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SANDWICH THEOREMS FOR RADIANT FUNCTIONS

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Dedicated to prof. I. Păvăloiu on the occasion of his 75th anniversary

Abstract. We show that between two graphs, one of a radiant function and the other of a coradiant, both defined on a real interval containing 0, there exists at least one line which separates the graphs. The conditions for the uniqueness of a separating linear function are also established.

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1. INTRODUCTION

The paper is concerned with the existence and uniqueness of a linear function whose graph separates the graphs of two real-valued functions defined on an interval in $\mathbb{R}$ containing zero (i.e. a radiant subset of $\mathbb{R}$), one of which being radiant and the other one coradiant. We show that, under some conditions, this sandwich-type problem has at least one solution. The uniqueness of the solution is also discussed. As application, one gives a sufficient of Hyers-Ulam type stability conditions for positively homogeneous functions.

Sandwich theorems for diverse classes of real-valued functions (monotonic, convex, quasiconvex) were considered in [1], [2], [9], [13] and for more general functions in [3], [12], [14], etc.

Let I be an interval in $\mathbb{R}$ containing 0. Then I is a radiant set, i.e. for every $x \in I$ and $\lambda \in [0,1]$ it follows $\lambda x \in I$.

A function $f : I \to \mathbb{R}$ is called a *radiant* function if

$$f(\lambda x) \leq \lambda f(x), \tag{1}$$

for all $x \in I$ and $\lambda \in [0,1]$.

A function $g : I \to \mathbb{R}$ is called a *coradiant* function if

$$g(\lambda x) \geq \lambda g(x), \tag{2}$$

for all $x \in I$ and $\lambda \in [0,1]$.

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Obviously, a function f is radiant iff $-f$ is coradiant. Every radiant function f verifies the inequality $f(0) \leq 0$ and every coradiant function g verifies $g(0) \geq 0$.

A function $h : I \to \mathbb{R}$ is called a Lipschitz function if there exists a constant $K \geq 0$ (depending on f and I), such that

$$|h(x) - h(y)| \leq K\,|x - y|\,, \tag{3}$$

for all $x, y \in I$.

The constant K is called a Lipschitz constant (for f) and the smallest Lipschitz constant is given by following expression:

$$\|f\|_I = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x \neq y,\ x, y \in I\right\}. \tag{4}$$

Denote by $Lip_0 I$ the real linear space of Lipschitz functions on I vanishing at 0, i.e.

$$Lip_0 I := \{f : I \to \mathbb{R},\ f(0) = 0 \text{ and } \|f\|_I < \infty\}. \tag{5}$$

The functional $\|\ \|_I : Lip_0 I \to [0, \infty)$ is a norm (called the Lipschitz norm) and $Lip_0 I$ is a Banach space with respect to this norm [4].

Denote

$$r\text{-}Lip_0 I := \{f \in Lip_0 I,\ f \textit{ radiant}\}, \tag{6}$$

$$cr\text{-}Lip_0 I := \{g \in Lip_0 I,\ g \textit{ coradiant}\} \tag{7}$$

and

$$o\text{-}Lip_0 I := \{h \in Lip_0 I : h(\lambda x) = \lambda h(x),\ \lambda \in [0, 1],\ x \in I\}. \tag{8}$$

Then $r\text{-}Lip_0 I$ and $cr\text{-}Lip_0 I$ are convex cones in the linear space $Lip_0 I$ and $o\text{-}Lip_0 I$ is a subspace of $Lip_0 I$. Also

$$o\text{-}Lip_0 I = (r\text{-}Lip_0) \cap (cr\text{-}Lip_0 I). \tag{9}$$

2. EXTENSIONS PRESERVING RADIANTNESS OF A LIPSCHITZ FUNCTIONS

The following extension result for real valued Lipschitz functions defined on a subset of a metric space was given by McShane [5]:

