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SEMILOCAL CONVEGENCE OF NEWTON-LIKE METHODS UNDER GENERAL CONDITIONS, WITH APPLICATIONS IN FRACTIONAL CALCULUS

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Abstract. We present a semilocal convergence study of Newton-like methods on a generalized Banach space setting to approximate a locally unique zero of an operator. Earlier studies such as [5, 6, 7, 14] require that the operator involved is Fréchet-differentiable. In the present study we assume that the operator is only continuous. This way we extend the applicability of Newton-like methods to include fractional calculus and problems from other areas. Some applications include fractional calculus involving the Riemann-Liouville fractional integral and the Caputo fractional derivative. Fractional calculus is very important for its applications in many applied sciences.

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1. INTRODUCTION

We present a semilocal convergence analysis for Newton-like methods on a generalized Banach space setting to approximate a zero of an operator. The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [5, 6, 7, 14] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In the present study we only assume the continuity of the operator. This may expand the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities

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and fixed points. In section 3 we present the semilocal convergence analysis of Newton-like methods. Finally, in the concluding sections 4-5, we present special cases and applications in fractional calculus.

2. GENERALIZED BANACH SPACES

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [5, 6, 7, 14], and the references there in.

Let X be a linear space. A subset C of X is called a cone if $C+C \subseteq C$ and $\alpha C \subseteq C$ for $\alpha > 0$. The cone C is proper if $C \cap (-C) = \{0\}$. The relation "$\leq$" defined by

$$x \leq y \text{ if and only if } y - x \in C$$

is a partial ordering on C which is compatible with the linear structure of this space. Two elements x and y of X are called comparable if either $x \leq y$ or $y \leq x$ holds. The space X endowed with the above relation is called a partially ordered linear space (POL-space). If X has a topology compatible with its linear structure and if the cone C is closed in that topology then X is called a partially ordered topological space (POTL-space).

We remark that in a POTL-space the intervals $[a,b] = \{x : a \leq x \leq b\}$ are closed sets. A stronger connection is considered in the following definitions:

Definition 2.1. *A POTL-space is called normal if, given a local base V for the topology, there exists a positive number η so that if $0 \leq z \in U \subseteq V$ then $[0,z] \subset \eta U$.*

Definition 2.2. *A POTL-space is called regular if every order bounded increasing sequence has a limit.*

If the topology of a POTL-space is given by a norm then this space is called a partially ordered normed space (PON-space). If a PON-space is complete with respect to its topology then it is called a partially ordered Banach space (POB-space). According to Definition 2.1 a PON-space is normal if and only if there exists a positive number α such that

$$\|x\| \leq \alpha \|y\|, \quad \text{for all } x, y \in X \text{ with } 0 \leq x \leq y.$$

Let us note that any regular POB-space is normal. The reverse is not true. For example, the space $C[0,1]$ of all continuous real functions defined on $[0,1]$, ordered by the cone of nonnegative functions, is normal but is not regular. All finite dimensional POTL-spaces are both normal and regular.

Definition 2.3. *A generalized Banach space is a triplet $(X, (E, K, \|\cdot\|), /\cdot/)$ such that*

(i) *X is a linear space over $\mathbb{R}(\mathbb{C})$.*
(ii) *$E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.*
(ii$_1$) *$(E, \|\cdot\|)$ is a real Banach space,*

(ii$_2$) *E is partially ordered by a closed convex cone K,*
(ii$_3$) *The norm $\|\cdot\|$ is monotone on K.*
(iii) *The operator $/\cdot/ : X \to K$ satisfies*

$$/x/ = 0 \Leftrightarrow x = 0, \ /\theta x/ = |\theta| \, /x/ \, ,$$

$$/x+y/ \leq /x/ + /y/ \, , \quad \text{for each } x, y \in X, \theta \in \mathbb{R}(\mathbb{C}).$$

(iv) *X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|/\cdot/\|$.*

Remark 2.4. The operator $/\cdot/$ is called a generalized norm. In view of (iii) and (ii$_3$) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm. □

Definition 2.5. *Let $L\left(X^j, Y\right)$ stand for the space of j-linear symmetric and bounded operators from X^j to Y, where X and Y are Banach spaces. For X, Y partially ordered $L_+\left(X^j, Y\right)$ stands for the subset of monotone operators P such that*

$$0 \leq a_i \leq b_i \Rightarrow P\left(a_1, ..., a_j\right) \leq P\left(b_1, ..., b_j\right). \tag{2.1}$$

