JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 44 (2015) no. 1, pp. 100–109
ictp.acad.ro/jnaat

SECOND ORDER DIFFERENTIABILITY OF THE INTERMEDIATE-POINT FUNCTION IN CAUCHY'S MEAN-VALUE THEOREM

BEATRIX-MIHAELA POP* and DOREL I. DUCA**

Dedicated to prof. I. Păvăloiu on the occasion of his 75th anniversary

Abstract. If the functions $f, g : I \to \mathbb{R}$ are differentiable on the interval $I \subseteq \mathbb{R}$, $a \in I$, then there exists a function $\overline{c} : I \to I$ such that

$$[f(x) - f(a)]\, g^{(1)}(\overline{c}(x)) = [g(x) - g(a)]\, f^{(1)}(\overline{c}(x)), \text{ for } x \in I.$$

In this paper we study the differentiability of the function $\overline{c}$, when

$$f^{(k)}(a)\, g^{(1)}(a) = f^{(1)}(a)\, g^{(k)}(a), \text{ for all } k \in \{1, ..., n-1\}$$

and

$$f^{(n)}(a)\, g^{(1)}(a) \neq f^{(1)}(a)\, g^{(n)}(a).$$

MSC 2010. 26A24.

Keywords. Cauchy's theorem, intermediate point, mean-value theorem.

1. INTRODUCTION

The mean value theorem is a cornerstone of the differential calculus. Cauchy's theorem is one of the generalizations of the mean value theorem.

The purpose of this note is to extend the results by D.I. Duca and O. Pop [1] concerning the mean value theorem to Cauchy's theorem. Also, the results can be considered to extend the results by D.I. Duca and O. Pop from [2].

Cauchy's theorem is usually presented in the following form:

THEOREM 1. (*A.L. Cauchy*) *Let I be an interval in $\mathbb{R}$, and let a be a point of I. If the functions $f, g : I \to \mathbb{R}$ are differentiable on I, then for each $x \in I \backslash \{a\}$ there exists a point c_x, from the interval with extremities x and a, such that*

$$[f(x) - f(a)]\, g^{(1)}(c_x) = [g(x) - g(a)]\, f^{(1)}(c_x).$$

* *Babeş-Bolyai* University of Cluj-Napoca, Faculty of Mathematics and Computer Science, M. Kogălniceanu, no. 1, Cluj-Napoca, Romania, e-mail: pop_mbeatrix@yahoo.com.

** *Babeş-Bolyai* University of Cluj-Napoca, Faculty of Mathematics and Computer Science, M. Kogălniceanu, no. 1, Cluj-Napoca, Romania, e-mail: dduca@math.ubbcluj.ro; dorelduca@yahoo.com.

If, in addition, $g^{(1)}(x) \neq 0$, *for all* $x \in \text{int} I$, *then* $g(x) \neq g(a)$, *for all* $x \in I \backslash \{a\}$, *and*

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(c_x)}{g^{(1)}(c_x)}, \quad \textit{for all } x \in I \backslash \{a\}. \tag{1}$$

If the function $f^{(1)}/g^{(1)}$ is injective on I, then for each $x \in I \backslash \{a\}$ there exists a unique point c_x, from the interval with the extremities x and a, such that (1) holds. In this case, we can define the function $c : I \backslash \{a\} \to I \backslash \{a\}$ by

$$c(x) = c_x, \quad \text{for all } x \in I \backslash \{a\}. \tag{2}$$

The function c has the property that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))}, \quad \text{for all } x \in I \backslash \{a\}. \tag{3}$$

If the function $f^{(1)}/g^{(1)}$ is not injective on I, then for some $x \in I \backslash \{a\}$ there exist several points c_x, from the interval with the extremities x and a, such that (1) is true. If for each $x \in I \backslash \{a\}$ we choose one c_x from the interval with the extremities x and a which satisfies (1), then we can also define the function $c : I \backslash \{a\} \to I \backslash \{a\}$ by formula (2). This function c satisfies (3), too (see [2]).