THEOREM 1. *Let (X, d) be a metric space, Y a nonvoid subset of X and $f : Y \to \mathbb{R}$ be a Lipschitz function having the Lipschitz constant $K(f)$(on Y). Then there exists a Lipschitz function $F : X \to \mathbb{R}$ such that*

$$F|_Y = f \quad \textit{and} \quad K(F) = K(f). \tag{10}$$

Such a function F is called a Lipschitz extension of f, preserving the Lipschitz constant. In the proof of this theorem one shows that the following two functions

$$F(f)(x) := \inf_{y \in Y}\{f(y) + K(f)d(x, y)\},\ x \in X, \tag{11}$$

and

$$G(f)(x) := \inf_{y \in Y}\{f(y) - K(f)d(x,y)\},\ x \in X \tag{12}$$

are Lipschitz extensions of f, preserving the constant $K(f)$, and every other extension H verifies the inequalities:

$$F(x) \geq H(x) \geq G(x),\ x \in X \tag{13}$$

(see also [7], [8] for a more general situation).

In the framework considered above we obtain the following result:

THEOREM 2. *Let $I \subset \mathbb{R}$ such that $0 \in I$ and $f : I \to \mathbb{R}$.*

a) *If $f \in r\text{-}Lip_0 I$ then the greatest extension of f, namely*

$$F(f)(x) := \inf_{y \in I}\{f(y) + \|f\|_I\,|x - y|\},\ x \in \mathbb{R} \tag{14}$$

is a radiant function on $\mathbb{R}$,

b) *If $g \in cr\text{-}Lip_0 I$, then the smallest extension of g, namely*

$$G(g)(x) := \sup_{y \in I}\{g(y) - \|g\|_I\,|x - y|\},\ x \in \mathbb{R} \tag{15}$$

is a coradiant function on $\mathbb{R}$.

c) *If $h \in o\text{-}Lip_0 I$ then the extension $F(h)$ defined by (14) is in the cone $r\text{-}Lip_0\mathbb{R}$, and $G(h)$ defined by (15) is in the cone $cr\text{-}Lip_0\mathbb{R}$.*

d) *If $f \in Lip_0\mathbb{R}$ and there exists $I \subset \mathbb{R}$ with $0 \in I$ such that $\|f|_I\|_I = \|f\|_R$ and $f|_I \in o\text{-}Lip_0 I$, then*

$$G(f|_I)(x) \leq f(x) \leq F(f|_I)(x),\ x \in \mathbb{R}. \tag{16}$$

Proof. The proof is similar to that in [8]. For the sake of completeness we sketch the proof.

a) Let $f \in r\text{-}Lip_0 I$. Then, for all $\lambda \in [0,1]$ and $y \in I$,

$$\begin{aligned} F(f)(\lambda x) &= \inf_{y \in I}\{f(y) + \|f\|_I\,|\lambda x - y|\} \\ &\leq \inf_{y \in I}\{f(\lambda y) + \|f\|_I\,|\lambda x - \lambda y|\} \\ &\leq \inf_{y \in I}\{\lambda f(y) + \lambda\,\|f\|_I\,|x - y|\} \\ &= \lambda \inf_{y \in I}\{f(y) + \|f\|_I\,|x - y|\} \\ &= \lambda F(f)(x), \end{aligned}$$

for every $x \in \mathbb{R}$. It follows

$$F(f)(\lambda x) \leq \lambda F(f)(x),$$

for all $\lambda \in [0,1]$ and all $x \in \mathbb{R}$.

b) Let $g \in cr\text{-}Lip_0 I$ and $x \in \mathbb{R}$. Then for all $\lambda \in [0,1]$ and $y \in I$,

$$\begin{aligned} G(g)(\lambda x) &= \sup_{y\in I}\{g(y) - \|g\|_I\, |\lambda x - y|\} \\ &\geq \sup_{y\in I}\{g(\lambda y) - \|g\|_I\, |\lambda x - \lambda y|\} \\ &\geq \lambda \sup_{y\in I}\{g(y) - \|g\|_I\, |x - y|\} \\ &= \lambda G(g)(x) \end{aligned}$$

Consequently $G(g)(\lambda x) \geq \lambda G(g)(x)$ for all $\lambda \in [0,1]$ and all $x \in \mathbb{R}$.

