the fact that:

$$l_{ij}^{(s)}(f)(x_i)=\delta_{sj}f^{(j)}(x_i),$$

if in the equality (58) we replace $x=x_i$ we obtain:

$$j!B(A_jt^j,h_{ij}^{(k)}(x_i)t^k)=\left\{([\widetilde{g}_i(x)]^{-1})_{x=x_i}^{(k)}t^k\right\}f^{(j)}(x_i)t^j. \tag{59}$$

If we introduce:

$$\widetilde{\widetilde{g}}_i(x)=\omega\begin{pmatrix}x_1\\r_1\end{pmatrix},...,\begin{matrix}x_{i-1}\\r_{i-1}\end{matrix},\begin{matrix}x_{i+1}\\r_{i+1}\end{matrix},...,\begin{matrix}x_m\\r_m\end{matrix};x\Big)\in(X,Y)^*$$

it is obvious that $\widetilde{g}_i(x)t=\widetilde{\widetilde{g}}_i(x)U^{-1}(t)$.

On account of the Proposition 12 we deduce that:

$$\left\{\left(\left[\widetilde{\widetilde{g}}_i\left(x\right)\right]^{-1}\right)^{(k)} t^k\right\} u = \tag{60}$$
$$= (-1)^k\, k! \sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} U^{-1} A_{k+1}\left(\left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} u, t^k\right).$$

From $\widetilde{g}_i\left(x\right) = \widetilde{\widetilde{g}}_i\left(x\right) U^{-1}$ we have $\left[\widetilde{g}_i\left(x\right)\right]^{-1} = U\left[\widetilde{\widetilde{g}}_i\left(x\right)\right]^{-1}$ therefore from (60) we deduce that:

$$\left\{\left(\left[\widetilde{g}_i\left(x\right)\right]^{-1}\right)^{(k)} t^k\right\} u = \tag{61}$$
$$= (-1)^k\, k! \sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} A_{k+1}\left(\left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} u, t^k\right)$$

From (59) and (61) we immediately deduce that:

$$B\left(A_j t^j, h_{ij}^{(k)}\left(x_i\right) t^k\right) = (-1)^k \tfrac{k!}{j!} \sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} \times$$
$$\times A_{k+1}\left(\left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} f^{(j)}\left(x_i\right) t^j, t^k\right) \tag{62}$$

Replacing now in (62) $t = x - x_i$ we obtain:

$$B\left(A_j^j\left(x-x_i\right), h_{ij}^{(k)}\left(x_i\right)\left(x-x_i\right)^k\right) =$$
$$= (-1)^k \tfrac{k!}{j!} \sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} \times$$
$$\times A_{k+1}\left(\left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} f^{(j)}\left(x_i\right)\left(x-x_i\right)^j, \left(x-x_i\right)^k\right) \tag{63}$$

and from the relations (56) and (63) results the equality:

$$l_{ij}\left(f\right)\left(x\right) = \tfrac{1}{j!} \sum_{k=0}^{r_i-j-1} (-1)^k \sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} \times$$
$$\times \Delta_{n,i,j,k}^{(r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_m)}\left(f\right), \tag{64}$$

where $\Delta_{n,i,j,k}^{(r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_m)}\left(f\right)$ is the value of the mapping A_{n-r_i+k+2} at the arguments:

$$\left(x-x_1\right)^{r_1}, \ldots, \left(x-x_{i-1}\right)^{r_{i-1}}, \left(x-x_i\right)^k, \left(x-x_{i+1}\right)^{r_{i+1}}, \ldots$$
$$\ldots, \left(x-x_m\right)^{r_m}, \left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} f^{(j)}\left(x_i\right)\left(x-x_i\right)^j.$$

Because:

$$\sum_{|J_i|=k} \prod_{s=1,\, s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} \left[\mathcal{W}_{m,i}^{(X_i,R_i,J_i)}\right]^{-1} f^{(j)}\left(x_i\right)\left(x-x_i\right)^j = Z_{n,i,j,k}\left(x\right)$$

evidently:

$$\sum_{|J_i|=k} \prod_{s=1,\ s\neq i}^{m} \binom{r_s+\alpha_s-1}{\alpha_s} \Delta_{n,i,j,k}^{(r_1,\ldots,r_{i-1},r_{i+1},\ldots,r_m)}(f) = \mathbf{Q}_{n,i,j,k}(f;x).$$

This last equality indicates that the theorem is proved. □

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Received by the editors: April 10, 2015.