Consequently, the following statement is true.

Theorem 2. *Let* I *be an interval in* $\mathbb{R}$, *and let* a *be a point of* I. *If the functions* $f, g : I \to \mathbb{R}$ *are differentiable on* I *and* $g^{(1)}(x) \neq 0$, *for all* $x \in I \backslash \{a\}$, *then there exists a function* $c : I \backslash \{a\} \to I \backslash \{a\}$ *such that (3) is true.*

Furthermore, if, in addition, $f^{(1)}/g^{(1)}$ *is injective, then the function* c *is unique.*

If $x \in I \backslash \{a\}$ tends to a, because $|c(x) - a| \leq |x - a|$, we have

$$\lim_{x \to a} c(x) = a.$$

Then the function $\overline{c} : I \to I$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in I \backslash \{a\} \\ a, & \text{if } x = a \end{cases} \tag{4}$$

is continuous at $x = a$.

The purpose of this paper is to establish under which circumstances the function $\overline{c}$ is twice differentiable at the point $x = a$ and to compute its derivatives $\overline{c}^{(1)}(a)$ and $\overline{c}^{(2)}(a)$. Do the derivatives $\overline{c}^{(1)}(a)$ and $\overline{c}^{(2)}(a)$ depend upon the functions f and g? If there exist several functions $\overline{c}$ which satisfy (3), do the derivatives of the function $\overline{c}$ at $x = a$ depend upon the function $\overline{c}$ we choose?

Since for $x \in I \backslash \{a\}$,

$$\frac{\overline{c}(x) - \overline{c}(a)}{x - a} = \frac{c(x) - a}{x - a},$$

if we denote by

$$\theta(x) = \frac{c(x) - a}{x - a},$$

then

$$\theta(x) \in]0, 1[$$

and

$$c(x) = a + (x - a)\theta(x)$$

and hence

$$[f(x) - f(a)]\, g^{(1)}(a + (x - a)\theta(x)) = [g(x) - g(a)]\, f^{(1)}(a + (x - a)\theta(x)).$$

Consequently, the following statement is true.

THEOREM 3. *Let I be an interval in $\mathbb{R}$, and let a be a point of I. If the functions $f, g : I \to \mathbb{R}$ are differentiable on I, and $g^{(1)}(x) \neq 0$, for all $x \in \text{int}I$, then*

$$g(x) \neq g(a), \text{ for all } x \in I \backslash \{a\},$$

and there exists a function $\theta : I \backslash \{a\} \to]0, 1[$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(a + (x - a)\theta(x))}{g^{(1)}(a + (x - a)\theta(x))}, \text{ for all } x \in I \backslash \{a\}.$$

Furthermore, if, in addition, the function $f^{(1)}/g^{(1)}$ is injective, then the function θ is unique.

Obviously, the function $\bar{c} : I \to I$ defined by (4) is differentiable at $x = a$ if and only if the function $\theta : I \backslash \{a\} \to]0, 1[$ defined by

$$\theta(x) = \frac{\bar{c}(x) - \bar{c}(a)}{x - a} = \frac{c(x) - a}{x - a}, \text{ for all } x \in I \backslash \{a\}$$

has limit at the point $x = a$. Moreover, if the function $\bar{c}$ is differentiable at $x = a$, then

$$\bar{c}^{(1)}(a) = \lim_{x \to a} \theta(x).$$

In [2], one proves the following theorem

THEOREM 4. *Let I be an interval in $\mathbb{R}$, let a be a point of I and let $f, g : I \to \mathbb{R}$ be functions which satisfy the following conditions:*

(a) the functions f and g are $n \geq 2$ times differentiable on I,

(b) the functions $f^{(n)}$ and $g^{(n)}$ are continuous on I,

(c) $f^{(1)}(a)\, g^{(k)}(a) = f^{(k)}(a)\, g^{(1)}(a)$, for all $k \in \{1, ..., n - 1\}$,

(d) $f^{(1)}(a)\, g^{(n)}(a) \neq f^{(n)}(a)\, g^{(1)}(a)$.