c) If $h \in o\text{-}Lip_0 I = (r\text{-}Lip_0 I) \cap (cr\text{-}Lip_0 I)$, then the assertions from c) follow from a) and b).

d) Let $f \in Lip_0\mathbb{R}$ and suppose that there exists $I \subset \mathbb{R}$ with $0 \in I$ such that $f|_I \in o\text{-}Lip_0 I$ and $\|f|_I = \|f\|_R$. Then f is an extension of $f|_I$ preserving the smallest Lipschitz constant $\|\, f|_I\|$ and the assertion follows by (13) and c). $\square$

3. SANDWICH THEOREMS

The assertion d) in Theorem 2 suggests the following problem: Let $f,\ g$ be two functions, on an interval I $(0 \in I)$, f radiant (coradiant) and g coradiant (radiant), and $f(x) \leq g(x)$ for all $x \in I$. Is there a function $h : I \to \mathbb{R}$ verifying $h(\lambda x) = \lambda h(x)$, for all $x \in I$ and all $\lambda \in [0,1]$, and such that $f(x) \leq h(x) \leq g(x)$, $x \in I$? Problems of such type were considered for example in [1], [2], [9] and, more general in [3], [12], [14], etc. The studies in this direction are motivated by applications to the theory of optimization and numerical analysis.

We show that the answer is affirmative in both cases.

Firstly, we need the following Lemma.

LEMMA 3. *Let $f, g : [0,a] \to \mathbb{R}$, f radiant and g coradiant such that $f(x) \leq g(x)$, $x \in [0,a]$. Then the following assertions hold:*

a) *The function $p_f : (0,a] \to \mathbb{R}$ defined by*

$$p_f(x) = \frac{f(x)}{x}, \tag{17}$$

is nondecreasing on $(0,a]$, and the function $p_g : (0,a] \to \mathbb{R}$ defined by

$$p_g(x) = \frac{g(x)}{x}, \tag{18}$$

is nonincreasing on $(0,a]$;

b) *For every $x_0 \in (0,a]$,*

$$\frac{f(x)}{x} \leq \frac{f(x_0)}{x_0} \leq \frac{g(x_0)}{x_0} \leq \frac{g(x)}{x},\ x \in (0, x_0]. \tag{19}$$

Proof. a) Let $f : [0, a] \to \mathbb{R}$, f radiant. For every $x \in (0, a]$ let $\lambda \in [0, 1]$ be such that $x = \lambda a$. Then

$$p_f(x) = \frac{f(x)}{x} = \frac{f(\lambda a)}{\lambda a} \leq \frac{\lambda f(a)}{\lambda a} = \frac{f(a)}{a}.$$

Let $x_1, x_2 \in (0, a]$, $x_1 \leq x_2$. Then there exists $\lambda \in (0, 1]$ such that $x_1 = \lambda x_2$, so that

$$p_f(x_1) = \frac{f(x_1)}{x_1} = \frac{f(\lambda x_2)}{\lambda x_2} \leq \frac{\lambda f(x_2)}{\lambda x_2} = \frac{f(x_2)}{x_2} = p_f(x_2).$$

Also

$$p_g(x_1) = \frac{g(x_1)}{x_1} = \frac{g(\lambda x_2)}{\lambda x_2} \geq \frac{\lambda g(x_2)}{\lambda x_2} = \frac{g(x_2)}{x_2} = p_g(x_2).$$

Consequently p_f is nondecreasing and p_g is nonincreasing on $(0, a]$.

b) Taking into account the hypothesis and the conclusions in a), the inequalities (19) follow. □

Remark 4. The inequalities (17), (18) and (19) imply that every line passing through the points (0,0) and $(x, f(x))$ is above the graph of f on the interval $[0, x]$, and every line passing through the points (0,0) and $(x, g(x))$ is bellow the graph of g on $[0, x]$. Also, the line passing through (0,0) and $(x, g(x))$ is above the line passing through (0,0) and $(x, f(x))$. □

We obtain the main result of the paper.

