Multicentric calculus and the Riesz projection

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Appendices

A Mathematica code used for Remark 2.10

A.1 Degree 4 polynomial

```
xi = Solve[(z - E^(I (Pi/4 - Pi/70))) (z - E^(-I (Pi/4 - Pi/70)))
(z + E^(I (Pi/4 - Pi/70))) (z + E^(-I ( Pi/4 - Pi/70))) == w,
    z] /. w -> 0.992

xi1 = Part[xi, 1, 1, 2]
xi2 = Part[xi, 2, 1, 2]
xi3 = Part[xi, 3, 1, 2]
xi4 = Part[xi, 4, 1, 2]

(* solutions of p_epsilon *)
l = NSolve[(z - E^(I (Pi/4 - Pi/70))) (z - E^(-I (Pi/4 - Pi/70)))
  (z + E^(I (Pi/4 - Pi/70))) (z + E^(-I ( Pi/4 - Pi/70)))== 0,
   z]

l1 = N[Part[l, 1, 1, 2]]
l2 = N[Part[l, 2, 1, 2]]
l3 = N[Part[l, 3, 1, 2]]
l4 = N[Part[l, 4, 1, 2]]

(* derivative of p_epsilon *)
ped[z_] :=
  D[(z - E^(I (Pi/4 - Pi/70))) (z - E^(-I (Pi/4 - Pi/70)))
    (z + E^(I (Pi/4 - Pi/70))) (z + E^(-I ( Pi/4 - Pi/70))), z];
```

```
(* computations of delta_j(lambda_k,w) *)
d1l1 = 0.992/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.992/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.992/((ped[x] /. x -> xi3) (xi3 - l1))
d4l1 = 0.992/((ped[x] /. x -> xi4) (xi4 - l1))

d1l2 = 0.992/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.992/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.992/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.992/((ped[x] /. x -> xi4) (xi4 - l2))

d1l3 = 0.992/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.992/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.992/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.992/((ped[x] /. x -> xi4) (xi4 - l3))

d1l4 = 0.992/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.992/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.992/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.992/((ped[x] /. x -> xi4) (xi4 - l4))

Abs[d1l1] + Abs[d2l1] + Abs[d3l1] + Abs[d4l1] + Abs[d1l2] +
 Abs[d2l2] + Abs[d3l2] + Abs[d4l2] + Abs[d1l3] + Abs[d2l3] +
 Abs[d3l3] + Abs[d4l3] + Abs[d1l4] + Abs[d2l4] + Abs[d3l4] +
 Abs[d4l4]

2293.81
```

A.2 Degree 6 polynomial

```
(* solutions of p_epsilon=w *)
xi = Solve[(z + 1) (z - 1) (z - E^(I (Pi/3 - Pi/70)))
     (z + E^(-I (Pi/3 - Pi/70))) (z + E^(I (Pi/3 - Pi/70)))
       (z - E^(-I (Pi/3 - Pi/70))) == w, z] /. w -> 0.9922

xi1 = N[Part[xi, 1, 1, 2]]
xi2 = N[Part[xi, 2, 1, 2]]
xi3 = N[Part[xi, 3, 1, 2]]
xi4 = N[Part[xi, 4, 1, 2]]
xi5 = N[Part[xi, 5, 1, 2]]
xi6 = N[Part[xi, 6, 1, 2]]

(* Solutions of p_epsilon *)
l = NSolve[(z + 1) (z - 1) (z - E^(I (Pi/3 - Pi/70)))
(z + E^(-I (Pi/3 - Pi/70))) (z + E^(I (Pi/3 - Pi/70)))
(z - E^(-I (Pi/3 - Pi/70))) == 0, z] ;

l1 = N[Part[l, 1, 1, 2]]
l2 = N[Part[l, 2, 1, 2]]
l3 = N[Part[l, 3, 1, 2]]
```

```
l4 = N[Part[l, 4, 1, 2]]
l5 = N[Part[l, 5, 1, 2]]
l6 = N[Part[l, 6, 1, 2]]

(* Derivative of p_epsilon *)
ped[z_] :=  D[(z + 1) (z - 1) (z - E^(I (Pi/3 - Pi/70)))
    (z + E^(-I (Pi/3 - Pi/70))) (z + E^(I (Pi/3 - Pi/70)))
     (z - E^(-I (Pi/3 - Pi/70))), z];

(* computations for delta_j(lambda_k,w)) *)
d1l1 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l1))
d4l1 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l1))
d5l1 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l1))
d6l1 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l1))

d1l2 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l2))
d5l2 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l2))
d6l2 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l2))

d1l3 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l3))
d5l3 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l3))
d6l3 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l3))

d1l4 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l4))
d5l4 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l4))
d6l4 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l4))

d1l5 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l5))
d2l5 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l5))
d3l5 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l5))
d4l5 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l5))
d5l5 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l5))
d6l5 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l5))

d1l6 = 0.9922/((ped[x] /. x -> xi1) (xi1 - l6))
d2l6 = 0.9922/((ped[x] /. x -> xi2) (xi2 - l6))
d3l6 = 0.9922/((ped[x] /. x -> xi3) (xi3 - l6))
d4l6 = 0.9922/((ped[x] /. x -> xi4) (xi4 - l6))
d5l6 = 0.9922/((ped[x] /. x -> xi5) (xi5 - l6))
```

```
d6l6 = 0.9922/((ped[x] /. x -> xi6) (xi6 - l6))

Abs[d1l1] + Abs[d1l2] + Abs[d1l3] + Abs[d1l4] + Abs[d1l5] +
 Abs[d1l6] + Abs[d2l1] + Abs[d2l2] + Abs[d2l3] + Abs[d2l4] +
 Abs[d2l5] + Abs[d2l6] + Abs[d3l1] + Abs[d3l2] + Abs[d3l3] +
 Abs[d3l4] + Abs[d3l5] + Abs[d3l6] + Abs[d4l1] + Abs[d4l2] +
 Abs[d4l3] + Abs[d4l4] + Abs[d4l5] + Abs[d4l6] + Abs[d5l1] +
 Abs[d5l2] + Abs[d5l3] + Abs[d5l4] + Abs[d5l5] + Abs[d5l6] +
 Abs[d6l1] + Abs[d6l2] + Abs[d6l3] + Abs[d6l4] + Abs[d6l5] +
 Abs[d6l6]

6.51228
```