Definition 2.6. *The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, /\cdot/)$ is defined to be:*

$$B(Q) := \left\{P \in L_+(E, E), \ /Qx/ \leq P\,/x/ \ \text{for each } x \in X\right\}. \tag{2.2}$$

Let $D \subset X$ and $T : D \to D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T\left(x_n\right) = T^{n+1}\left(x_0\right) \tag{2.3}$$

is well defined. We write in case of convergence

$$T^{\infty}\left(x_0\right) := \lim\left(T^n\left(x_0\right)\right) = \lim_{n\to\infty} x_n. \tag{2.4}$$

We need some auxiliary results on inequations.

Lemma 2.7. *Let $(E, K, \|\cdot\|)$ be a partially ordered Banach space, $\xi \in K$ and $M, N \in L_+(E, E)$.*

(i) *Suppose there exists $r \in K$ such that*

$$R(r) := (M+N)\, r + \xi \leq r \tag{2.5}$$

and

$$(M+N)^k\, r \to 0 \quad as \quad k \to \infty. \tag{2.6}$$

Then, $b := R^{\infty}(0)$ is well defined, satisfies the equation $t = R(t)$ and is smaller than any solution of the inequality $R(s) \leq s$.

(ii) *Suppose there exists $q \in K$ and $\theta \in (0,1)$ such that $R(q) \leq \theta q$, then there exists $r \leq q$ satisfying* (i)*.*

Proof. (i) Define sequence $\{b_n\}$ by $b_n = R^n(0)$. Then, we have by (2.5) that $b_1 = R(0) = \xi \le r \Rightarrow b_1 \le r$. Suppose that $b_k \le r$ for each $k = 1, 2, ..., n$. Then, we have by (2.5) and the inductive hypothesis that $b_{n+1} = R^{n+1}(0) = R(R^n(0)) = R(b_n) = (M+N)b_n + \xi \le (M+N)r + \xi \le r \Rightarrow b_{n+1} \le r$. Hence, sequence $\{b_n\}$ is bounded above by r. Set $P_n = b_{n+1} - b_n$. We shall show that

$$P_n \le (M+N)^n r \quad \text{for each } n = 1, 2, ... \tag{2.7}$$

We have by the definition of P_n and (2.6) that

$$\begin{aligned} P_1 &= R^2(0) - R(0) = R(R(0)) - R(0) \\ &= R(\xi) - R(0) = \int_0^1 R'(t\xi)\,\xi dt \le \int_0^1 R'(\xi)\,\xi dt \\ &\le \int_0^1 R'(r)\,r dt \le (M+N)\,r, \end{aligned}$$

which shows (2.7) for $n = 1$. Suppose that (2.7) is true for $k = 1, 2, ..., n$. Then, we have in turn by (2.6) and the inductive hypothesis that

$$\begin{aligned} P_{k+1} &= R^{k+2}(0) - R^{k+1}(0) = R^{k+1}(R(0)) - R^{k+1}(0) \\ &= R^{k+1}(\xi) - R^{k+1}(0) = R\left(R^k(\xi)\right) - R\left(R^k(0)\right) \\ &= \int_0^1 R'\left(R^k(0) + t\left(R^k(\xi) - R^k(0)\right)\right)\left(R^k(\xi) - R^k(0)\right) dt \le \\ &\le R'\left(R^k(\xi)\right)\left(R^k(\xi) - R^k(0)\right) = R'\left(R^k(\xi)\right)\left(R^{k+1}(0) - R^k(0)\right) \\ &\le R'(r)\left(R^{k+1}(0) - R^k(0)\right) \le (M+N)(M+N)^k r = (M+N)^{k+1} r, \end{aligned}$$

which completes the induction for (2.7). It follows that $\{b_n\}$ is a complete sequence in a Banach space and as such it converges to some b. Notice that $R(b) = R\left(\lim_{n\to\infty} R^n(0)\right) = \lim_{n\to\infty} R^{n+1}(0) = b \Rightarrow b$ solves the equation $R(t) = t$. We have that $b_n \le r \Rightarrow b \le r$, where r a solution of $R(r) \le r$. Hence, b is smaller than any solution of $R(s) \le s$.