Then the following statements are true:

1^0 *If $\theta : I \backslash \{a\} \to]0, 1[$ is a function such that*

$$[f(x) - f(a)]\, g^{(1)}(a + (x - a)\theta(x)) = [g(x) - g(a)]\, f^{(1)}(a + (x - a)\theta(x)),$$

for all $x \in I \backslash \{a\}$, *then there exists the limit*

$$\lim_{x \to a} \theta(x) = \begin{cases} \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{\sqrt[n-1]{n}}, & \text{if } n \geq 3. \end{cases}$$

2^0 *If* $c : I \backslash \{a\} \to I$ *is a function such that*

$$[f(x) - f(a)]\, g^{(1)}(c(x)) = [g(x) - g(a)]\, f^{(1)}(c(x)),$$

for all $x \in I \backslash \{a\}$, *then the function* $\overline{c} : I \to I$ *defined by (4) is differentiable at* $x = a$ *and*

$$\overline{c}^{(1)}(a) = \begin{cases} \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{\sqrt[n-1]{n}}, & \text{if } n \geq 3. \end{cases}$$

Related to the higher order differentiability of the intermediate point function in Cauchy's mean-value theorem one know the following result (see [3])

Theorem 5. *Let* I *be an interval in* $\mathbb{R}$, *let* a *be a point of* I *and let* $f, g : I \to \mathbb{R}$ *be functions which satisfy the following conditions:*

(*a*) *the functions* f *and* g *are* 3 *times differentiable on* I,
(*b*) *the functions* $f^{(3)}$ *and* $g^{(3)}$ *are continuous at* $x = a$,
(*c*) $g^{(1)}(x) \neq 0$, *for all* $x \in \operatorname{int} I$,
(*d*) $f^{(1)}(a)\, g^{(2)}(a) \neq f^{(2)}(a)\, g^{(1)}(a)$.

Then the following statements are true:

1^0 *There exists a real number* $\delta > 0$ *such that:*
i) $]a - \delta, a + \delta[\subseteq I$;
ii) $f^{(1)}(x) g^{(2)}(x) \neq f^{(2)}(x) g^{(1)}(x)$, *for all* $x \in]a - \delta, a + \delta[$;
iii) *the function* $f^{(1)}/g^{(1)}$ *is injective on* $]a - \delta, a + \delta[$.

2^0 *There exists a unique function* $c :]a - \delta, a + \delta[\backslash \{a\} \to]a - \delta, a + \delta[\backslash \{a\}$ *such that*

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))}, \tag{5}$$

for all $x \in]a - \delta, a + \delta[\backslash \{a\}$.

3^0 *There exists a unique function* $\theta :]a - \delta, a + \delta[\backslash \{a\} \to]0, 1[$ *such that*

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(a + (x - a)\theta(x))}{g^{(1)}(a + (x - a)\theta(x))}, \tag{6}$$

for all $x \in I \backslash \{a\}$.

4^0 *The function* θ *has limit at* $x = a$ *and*

$$\lim_{x \to a} \theta(x) = \tfrac{1}{2}.$$

5^0 *The function* $\overline{\theta} :]a - \delta, a + \delta[\to]0, 1[$ *defined by*

$$\overline{\theta}(x)=\begin{cases}\theta(x), & \text{if } x\in]a-\delta,a+\delta[\setminus\{a\}\\ 1/2, & \text{if } x=a,\end{cases}$$

is differentiable at $x=a$ and

$$\overline{\theta}^{(1)}(a)=\frac{f^{(1)}(a)\,g^{(3)}(a)-f^{(3)}(a)\,g^{(1)}(a)}{24\left[f^{(1)}(a)\,g^{(2)}(a)-f^{(2)}(a)\,g^{(1)}(a)\right]}.$$