Theorem 5. *Let $f, g : [0, a] \to \mathbb{R}$, f radiant and g coradiant, and $f(x) \leq g(x)$, for all $x \in [0, a]$. Then there exists at least a function $h : [0, a] \to \mathbb{R}$ having the form $h(x) = \alpha x$, $\alpha \in \mathbb{R}$ and such that*

$$f(x) \leq h(x) \leq g(x),\ x \in [0, a]. \tag{20}$$

If there exists $x_0 \in (0, a]$ such that $f(x_0) = g(x_0)$ then the unique function $h : [0, a] \to \mathbb{R}$ verifying (20) is $h(x) = \frac{f(a)}{a} x$.

Proof. By Lemma 3, taking into account (19), one obtains

$$\frac{f(x)}{x} \leq \frac{f(a)}{a} \leq \frac{g(a)}{a} \leq \frac{g(x)}{x}, x \in (0, a].$$

Then, for $\alpha \in [\frac{f(a)}{a}, \frac{g(a)}{a}]$ the function $h(x) = \alpha x$, $x \in [0, a]$ verifies the properties from the conclusions of first part of theorem.

Now, if $f(x_0) = g(x_0)$ at $x_0 \in (0, a]$ one obtains

$$\frac{f(x_0)}{x_0} \leq \frac{f(a)}{a} \leq \frac{g(a)}{a} \leq \frac{g(x_0)}{x_0}$$

and then $f(a) = g(a)$. For $x \in [x_0, a]$ one obtains

$$\frac{f(x)}{x} = \frac{g(x)}{x}$$

i.e $f(x) = g(x) = \frac{f(a)}{a} x$. Then $h(x) = \frac{f(a)}{a} x$ verifies $f(x) \leq h(x) \leq g(x), x \in [0, a]$, and the second part of the theorem follows. □

In the case $I = [0, \infty)$ one obtains.

THEOREM 6. *Let $f, g : [0, \infty) \to \mathbb{R}$, f radiant and g coradiant, and $f(x) \leq g(x)$ for all $x \in [0, \infty)$. Then there exists at least a function $h : [0, \infty) \to \mathbb{R}$, of the form $h(x) = \alpha x$, $\alpha \in \mathbb{R}$ fixed such that $f(x) \leq h(x) \leq g(x)$, $x \in [0, \infty)$.*

If there exists $x_0 > 0$ such that $f(x_0) = g(x_0)$ then the unique function $h : [0, \infty) \to \mathbb{R}$ satisfying $f(x) \leq h(x) \leq g(x)$ is $h(x) = \frac{f(x_0)}{x_0} x$.

Proof. By Lemma 3

$$\frac{f(x)}{x} \leq \frac{f(a)}{a} \leq \frac{g(a)}{a} \leq \frac{g(x)}{x}, x \in (0, a],$$

for every $a \in (0, \infty)$,

Then

$$\sup_{0<x\leq a} \frac{f(x)}{x} \leq \inf_{0<x\leq a} \frac{g(x)}{x},$$

and

$$\lim_{a\to\infty} \sup_{0<x\leq a} \frac{f(x)}{x} \leq \lim_{a\to\infty} \inf_{0<x\leq a} \frac{g(x)}{x}.$$

By considering $\alpha \in [\lim_{a\to\infty} \sup_{0<x\leq a} \frac{f(x)}{x}, \lim_{a\to\infty} \inf_{0<x\leq a} \frac{g(x)}{x}]$ one obtains

$$\frac{f(x)}{x} \leq \alpha \leq \frac{g(x)}{x}, x \in (0, \infty),$$

and consequently

$$f(x) \leq \alpha x \leq g(x),\ x \in [0, \infty).$$

Then $h(x) = \alpha x, x \in [0, \infty)$, satisfies the first conclusion of the theorem.

Let $x_0 > 0$ such that $f(x_0) = g(x_0)$. For every $x > x_0$ we have

$$\frac{f(x_0)}{x_0} \leq \frac{f(x)}{x} \leq \frac{g(x)}{x} \leq \frac{g(x_0)}{x_0}$$

It follows $f(x) = g(x) = \frac{f(x_0)}{x_0} x$, for all $x > x_0$ and consequently, for $x \in [0, \infty)$ the function $h(x) = \frac{f(x_0)}{x_0} x$ verifies

$$f(x) \leq h(x) \leq g(x).$$

□

Now we consider $I = [a, 0]$.