(ii) Define sequences $\{v_n\}$, $\{w_n\}$ by $v_0 = 0$, $v_{n+1} = R(v_n)$, $w_0 = q$, $w_{n+1} = R(w_n)$. Then, we have that

$$\begin{gathered} 0 \le v_n \le v_{n+1} \le w_{n+1} \le w_n \le q, \\ w_n - v_n \le \theta^n (q - v_n) \end{gathered} \tag{2.8}$$

and sequence $\{v_n\}$ is bounded above by q. Hence, it converges to some r with $r \le q$. We also get by (2.8) that $w_n - v_n \to 0$ as $n \to \infty \Rightarrow w_n \to r$ as $n \to \infty$. □

We also need the auxiliary result for computing solutions of fixed point problems.

Lemma 2.8. *Let* $(X, (E, K, \|\cdot\|), /\cdot/)$ *be a generalized Banach space, and* $P \in B(Q)$ *be a bound for* $Q \in L(X, X)$. *Suppose there exists* $y \in X$ *and* $q \in K$ *such that*

$$Pq + /y/ \leq q \quad \text{and} \quad P^k q \to 0, \quad \text{as } k \to \infty. \tag{2.9}$$

Then, $z = T^\infty(0)$, $T(x) := Qx + y$ *is well defined and satisfies:* $z = Qz + y$ *and* $/z/ \leq P/z/ + /y/ \leq q$. *Moreover,* z *is the unique solution in the subspace* $\{x \in X | \exists\, \theta \in \mathbb{R} : \{x\} \leq \theta q\}$.

The proof can be found in [14, Lemma 3.2].

3. SEMILOCAL CONVERGENCE

Let $(X, (E, K, \|\cdot\|), /\cdot/)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \to Y$ a continuous operator and $A(\cdot) : D \to L(X, Y)$. A zero of operator G is to be determined by a Newton-like method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$\begin{aligned} x_{n+1} &= x_n + y_n, \;\; A(x_n) y_n + F(x_n) = 0 \\ &\Leftrightarrow y_n = T(y_n) := (I - A(x_n)) y_n - F(x_n). \end{aligned} \tag{3.1}$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we present the semilocal convergence analysis of Newton-like method (3.1) using the preceding notation.

Theorem 3.1. *Let* $F : D \subset X \to X$, $A(\cdot) : D \to L(X, X)$ *and* $x_0 \in D$ *be as defined previously. Suppose:*

(H_1) *There exists an operator* $M \in B(I - A(x))$ *for each* $x \in D$.

(H_2) *There exists an operator* $N \in L_+(E, E)$ *satisfying for each* $x, y \in D$

$$/F(y) - F(x) - A(x)(y - x)/ \leq N/y - x/.$$

(H_3) *There exists a solution* $r \in K$ *of*

$$R_0(t) := (M + N)t + /F(x_0)/ \leq t.$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \to 0$ *as* $k \to \infty$.

Then, the following hold:

(C_1) *The sequence* $\{x_n\}$ *defined by*

$$\begin{aligned} x_{n+1} &= x_n + T_n^\infty(0), \\ T_n(y) &:= (I - A(x_n)) y - F(x_n) \end{aligned} \tag{3.2}$$

is well defined, remains in $U(x_0, r)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to the unique zero of operator* F *in* $U(x_0, r)$.

(C_2) *An apriori bound is given by the null-sequence* $\{r_n\}$ *defined by* $r_0 := r$ *and for each* $n = 1, 2, ...$

$$r_n = P_n^{\infty}(0), \quad P_n(t) = Mt + Nr_{n-1}.$$

(C_3) *An a posteriori bound is given by the sequence* $\{s_n\}$ *defined by*

$$s_n := R_n^{\infty}(0), \quad R_n(t) = (M+N)t + Na_{n-1},$$
$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/, \quad \textit{for each } n = 1, 2, ...$$

Proof. Let us define for each $n \in \mathbb{N}$ the statement:
(I_n) $x_n \in X$ and $r_n \in K$ are well defined and satisfy

$$r_n + a_{n-1} \leq r_{n-1}.$$

We use induction to show (I_n). The statement (I_1) is true: By Lemma 2.7 and (H_3), (H_5) there exists $q \leq r$ such that:

$$Mq + /F(x_0)/ = q \text{ and } M^k q \leq M^k r \to 0 \text{ as } k \to \infty.$$