6^0 *The function $\overline{c}:]a-\delta,a+\delta[\to]a-\delta,a+\delta[$ defined by*

$$\overline{c}(x)=\begin{cases}c(x), & \text{if } x\in]a-\delta,a+\delta[\setminus\{a\}\\ a, & \text{if } x=a,\end{cases}$$

is second order differentiable at the point $x=a$ and

$$\overline{c}^{(1)}(a)=\frac{1}{2},\quad c^{(2)}(a)=\frac{f^{(1)}(a)\,g^{(3)}(a)-f^{(3)}(a)\,g^{(1)}(a)}{12\left[f^{(1)}(a)\,g^{(2)}(a)-f^{(2)}(a)\,g^{(1)}(a)\right]}.$$

There are no known result related to the higher order differentiability of the intermediate-point function in Cauchy's mean-value theorem if $f^{(1)}(a)g^{(2)}(a)$ $=f^{(2)}(a)g^{(1)}(a)$

In this paper, we establish under which circumstances the intermediate-point function in Cauchy's mean-value theorem is twice differentiable at the point $x=a$ and to compute its derivatives, $\overline{c}^{(1)}(a)$ and $c^{(2)}(a)$, if

$$f^{(1)}(a)g^{(k)}(a)=f^{(k)}(a)g^{(1)}(a)\,,\ \text{for all } k\in\{1,...,n-1\}$$

and

$$f^{(1)}(a)g^{(n)}(a)\neq f^{(n}(a)g^{(1)}(a)\,.$$

2. MAIN RESULT

The following statement is true.

THEOREM 6. *Let I be an interval in $\mathbb{R}$, let a be a point of I and let $f,g:I\to\mathbb{R}$ be functions which satisfy the following conditions:*

(a) the functions f and g are $n+1\geq 3$ times differentiable on I,

(b) the functions $f^{(n+1)}$ and $g^{(n+1)}$ are continuous at $x=a$,

(c) $f^{(1)}(a)\,g^{(k)}(a)=f^{(k)}(a)\,g^{(1)}(a)$, for all $k\in\{1,...,n-1\}$,

(d) $f^{(1)}(a)\,g^{(n)}(a)\neq f^{(n)}(a)\,g^{(1)}(a)$.

1^0 *If $\theta:I\setminus\{a\}\to]0,1[$ is a function with property that*

(7)

$$[f(x)-f(a)]\,g^{(1)}(a+(x-a)\,\theta(x))=[g(x)-g(a)]\,f^{(1)}(a+(x-a)\,\theta(x))\,,$$

for all $x\in I\setminus\{a\}$, then the function $\overline{\theta}:I\to]0,1[$ defined by

$$\overline{\theta}(x)=\begin{cases}\theta(x), & \text{if } x\in I\setminus\{a\}\\ \frac{1}{\sqrt[n-1]{n}}, & \text{if } x=a,\end{cases}$$

is differentiable at $x = a$ *and*

$$\overline{\theta}^{(1)}(x) = \frac{1}{(n-1)\sqrt[n-1]{n^2}D_{1,n}}\left[\frac{2-\sqrt[n-1]{n}}{2}D_{2,n} + \frac{n\sqrt[n-1]{n}-n-1}{(n+1)}D_{1,n+1}\right],$$

where

$$D_{k,j} = f^{(k)}(a)\,g^{(j)}(a) - f^{(j)}(a)\,g^{(k)}(a)\,,\ \textit{for all } k,j \in \{1,...,n+1\}.$$

2^0 *If* $c : I\backslash\{a\} \to I$ *is a function with property that*

$$[f(x) - f(a)]\,g^{(1)}(c(x)) = [g(x) - g(a)]\,f^{(1)}(c(x))\,,$$

for all $x \in I\backslash\{a\}$, *then the function* $\overline{c} : I \to I$ *defined by*

$$\overline{c}(x) = \begin{cases} c(x)\,, & \textit{if } x \in I\backslash\{a\} \\ a, & \textit{if } x = a, \end{cases}$$

is second order differentiable at $x = a$ *and*

$$\overline{c}^{(1)}(a) = \frac{1}{\sqrt[n-1]{n}},$$

$$\overline{c}^{(2)}(a) = \frac{2}{(n-1)\sqrt[n-1]{n^2}D_{1,n}}\left[\frac{2-\sqrt[n-1]{n}}{2}D_{2,n} + \frac{n\sqrt[n-1]{n}-n-1}{(n+1)}D_{1,n+1}\right].$$