THEOREM 7. *Let $f, g : [a, 0] \to \mathbb{R}$ $(a < 0)$, f radiant and g coradiant, and $f(x) \leq g(x)$, for all $x \in [a, 0]$. Then there exists at least a function $h : [a, 0] \to \mathbb{R}$ having the form $h(x) = \alpha x$, $\alpha \in \mathbb{R}$ fixed, and verifying the inequalities:*

$$f(x) \leq h(x) \leq g(x),\ x \in [a, 0]. \tag{21}$$

If there exists $x_0 \in [a, 0)$ such that $f(x_0) = g(x_0)$ then the only function $h : [a, 0] \to \mathbb{R}$ such that $f(x) \leq h(x) \leq g(x)$ is $h(x) = \frac{f(a)}{a} x$.

Proof. Observe that for all $x \in [a, 0]$ one obtains $f(x) \le \frac{f(a)}{a}x$. Indeed, if $x \in [a, 0)$, there exists $\lambda \in [0, 1]$ such that $x = \lambda a$. Then

$$f(x) = f(\lambda a) \le \lambda f(a) = \frac{f(a)}{a}(\lambda a) = \frac{f(a)}{a}x.$$

Analogously,

$$g(x) = g(\lambda a) \ge \lambda g(a) = \frac{g(a)}{a}(\lambda a) = \frac{g(a)}{a}x.$$

But $g(a) \ge f(a)$ implies $\frac{g(a)}{a} \le \frac{f(a)}{a}$, and for $\alpha \in \left[\frac{g(a)}{a}, \frac{f(a)}{a}\right]$ it follows

$$\frac{g(a)}{a}x \ge \alpha x \ge \frac{f(a)}{a}x,\ x \in [a, 0]$$

and, consequently $f(x) \le \alpha x \le g(x)$, $x \in [0, a]$.

Now, let $x_0 \in [a, 0)$ such that $f(x_0) = g(x_0)$.

Then

$$\frac{f(a)}{a} \le \frac{f(x_0)}{x_0} = \frac{g(x_0)}{x_0} \le \frac{g(a)}{a}.$$

We are lead to $f(a) \ge g(a)$, and because of hypothesis $f(a) \le g(a)$ it follows $\frac{f(a)}{a} = \frac{g(a)}{a}$.

Then $h(x) = \frac{f(a)}{a}x$ is the only function having the graph between the graphs of f and g on $[a, 0]$. □

Suppose now that f is radiant, g is coradiant and $f(x) \ge g(x)$ for all $x \in [0, a]$. Because $f(0) \le 0$ and $g(0) \ge 0$ it follows $f(0) = g(0) = 0$.

The following theorem holds.

Theorem 8. *Let $f, g : [0, a] \to \mathbb{R}$, f radiant, g coradiant and $f(x) \ge g(x)$, $x \in [0, a]$. Then there exists at least a function $h : [0, a] \to \mathbb{R}$ having the form $h(x) = \alpha x$, $\alpha \in \mathbb{R}$ fixed, and such that*

$$f(x) \ge h(x) \ge g(x),\ x \in [0, a]. \tag{22}$$

If there exists $x_0 \in (0, a]$ such that $f(x_0) = g(x_0)$ then $f(x) = g(x)$ for every $x \in [0, x_0]$ and the only function $h : [0, a\} \to \mathbb{R}$ verifying (22) is $h(x) = \frac{f(x_0)}{x_0}x$.

Proof. Let $x_0 \in (0, a]$ be such that $f(x_0) = g(x_0)$. If $x \in (0, x_0]$, by Lemma 3 one obtains

$$\frac{g(x)}{x} \ge \frac{g(x_0)}{x_0} = \frac{f(x_0)}{x_0} \ge \frac{f(x)}{x}.$$

It follows $g(x) \ge f(x)$, $x \in [0, x_0]$ and because $f(x) \ge g(x)$ (by hypothesis) one obtains $f(x) = g(x)$ for every $x \in [0, x_0]$.