Hence, by Lemma 2.8 x_1 is well defined and we have $a_0 \leq q$. Then, we get the estimate

$$\begin{aligned} P_1(r-q) &= M(r-q) + Nr_0 \\ &\leq Mr - Mq + Nr = R_0(r) - q \\ &\leq R_0(r) - q = r - q. \end{aligned}$$

It follows with Lemma 2.7 that r_1 is well defined and

$$r_1 + a_0 \leq r - q + q = r = r_0.$$

Suppose that (I_j) is true for each $j = 1, 2, ..., n$. We need to show the existence of x_{n+1} and to obtain a bound q for a_n. To achieve this notice that:

$$Mr_n + N(r_{n-1} - r_n) = Mr_n + Nr_{n-1} - Nr_n = P_n(r_n) - Nr_n \leq r_n.$$

Then, it follows from Lemma 2.7 that there exists $q \leq r_n$ such that

$$q = Mq + N(r_{n-1} - r_n) \text{ and } (M+N)^k q \to 0, \text{ as } k \to \infty. \tag{3.3}$$

By (I_j) it follows that

$$b_n = /x_n - x_0/ \leq \sum_{j=0}^{n-1} a_j \leq \sum_{j=0}^{n-1} (r_j - r_{j+1}) = r - r_n \leq r.$$

Hence, $x_n \in U(x_0, r) \subset D$ and by (H_1) M is a bound for $I - A(x_n)$.

We can write by (H_2) that

$$\begin{aligned} /F(x_n)/ &= /F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})/ \\ &\leq Na_{n-1} \leq N(r_{n-1} - r_n). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$Mq + /F(x_n)/ \leq q.$$

By Lemma 2.8, x_{n+1} is well defined and $a_n \leq q \leq r_n$. In view of the definition of r_{n+1} we have that

$$P_{n+1}(r_n - q) = P_n(r_n) - q = r_n - q,$$

so that by Lemma 2.7, r_{n+1} is well defined and

$$r_{n+1} + a_n \leq r_n - q + q = r_n,$$

which proves (I_{n+1}). The induction for (I_n) is complete. Let $m \geq n$, then we obtain in turn that

$$/x_{m+1} - x_n/ \leq \sum_{j=n}^{m} a_j \leq \sum_{j=n}^{m} (r_j - r_{j+1}) = r_n - r_{m+1} \leq r_n. \tag{3.5}$$

Moreover, we get inductively the estimate

$$r_{n+1} = P_{n+1}(r_{n+1}) \leq P_{n+1}(r_n) \leq (M+N) r_n \leq ... \leq (M+N)^{n+1} r.$$

It follows from (H_5) that $\{r_n\}$ is a null-sequence. Hence, $\{x_n\}$ is a complete sequence in a Banach space X by (3.5) and as such it converges to some $x^* \in X$. By letting $m \to \infty$ in (3.5) we deduce that $x^* \in U(x_n, r_n)$. Furthermore, (3.4) shows that x^* is a zero of F. Hence, (C_1) and (C_2) are proved.

In view of the estimate

$$R_n(r_n) \leq P_n(r_n) \leq r_n$$

the apriori, bound of (C_3) is well defined by Lemma 2.7. That is s_n is smaller in general than r_n. The conditions of Theorem 3.1 are satisfied for x_n replacing x_0. A solution of the inequality of (C_2) is given by s_n (see (3.4)). It follows from (3.5) that the conditions of Theorem 3.1 are easily verified. Then, it follows from (C_1) that $x^* \in U(x_n, s_n)$ which proves (C_3). □

In general the a posteriori estimate is of interest. Then, condition (H_5) can be avoided as follows:

PROPOSITION 3.2. *Suppose: condition* (H_1) *of Theorem* 3.1 *is true.*

(H_3') *There exists* $s \in K$, $\theta \in (0,1)$ *such that*

$$R_0(s) = (M+N)s + /F(x_0)/ \leq \theta s.$$

(H_4') $U(x_0, s) \subset D$.

Then, there exists $r \leq s$ *satisfying the conditions of Theorem* 3.1. *Moreover, the zero* x^* *of* F *is unique in* $U(x_0, s)$.

REMARK 3.3. (i) Notice that by Lemma 2.7 $R_n^{\infty}(0)$ is the smallest solution of $R_n(s) \leq s$. Hence any solution of this inequality yields on upper estimate for $R_n^{\infty}(0)$. Similar inequalities appear in (H_2) and (H_2').