Proof. By Taylor's formula, for each $x \in I\backslash\{a\}$ there exist two real numbers $\widehat{\theta}_{f,x}$, $\widehat{\theta}_{g,x} \in]0,1[$ such that

$$f(x) = \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{f^{(n+1)}(A_f(x))}{(n+1)!}(x-a)^{n+1}\,, \tag{8}$$

and

$$g(x) = \sum_{i=0}^{n}\frac{g^{(i)}(a)}{i!}(x-a)^i + \frac{g^{(n+1)}(A_g(x))}{(n+1)!}(x-a)^{n+1}\,, \tag{9}$$

where

$$A_f(x) = a + (x-a)\,\widehat{\theta}_{f,x} \quad \text{and} \quad A_g(x) = a + (x-a)\,\widehat{\theta}_{g,x}.$$

On the other hand, by Taylor's formula applied to the functions $f^{(1)}$ and $g^{(1)}$, for each $x \in I\backslash\{a\}$ there exist two real numbers $\widetilde{\theta}_{f,x}$, $\widetilde{\theta}_{g,x} \in]0,1[$ such that

$$\begin{aligned} (10) \quad & f^{(1)}(a + (x-a)\,\theta(x)) = \\ & = \sum_{k=1}^{n}\frac{f^{(k)}(a)}{(k-1)!}(x-a)^{k-1}\theta^{k-1}(x) + \frac{f^{(n+1)}(B_f(x))}{n!}(x-a)^n\,\theta^n(x)\,, \end{aligned}$$

and

$$g^{(1)}\left(a+(x-a)\,\theta(x)\right)= \tag{11}$$
$$=\sum_{k=1}^{n}\frac{g^{(k)}(a)}{(k-1)!}(x-a)^{k-1}\,\theta^{k-1}(x)+\frac{g^{(n+1)}\left(B_g(x)\right)}{n!}(x-a)^{n}\,\theta^{n}(x),$$

where

$$B_f(x)=a+(x-a)\,\widetilde{\theta}_{f,x}\quad\text{and}\quad B_g(x)=a+(x-a)\,\widetilde{\theta}_{g,x}.$$

Substituting $(8)-(11)$ in (7), we obtain that,

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}D_{i,k}\left[\frac{\theta^{k-1}(x)}{i!(k-1)!}-\frac{\theta^{i-1}(x)}{k!(i-1)!}\right](x-a)^{k+i-1} \tag{12}$$
$$+\sum_{i=1}^{n}\frac{f^{(i)}(a)g^{(n+1)}(B_g(x))-f^{(n+1)}\big(B_f(x)\big)g^{(i)}(a)}{i!n!}(x-a)^{n+i}\,\theta^{n}(x)$$
$$+\sum_{k=1}^{n}\frac{f^{(n+1)}\big(A_f(x)\big)g^{(k)}(a)-f^{(k)}(a)g^{(n+1)}(A_g(x))}{(n+1)!(k-1)!}(x-a)^{n+k}\,\theta^{k-1}(x)$$
$$+\frac{f^{(n+1)}\big(A_f(x)\big)g^{(n+1)}(B_g(x))-f^{(n+1)}\big(B_f(x)\big)g^{(n+1)}(A_g(x))}{n!(n+1)!}(x-a)^{2n+1}\,\theta^{n}(x)$$
$$=0,$$

for all $x\in I\backslash\{a\}$.