The line $h(x) = \frac{f(x_0)}{x_0}x$ $(= \frac{g(x_0)}{x_0}x)$ is between the graphs of f and g over the interval $[0, x_0]$.

For $x > x_0$, $x \le a$

$$\frac{f(x_0)}{x_0} \le \frac{f(x)}{x} \text{ and } \frac{f(x_0)}{x_0} \le \frac{g(x)}{x}$$

and consequently the function $h(x) = \frac{f(x_0)}{x_0}x$, $x \in [0,a]$ verifies

$$f(x) \geq h(x) \geq g(x),\ x \in [0,a].$$

Consider now the remaining case $f(x) > g(x)$, $x \in (0,a]$. Because

$$\frac{g(a)}{a} \leq \frac{g(x)}{x} < \frac{f(x)}{x} \leq \frac{f(a)}{a},\ x \in (0,a]$$

it follows that $\inf\left\{\frac{f(x)}{x} : x \in (0,a]\right\}$ and $\sup\left\{\frac{g(x)}{x} : x \in (0,a]\right\}$ are finite, and

$$\sup_{0<x\leq a} \frac{g(x)}{x} \leq \inf_{0<x\leq a} \frac{f(x)}{x}.$$

By considering $\alpha \in \left[\sup\limits_{0<x\leq a} \frac{g(x)}{x}, \inf\limits_{0<x\leq a} \frac{f(x)}{x}\right]$, the line $h(x) = \alpha x$, $x \in [0,a]$ lies between the graphs of f and g, i.e.

$$f(x) \geq h(x) \geq g(x),\ x \in [0,a].$$

$\square$

Remark 9. The result in Theorem 8 is valid also if $f, g : (-\infty, 0] \to \mathbb{R}$. In this case there exists $\alpha \in [\lim\limits_{a\to-\infty} \sup_{a<x\leq 0} \frac{g(x)}{x}, \lim\limits_{a\to-\infty} \inf_{a<x\leq 0} \frac{f(x)}{x}]$ such that the function $h(x) = \alpha x$ satisfies the inequalities

$$f(x) \leq h(x) \leq g(x),\ x \in (-\infty, 0]. \quad \square$$

Now consider $I = [a,b]$ where $a < 0 < b$, or $I = \mathbb{R}$.

By the above results it follows:

Corollary 10. a) *Let $f, g : [a,b] \to \mathbb{R}$, $a < 0 < b$, f radiant, g coradiant and such that $f(x) \leq g(x)$, $x \in [a,b]$. Then there exists $h : [a,b] \to \mathbb{R}$,*

$$h(x) = \begin{cases} \alpha x,\ x \in [a,0],\ \alpha \in \left[\frac{g(a)}{a}, \frac{f(a)}{a}\right], \\ \beta x,\ x \in (0,b],\ \beta \in \left[\frac{f(b)}{b}, \frac{g(b)}{b}\right]. \end{cases}$$

such that $f(x) \leq h(x) \leq g(x)$, $x \in [a,b]$.

b) *Let $f, g : \mathbb{R} \to \mathbb{R}$, f radiant, g coradiant, $f(x) \leq g(x)$, for all $x \in \mathbb{R}$. Then there exists $h : \mathbb{R} \to \mathbb{R}$ of the form*

$$h(x) = \begin{cases} \alpha x,\ x \in (-\infty, 0], \\ \beta x,\ x \in (0, \infty), \end{cases}$$

where

$$\alpha \in \left[\lim_{a\to-\infty} \sup\nolimits_{x\geq a} \frac{g(x)}{x}, \lim_{a\to-\infty} \inf\nolimits_{x\geq a} \frac{f(x)}{x}\right]$$

and

$$\beta \in \left[\lim_{a\to\infty} \sup\nolimits_{x\leq a} \frac{f(x)}{x}, \lim_{a\to\infty} \inf\nolimits_{x\leq a} \frac{g(x)}{x}\right],$$

verifying the inequalities:

$$f(x) \le h(x) \le g(x),\ x \in \mathbb{R}.$$

A similar result is valid if $f(x) \ge g(x)$, f radiant and g coradiant on $[a,b]$, $a<0<b$, respectively on $\mathbb{R}$.