(ii) The weak assumptions of Theorem 3.1 do not imply the existence of $A(x_n)^{-1}$. In practice the computation of $T_n^{\infty}(0)$ as a solution of a linear equation is no problem and the computation of the expensive or impossible to compute in general $A(x_n)^{-1}$ is not needed.

(iii) We can use the following result for the computation of the a posteriori estimates. The proof can be found in [14, Lemma 4.2] by simply exchanging the definitions of R. □

LEMMA 3.4. *Suppose that the conditions of Theorem* 3.1 *are satisfied. If* $s \in K$ *is a solution of* $R_n(s) \leq s$*, then* $q := s - a_n \in K$ *and solves* $R_{n+1}(q) \leq q$*. This solution might be improved by* $R_{n+1}^k(q) \leq q$ *for each* $k = 1, 2, \ldots$.

4. SPECIAL CASES AND APPLICATIONS

APPLICATION 4.1. The results obtained in earlier studies such as [5, 6, 7, 14] require that operator F (i.e. G) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that F is a continuous operator. Hence, we have extended the applicability of Newton-like methods to classes of operators that are only continuous. If $A(x) = F'(x)$ Newton-like method (3.1) reduces to Newton's method considered in [14]. □

EXAMPLE 4.2. The j-dimensional space $\mathbb{R}^j$ is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 3.1 look like. □

THEOREM 4.3. *Assume:*

(H$_1$) $\|I - A(x)\| \leq M$ *for some* $M \geq 0$.

(H$_2$) $\|F(y) - F(x) - A(x)(y - x)\| \leq N \|y - x\|$ *for some* $N \geq 0$.

(H$_3$) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \tag{4.1}$$

(H$_4$) $U(x_0, r) \subseteq D$.

(H$_5$) $(M + N)^k r \to 0$ *as* $k \to \infty$*, where* r *is given by* (4.1).

Then, the conclusions of Theorem 3.1 *hold.*

5. APPLICATION TO FRACTIONAL CALCULUS

Our presented earlier semilocal convergence Newton-type general methods, see Theorem 4.3, apply in the next two fractional settings given that the following inequalities are fulfilled:

$$\|1 - A(x)\|_\infty \leq \gamma_0 \in (0, 1), \tag{5.1}$$

and

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \tag{5.2}$$

where $\gamma_0, \gamma_1 \in (0,1)$, furthermore

$$\gamma = \gamma_0 + \gamma_1 \in (0,1), \tag{5.3}$$

for all $x, y \in [a, b^*]$.

Here we consider $a < b^* < b$.

The specific functions $A(x)$, $F(x)$ will be described next.

I) Let $\alpha > 0$ and $f \in L_\infty([a,b])$. The right Riemann-Liouville integral [4, pp. 333–354] is given by

$$\left(J_b^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x \in [a,b]. \tag{5.4}$$

Then

$$\begin{aligned} \left|\left(J_b^\alpha f\right)(x)\right| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (t-x)^{\alpha-1} |f(t)|\, dt \right) \\ &\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (t-x)^{\alpha-1}\, dt \right) \|f\|_\infty = \frac{1}{\Gamma(\alpha)} \frac{(b-x)^\alpha}{\alpha} \|f\|_\infty \\ &= \frac{(b-x)^\alpha}{\Gamma(\alpha+1)} \|f\|_\infty = (\xi_1). \end{aligned} \tag{5.5}$$

Clearly

$$\left(J_b^\alpha f\right)(b) = 0. \tag{5.6}$$

$$(\xi_1) \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_\infty. \tag{5.7}$$

That is

$$\left\|J_b^\alpha f\right\|_{\infty,[a,b]} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_\infty < \infty, \tag{5.8}$$

i.e. J_b^α is a bounded linear operator.