Since

$$\frac{f^{(1)}(a)}{g^{(1)}(a)}=\frac{f^{(2)}(a)}{g^{(2)}(a)}=\cdots=\frac{f^{(n-1)}(a)}{g^{(n-1)}(a)}$$

and

$$\frac{f^{(1)}(a)}{g^{(1)}(a)}\neq\frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

we have

$$D_{k,j}=0,\ \text{for all } k,j\in\{1,...,n-1\},$$

and

$$D_{1,n}\neq 0.$$

Then (12) becomes

$$\frac{1}{(n-1)!}\sum_{k=1}^{n-1}\frac{D_{k,n}}{(k-1)!}\left[\frac{\theta^{n-1}(x)}{k}-\frac{\theta^{k-1}(x)}{n}\right](x-a)^{k-1} \tag{13}$$
$$+\ \frac{1}{n!}\sum_{k=1}^{n}\frac{f^{(k)}(a)g^{(n+1)}(B_g(x))-f^{(n+1)}\big(B_f(x)\big)g^{(k)}(a)}{k!}(x-a)^{k}+$$

$$+\frac{1}{(n+1)!}\sum_{k=1}^{n}\frac{f^{(n+1)}\left(A_f(x)\right)g^{(k)}(a)-f^{(k)}(a)g^{(n+1)}(A_g(x))}{(k-1)!}\theta^{k-1}(x)(x-a)^k$$
$$+\frac{f^{(n+1)}\left(A_f(x)\right)g^{(n+1)}(B_g(x))-f^{(n+1)}\left(B_f(x)\right)g^{(n+1)}(A_g(x))}{n!(n+1)!}\theta^{n}(x)(x-a)^{n+1}$$
$$=0,$$

for all $x\in I\backslash\{a\}$.

Taking $x\to a$, we obtain

$$\lim_{x\to a}\left[\frac{\theta^{n-1}(x)}{1}-\frac{1}{n}\right]=0,$$

hence

$$\overline{\theta}^{n-1}(a)=\lim_{x\to a}\theta^{n-1}(x)=\frac{1}{n}.$$

Consequently, the function $\overline{\theta}$ is continuous at $x=a$.

From (13), we deduce that

$$D_{1,n}\left[\theta^{n-1}(x)-\tfrac{1}{n}\right]+$$
$$+\sum_{k=2}^{n}\frac{D_{k,n}}{(k-1)!}\left[\frac{\theta^{n-1}(x)}{k}-\frac{\theta^{k-1}(x)}{n}\right](x-a)^{k-1}$$
$$+\frac{\theta^{n}(x)}{n}\sum_{k=1}^{n}\frac{f^{(k)}(a)g^{(n+1)}(B_g(x))-f^{(n+1)}\left(B_f(x)\right)g^{(k)}(a)}{k!}(x-a)^k$$
$$+\frac{1}{n(n+1)}\sum_{k=1}^{n}\frac{f^{(n+1)}\left(A_f(x)\right)g^{(k)}(a)-f^{(k)}(a)g^{(n+1)}(A_g(x))}{(k-1)!}(x-a)^k\theta^{k-1}(x)$$
$$+\frac{f^{(n+1)}\left(A_f(x)\right)g^{(n+1)}(B_g(x))-f^{(n+1)}\left(B_f(x)\right)g^{(n+1)}(A_g(x))}{(n+1)!n}(x-a)^{n+1}\theta^{n}(x)=$$
$$=0,$$

for all $x\in I\backslash\{a\}$.

Dividing by $(x-a)\,D_{1,n}$ we obtain

$$(14)\quad \frac{1}{x-a}\left[\theta^{n-1}(x)-\frac{1}{n}\right]+\frac{D_{2,n}}{D_{1,n}}\left[\frac{\theta^{n-1}(x)}{2}-\frac{\theta(x)}{n}\right]+$$
$$+\frac{f^{(1)}(a)\,g^{(n+1)}(B_g(x))-f^{(n+1)}(B_f(x))\,g^{(1)}(a)}{nD_{1,n}}\theta^{n}(x)$$
$$+\frac{f^{(n+1)}(A_f(x))\,g^{(1)}(a)-f^{(1)}(a)\,g^{(n+1)}(A_g(x))}{n(n+1)\,D_{1,n}}+o(x-a)=0,$$

for all $x\in I\backslash\{a\}$.