In Corollary 11 a), if $\left[\frac{g(a)}{a}, \frac{f(a)}{a}\right] \cap \left[\frac{f(b)}{b}, \frac{g(b)}{b}\right] \ne \emptyset$ and μ is a number from this set, then $h(x) = \mu x$, $x \in [a,b]$ verify

$$f(x) \le \mu x \le g(x),\ x \in [a,b].$$

A similar result follows in the case b).

If $f : I \to \mathbb{R}\ \ (0 \in I)$, and f is a convex (concave) function on I, $f(0) = 0$, then f is radiant (coradiant). The above results may be enounced for convex and concave function defined on I, vanishing at zero.

Examples. 1^0 Let $f_m, g : [0,2] \to \mathbb{R}$ be the functions defined by

$$f_m(x) = \begin{cases} mx^3, & x \in [0,1] \\ mx, & x \in (1,2] \end{cases},\ m \in \mathbb{R},$$

$$g(x) = \begin{cases} -x^2+2, & x \in [0,1], \\ x, & x \in (1,2]. \end{cases}$$

Then, for $m > 0$ the function f_m is radiant, g is coradiant and $f_m(x) \le g(x)$, $x \in [0,2]$. Every function $h : [0,2] \to \mathbb{R}$, $h(x) = \alpha x$, where $\alpha \in [m,1]$ verifies

$$f_m(x) \le h(x) \le g(x),\ x \in [0,2]$$

Also, for $m = 1$, $f_1(x) \le g(x)$, $x \in [0,2]$ and because $f_1(1) = g(1) = 1$ one obtains that $h(x) = x$ is the unique function verifying $f_1(x) \le h(x) \le g(x)$, $x \in [0,2]$. Consequently Theorem 6 is fulfilled.

2^0 Let $a > 1$, $m \in (0,1]$ and let f_m, $g_a : [0,\infty) \to \mathbb{R}$, be the functions:

$$f_m(x) = \begin{cases} \frac{mx^3}{a^2}, & x \in [0,a], \\ mx, & x \in (a+\infty), \end{cases}$$

$$g_a(x) = \begin{cases} -x^2+(1+a)x, & x \in [0,a] \\ x, & x \in (a,+\infty) \end{cases}$$

Then $f_m(x) \le g_a(x)$, for $x \in [0,\infty)$, f_m is radiant and g_a is coradiant.

For every $\alpha \in [m,1]$ one obtains

$$f_m(x) \le h(x) = \alpha x \le g(x),\ x \in [0,\infty)$$

For $m = 1$, $f_1(a) = g_a(a) = a$ and consequently $h(x) = x$ is the only function verifying $f_1(x) \le h(x) = x \le g(x)$, $x \in [0,\infty)$. Theorem 7 is fulfilled.

3^0 Let $f, g : [-2,0] \to \mathbb{R}$, $f(x) = x^2 + x$ and $g(x) = -x^2 - 4x$. Then f is radiant, g is coradiant and $f(x) \le g(x)$, $x \in [-2,0]$. Every line $h(x) = \alpha x$, where $\alpha \in [-2,-1]$ has the graph between the graphs of f and g.

Now let $f, g : [-3, 0] \to \mathbb{R}$ be the functions

$$f(x) = \begin{cases} \frac{-x}{2}, & x \in [-3, -1] \\ x^2 + \frac{x}{2}, & x \in (-1, 0], \end{cases}$$

$$g(x) = \begin{cases} \frac{-x}{2}, & x \in [-3, -1] \\ -4x^2 \frac{9}{2}x, & x \in (-1, 0]. \end{cases}$$

Then f is radiant, g is coradiant and $f(x) \le g(x),\ x \in [-3, 0]$.

The only function h such that $f(x) \le h(x) \le g(x)$, $x \in [-3, 0]$ is $h(x) = -\frac{1}{2}x$. Thus Theorem 8 is fulfilled.