By [3] we get that $\left(J_b^\alpha f\right)$ is a continuous function over $[a,b]$ and in particular over $[a,b^*]$. Thus there exist $x_1, x_2 \in [a,b^*]$ such that

$$\begin{aligned} \left(J_b^\alpha f\right)(x_1) &= \min \left(J_b^\alpha f\right)(x), \\ \left(J_b^\alpha f\right)(x_2) &= \max \left(J_b^\alpha f\right)(x), \ \ x \in [a,b^*]. \end{aligned} \tag{5.9}$$

We assume that

$$\left(J_b^\alpha f\right)(x_1) > 0. \tag{5.10}$$

Hence

$$\left\|J_b^\alpha f\right\|_{\infty,[a,b^*]} = \left(J_b^\alpha f\right)(x_2) > 0. \tag{5.11}$$

Here it is

$$J(x) = mx, \ \ m \neq 0. \tag{5.12}$$

Therefore the equation

$$Jf(x) = 0, \quad x \in [a, b^*], \tag{5.13}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left(J_b^\alpha f\right)(x_2)} = 0, \quad x \in [a, b^*]. \tag{5.14}$$

Notice that

$$J_b^\alpha \left(\frac{f}{2\left(J_b^\alpha f\right)(x_2)}\right)(x) = \frac{\left(J_b^\alpha f\right)(x)}{2\left(J_b^\alpha f\right)(x_2)} \leq \frac{1}{2} < 1, \quad x \in [a, b^*]. \tag{5.15}$$

Call

$$A(x) := \frac{\left(J_b^\alpha f\right)(x)}{2\left(J_b^\alpha f\right)(x_2)}, \quad \forall\, x \in [a, b^*]. \tag{5.16}$$

We notice that

$$0 < \frac{\left(J_b^\alpha f\right)(x_1)}{2\left(J_b^\alpha f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \ \ \forall\, x \in [a, b^*]. \tag{5.17}$$

Hence the first condition (5.1) is fulfilled

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left(J_b^\alpha f\right)(x_1)}{2\left(J_b^\alpha f\right)(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*]. \tag{5.18}$$

Clearly $\gamma_0 \in (0, 1)$.

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \text{all } x, y \in [a, b^*], \tag{5.19}$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left(J_b^\alpha f\right)(x_2) |x - y|, \quad \text{all } x, y \in [a, b^*]. \tag{5.20}$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y - x)| &\leq |F(y) - F(x)| + |A(x)| |y - x| \\ &\leq \lambda |y - x| + |A(x)| |y - x| \\ &= (\lambda + |A(x)|) |y - x| \\ &=: (\psi_1), \ \ \forall\, x, y \in [a, b^*]. \end{aligned} \tag{5.21}$$

We have that

$$\left|\left(J_b^\alpha f\right)(x)\right| \leq \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \|f\|_\infty < \infty, \quad \forall\, x \in [a, b^*]. \tag{5.22}$$

Hence

$$|A(x)| = \frac{\left|\left(J_b^\alpha f\right)(x)\right|}{2\left(J_b^\alpha f\right)(x_2)} \leq \frac{(b - a)^\alpha \|f\|_\infty}{2\Gamma(\alpha + 1)\left(\left(J_b^\alpha f\right)(x_2)\right)} < \infty, \quad \forall\, x \in [a, b^*]. \tag{5.23}$$

Therefore we get

$$(5.24)\qquad (\psi_1) \leq \left(\lambda + \frac{(b-a)^a \|f\|_\infty}{2\Gamma(\alpha+1)\left(\left(J_b^\alpha f\right)(x_2)\right)}\right)|y-x|, \quad \forall\, x,y \in [a,b^*].$$

Call

$$(5.25)\qquad 0 < \gamma_1 := \lambda + \frac{(b-a)^a \|f\|_\infty}{2\Gamma(\alpha+1)\left(\left(J_b^\alpha f\right)(x_2)\right)},$$

choosing $(b-a)$ small enough we can make $\gamma_1 \in (0,1)$, fulfilling (5.2).

Next we call and we need that

$$(5.26)\quad 0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{\left(J_b^\alpha f\right)(x_1)}{2\left(J_b^\alpha f\right)(x_2)} + \lambda + \frac{(b-a)^a \|f\|_\infty}{2\Gamma(\alpha+1)\left(\left(J_b^\alpha f\right)(x_2)\right)} < 1,$$

equivalently,

$$(5.27)\qquad \lambda + \frac{(b-a)^a \|f\|_\infty}{2\Gamma(\alpha+1)\left(\left(J_b^\alpha f\right)(x_2)\right)} < \frac{\left(J_b^\alpha f\right)(x_1)}{2\left(J_b^\alpha f\right)(x_2)},$$

equivalently,

$$(5.28)\qquad 2\lambda\left(J_b^\alpha f\right)(x_2) + \frac{(b-a)^a \|f\|_\infty}{\Gamma(\alpha+1)} < \left(J_b^\alpha f\right)(x_1),$$

which is possible for small λ, $(b-a)$. That is $\gamma \in (0,1)$, fulfilling (5.3). So our numerical method converges and solves (5.13).