Evidently,

$$\frac{1}{x-a}\left[\theta^{n-1}(x)-\frac{1}{n}\right]=\frac{1}{x-a}\left[\theta(x)-\frac{1}{\sqrt[n-1]{n}}\right]\sum_{k=0}^{n-2}\theta^{n-2-k}(x)\left(\frac{1}{\sqrt[n-1]{n}}\right)^{k},$$

for all $x\in I\backslash\{a\}$.

Since

$$\lim_{x\to a}\theta(x)=\frac{1}{\sqrt[n-1]{n}}=\overline{\theta}(a),$$

$$\lim_{x\to a}A_f(x)=\lim_{x\to a}A_g(x)=\lim_{x\to a}B_f(x)=\lim_{x\to a}B_g(x)=a,$$

$$\lim_{x\to a}\left[f^{(1)}(a)\,g^{(n+1)}(B_g(x))-f^{(n+1)}(B_f(x))\,g^{(n)}(a)\right]=D_{1,n+1},$$

$$\lim_{x\to a}\left[f^{(n+1)}(A_f(x))\,g^{(1)}(a)-f^{(1)}(a)\,g^{(n)}(A_g(x))\right]=-D_{1,n+1},$$

from (14), we deduce that there exists the limit

$$\lim_{x\to a}\left(\frac{1}{x-a}\left[\theta(x)-\frac{1}{\sqrt[n-1]{n}}\right]\right)=\lim_{x\to a}\frac{\overline{\theta}(x)-\overline{\theta}(a)}{x-a}=\overline{\theta}'(a),$$

and

$$(n-1)\,\overline{\theta}^{n-2}(a)\,\overline{\theta}'(a)+\frac{D_{2,n}}{D_{1,n}}\left[\frac{\overline{\theta}^{n-1}(a)}{2}-\frac{\overline{\theta}(a)}{n}\right]+\frac{D_{1,n+1}}{D_{1,n}}\left[\frac{\overline{\theta}^{n}(a)}{n}-\frac{1}{n(n+1)}\right]=0.$$

Consequently,

$$\overline{\theta}'(a)=\frac{1}{(n-1)\sqrt[n-1]{n^2}D_{1,n}}\left[\frac{2-\sqrt[n-1]{n}}{2}D_{2,n}+\frac{n.\sqrt[n-1]{n^n}-n-1}{n(n+1)}D_{1,n+1}\right].$$

2^0 The statement 2^0 follows from the statement 1^0. □

Remark 7. *If* $g=1_I$, *then*

$$\overline{\theta}'(a)=\frac{n\sqrt[n-1]{n}-n-1}{n(n^2-1)\sqrt[n-1]{n^2}}\frac{f^{(n+1)}(a)}{f^{(n)}(a)},$$

and

$$\overline{c}''(a)=\frac{2\left(n\sqrt[n-1]{n}-n-1\right)}{n(n^2-1)\sqrt[n-1]{n^2}}\frac{f^{(n+1)}(a)}{f^{(n)}(a)}.$$

REFERENCES

[1] D.I. Duca and O.T. Pop, *Concerning the intermediate point in the mean value theorem*, Mathematical Inequalities & Applications, **12** (2009), no. 3, pp. 499–512.

[2] D.I. Duca and O. Pop, *On the intermediate point in Cauchy's mean-value theorem*, Mathematical Inequalities & Applications, **9** (2006), no. 3, pp. 375–389.

[3] B.-M. Pop and D.I. Duca, *The derivability of the intermediate point function in Cauchy's mean-value theorem*, Didactica Mathematica, **32** (2014), pp. 87–100 (in Romanian).

[4] T. Trif, *Asymptotic behavior of intermediate point in certain mean value theorems*, J. Math. Ineq., **2** (2008), pp. 151–161.

Received by the editors: February 20, 2015.