4^0 Let $f_m, g : [0, a] \to \mathbb{R}$, $(a > 0,\ m \ge 0)$ be the functions

$$f_m(x) = x^2 + mx,$$
$$g(x) = -x^3.$$

Then f_m is radiant, g is coradiant and $f(x) \ge g(x),\ x \in [0, a]$. The function $h(x) = \alpha x$, where $\alpha \in [0, m]$ is such that

$$f(x) \ge h(x) \ge g(x),\ x \in [0, a].$$

For $m = 0$, the function $h(x) = 0$ verify

$$f(x) \ge h(x) \ge g(x),\ x \in [0, a].$$

5^0 Let $f, g : [0, 2] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x, & x \in [0, 1], \\ 2(x-1), & x \in (1, 2]. \end{cases}$$

$$g(x) = \begin{cases} x, & x \in [0, 1] \\ -3(x-1), & x \in (1, 2]. \end{cases}$$

Then f is radiant, g is coradiant, $f(1) = g(1)$ and $f(x) \ge g(x)$, $x \in [0, 2]$. The function $h(x) = x$, $x \in [0, 2]$ is the only function having the graph between graphs of f and g. Thus Theorem 8 is fulfilled.

4. APPLICATIONS

Let $I = [0, a]$ and let $f : I \to \mathbb{R}$. The function f is called ε-positively homogeneous if $|f(\lambda x) - \lambda f(x)| < \varepsilon$, for all $x \in [0, a]$ and $\lambda \in [0, 1]$.

The function $f : [0, a] \to \mathbb{R}$ is both radiant and coradiant iff f is positively homogeneous, i.e., $f(\lambda x) = \lambda f(x)$, for all $x \in [0, a]$ and $\lambda \in [0, 1]$.

The above results gives sufficient stability conditions of Hyers-Ulam type for positively homogeneous functions.

Corollary 11. *Let $f : [0, a] \to \mathbb{R}$ such that f is radiant or coradiant. Let $\varepsilon > 0$ be a real number. In order to obtain*

$$|f(\lambda x) - \lambda f(x)| < \varepsilon$$

for all $x \in [0,a]$ and $\lambda \in [0,1]$ it is sufficient that

$$\left| f(x) - \frac{f(a)}{a}x \right| < \varepsilon, \; x \in [0,a].$$

Proof. Let $f : [0,a] \to \mathbb{R}$ be radiant. By Lemma 3 and Theorem 6 it follows $f(x) \leq \frac{f(a)}{a}x$, for all $x \in [0,a]$.

Then

$$\begin{aligned} 0 \leq \lambda f(x) - f(\lambda x) &\leq \lambda \frac{f(a)}{a}x - f(\lambda x) = \\ &= \frac{f(a)}{a}(\lambda x) - f(\lambda x) = \\ &= \left| \frac{f(a)}{a}(\lambda x) - f(\lambda x) \right| < \varepsilon. \end{aligned}$$

Consequently

$$\lambda f(x) - f(\lambda x) = |f(\lambda x) - \lambda f(x)| < \varepsilon.$$

Now, if f is coradiant, then by Theorem 6 it follows $f(x) \geq \frac{f(a)}{a}x$ and

$$\begin{aligned} 0 \leq f(\lambda x) - \lambda f(x) &\leq f(\lambda x) - \lambda \frac{f(a)}{a}x \\ &= f(\lambda x) - \frac{f(a)}{a}\lambda x = \left| f(\lambda x) - \frac{f(a)}{a}\lambda x \right| < \varepsilon. \end{aligned}$$

Consequently

$$f(\lambda x) - \lambda f(x) = |f(\lambda x) - \lambda f(x)| < \varepsilon$$

if

$$\left| f(x) - \frac{f(a)}{a}x \right| < \varepsilon.$$

□

By Theorem 6, it follows that if $f : [0,a] \to \mathbb{R}$ is positively homogeneous, $x_0 \in (0,a]$ and $f(x_0)$ is given, then the function is exactly $f(x) = \frac{f(x_0)}{x_0}x, \; x \in [0,a]$.

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