II) Let again $a < b^* < b$, $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling function), $\alpha \notin \mathbb{N}$, $G \in C^{m-1}([a,b])$, $0 \neq G^{(m)} \in L_\infty([a,b])$. Here we consider the right Caputo fractional derivative (see [4, p. 337]),

$$(5.29)\qquad D_{b-}^\alpha G(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (t-x)^{m-\alpha-1} G^{(m)}(t)\, dt.$$

By [3] $D_{b-}^\alpha G$ is a continuous function over $[a,b]$ and in particular continuous over $[a,b^*]$. Notice that by [4, p. 358], we have that $D_{b-}^\alpha G(b) = 0$.

Therefore there exist $x_1, x_2 \in [a,b^*]$ such that $D_{b-}^\alpha G(x_1) = \min D_{b-}^\alpha G(x)$, and $D_{b-}^\alpha G(x_2) = \max D_{b-}^\alpha G(x)$, for $x \in [a,b^*]$.

We assume that

$$D_{b-}^\alpha G(x_1) > 0.$$

(i.e. $D_{b-}^\alpha G(x) > 0$, $\forall\, x \in [a,b^*]$).

Furthermore

$$\left\|D_{b-}^\alpha G\right\|_{\infty,[a,b^*]} = D_{b-}^\alpha G(x_2).$$

Here it is

$$J(x) = mx,\ m \neq 0.$$

The equation

$$JG(x) = 0, \quad x \in [a,b^*],$$

has the same set of solutions as the equation

$$F(x) := \frac{JG(x)}{2D_{b-}^{\alpha}G(x_2)} = 0, \quad x \in [a, b^*].$$

Notice that

$$D_{b-}^{\alpha}\left(\frac{G(x)}{2D_{b-}^{\alpha}G(x_2)}\right) = \frac{D_{b-}^{\alpha}G(x)}{2D_{b-}^{\alpha}G(x_2)} \leq \frac{1}{2} < 1, \quad \forall\, x \in [a, b^*].$$

We call

$$A(x) := \frac{D_{b-}^{\alpha}G(x)}{2D_{b-}^{\alpha}G(x_2)}, \quad \forall\, x \in [a, b^*].$$

We notice that

$$0 < \frac{D_{b-}^{\alpha}G(x_1)}{2D_{b-}^{\alpha}G(x_2)} \leq A(x) \leq \frac{1}{2}.$$

Hence the first condition (5.1) is fulfilled

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D_{b-}^{\alpha}G(x_1)}{2D_{b-}^{\alpha}G(x_2)} =: \gamma_0, \quad \forall\, x \in [a, b^*].$$

Clearly $\gamma_0 \in (0, 1)$.

Next we assume that $F(x)$ is a contraction over $[a, b^*]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall\, x, y \in [a, b^*],$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D_{b-}^{\alpha}G(x_2)\right)|x - y|, \quad \forall\, x, y \in [a, b^*].$$

We observe that

$$\begin{aligned}
|F(y) - F(x) - A(x)(y - x)| &\leq |F(y) - F(x)| + |A(x)||y - x| \\
&\leq \lambda |y - x| + |A(x)||y - x| \\
&= (\lambda + |A(x)|)|y - x| \\
&=: (\xi_2), \quad \forall\, x, y \in [a, b^*].
\end{aligned}$$

We observe that

$$\begin{aligned}
\left|D_{b-}^{\alpha}G(x)\right| &\leq \tfrac{1}{\Gamma(m-\alpha)} \int_x^b (t - x)^{m-\alpha-1} \left|G^{(m)}(t)\right| dt \\
&\leq \tfrac{1}{\Gamma(m-\alpha)} \Big(\int_x^b (t - x)^{m-\alpha-1}\, dt\Big) \left\|G^{(m)}\right\|_{\infty} \\
&= \tfrac{1}{\Gamma(m-\alpha)} \tfrac{(b-x)^{m-\alpha}}{(m-\alpha)} \left\|G^{(m)}\right\|_{\infty} \\
&= \tfrac{1}{\Gamma(m-\alpha+1)} (b - x)^{m-\alpha} \left\|G^{(m)}\right\|_{\infty} \leq \tfrac{(b-a)^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\|G^{(m)}\right\|_{\infty}.
\end{aligned}$$

That is

$$\left|D_{b-}^{\alpha}G(x)\right| \leq \tfrac{(b-a)^{m-\alpha}}{\Gamma(m-\alpha+1)} \left\|G^{(m)}\right\|_{\infty} < \infty, \quad \forall\, x \in [a, b].$$

Hence, $\forall\ x \in [a, b^*]$ we get that

$$|A(x)| = \frac{\left|D_{b-}^{\alpha} G(x)\right|}{2 D_{b-}^{\alpha} G(x_2)} \leq \frac{(b-a)^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\|G^{(m)}\|_{\infty}}{D_{b-}^{\alpha} G(x_2)} < \infty.$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(b-a)^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\|G^{(m)}\|_{\infty}}{D_{b-}^{\alpha} G(x_2)}\right) |y-x|, \quad \forall\ x, y \in [a, b^*].$$

Call

$$0 < \gamma_1 := \lambda + \frac{(b-a)^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\|G^{(m)}\|_{\infty}}{D_{b-}^{\alpha} G(x_2)},$$

choosing $(b-a)$ small enough we can make $\gamma_1 \in (0,1)$. So (5.2) is fulfilled.

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{D_{b-}^{\alpha} G(x_1)}{2 D_{b-}^{\alpha} G(x_2)} + \lambda + \frac{(b-a)^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\|G^{(m)}\|_{\infty}}{D_{b-}^{\alpha} G(x_2)} < 1,$$

equivalently we find,

$$\lambda + \frac{(b-a)^{m-\alpha}}{2\Gamma(m-\alpha+1)} \frac{\|G^{(m)}\|_{\infty}}{D_{b-}^{\alpha} G(x_2)} < \frac{D_{b-}^{\alpha} G(x_1)}{2 D_{b-}^{\alpha} G(x_2)},$$

equivalently,

$$2\lambda D_{b-}^{\alpha} G(x_2) + \frac{(b-a)^{m-\alpha}}{\Gamma(m-\alpha+1)} \|G^{(m)}\|_{\infty} < D_{b-}^{\alpha} G(x_1),$$

which is possible for small λ, $(b-a)$.

That is $\gamma \in (0,1)$, fulfilling (5.3). Hence equation (5) can be solved with our presented numerical methods.

REFERENCES

[1] S. Amat, S. Busquier, S. Plaza, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Appl., **366** (2010) 1, pp. 24–32.

[2] G. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[3] G.A. Anastassiou, *Fractional Representation Formulae and Right Fractional Inequalities*, Mathematical and Computer Modelling, **54** (2011) 11–12, pp. 3098–3115.

[4] G. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, 2011.

[5] I.K. Argyros, *Newton-like methods in partially ordered linear spaces*, Approx. Theory Appl., **9** (1993) 1, pp. 1–9.

[6] I.K. Argyros, *Results on controlling the residuals of perturbed Newton-like methods on Banach spaces with a convergence structure*, Southwest J. Pure Appl. Math., **1** (1995), pp. 32–38.

[7] I.K. Argyros, *Convergence and Applications of Newton-type iterations*, Springer-Verlag Publ., New York, 2008.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1st edition, Springer, New York, Heidelberg, 2010.

[9] J.A. Ezquerro, J.M. Gutierrez, M.Á. Hernández, N. Romero, M.J. Rubio, *The Newton method: From Newton to Kantorovich* (Spanish), Gac. R. Soc. Mat. Esp., **13** (2010), pp. 53–76.

[10] J.A. Ezquerro, M.Á. Hernández, *Newton-type methods of high order and domains of semilocal and global convergence*, Appl. Math. Comput., **214** (2009) 1, pp. 142–154.

[11] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1964.

[12] Á. A. Magreñán, *Different anomalies in a Jarratt family of iterative root finding methods*, Appl. Math. Comput., **233** (2014), pp. 29–38.

[13] Á. A. Magreñán, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput., 248 (2014), pp. 215–224.

[14] P.W. Meyer, *Newton's method in generalized Banach spaces*, Numer. Func. Anal. Optimiz., **9** (1987) 3–4, pp. 249–259.

[15] F.A. Potra, V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[16] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity, **26** (2010), 3–42.

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