A.3 Degree 8 polynomial

```
xi = Solve[(z - E^(I Pi/8)) (z - E^(I (3 Pi/8 - Pi/70)))
   (z - E^(I (5 Pi/8 - Pi/70))) (z - E^(I 7 Pi/8))
   (z - E^(I 9 Pi/8)) (z - E^(I (11 Pi/8 - Pi/70)))
   (z - E^(I (13 Pi/8 - Pi/70))) (z - E^(I 15 Pi/8))
   == 0.9962, z]

xi1 = N[Part[xi, 1, 1, 2]]
xi2 = N[Part[xi, 2, 1, 2]]
xi3 = N[Part[xi, 3, 1, 2]]
xi4 = N[Part[xi, 4, 1, 2]]
xi5 = N[Part[xi, 5, 1, 2]]
xi6 = N[Part[xi, 6, 1, 2]]
xi7 = N[Part[xi, 7, 1, 2]]
xi8 = N[Part[xi, 8, 1, 2]]

l = NSolve[(z - E^(I Pi/8)) (z - E^(I (3 Pi/8 - Pi/70)))
(z - E^(I (5 Pi/8 - Pi/70))) (z - E^(I 7 Pi/8))
(z - E^(I 9 Pi/8)) (z - E^(I (11 Pi/8 - Pi/70)))
(z - E^(I (13 Pi/8 - Pi/70))) (z - E^(I 15 Pi/8)) == 0, z]

l1 = N[Part[l, 1, 1, 2]]
l2 = N[Part[l, 2, 1, 2]]
l3 = N[Part[l, 3, 1, 2]]
l4 = N[Part[l, 4, 1, 2]]
l5 = N[Part[l, 5, 1, 2]]
l6 = N[Part[l, 6, 1, 2]]
l7 = N[Part[l, 7, 1, 2]]
l8 = N[Part[l, 8, 1, 2]]

ped[z_] :=
 D[(z - E^(I Pi/8)) (z - E^(I (3 Pi/8 - Pi/70)))
   (z - E^(I (5 Pi/8 - Pi/70))) (z - E^(I 7 Pi/8))
   (z - E^(I 9 Pi/8)) (z - E^(I (11 Pi/8 - Pi/70)))
   (z - E^(I (13 Pi/8 - Pi/70))) (z - E^(I 15 Pi/8)), z]
```

```
d1l1 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l1))
d4l1 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l1))
d5l1 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l1))
d6l1 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l1))
d7l1 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l1))
d8l1 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l1))

d1l2 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l2))
d5l2 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l2))
d6l2 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l2))
d7l2 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l2))
d8l2 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l2))

d1l3 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l3))
d5l3 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l3))
d6l3 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l3))
d7l3 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l3))
d8l3 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l3))

d1l4 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l4))
d5l4 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l4))
d6l4 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l4))
d7l4 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l4))
d8l4 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l4))

d1l5 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l5))
d2l5 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l5))
d3l5 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l5))
d4l5 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l5))
d5l5 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l5))
d6l5 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l5))
d7l5 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l5))
d8l5 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l5))

d1l6 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l6))
d2l6 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l6))
d3l6 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l6))
d4l6 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l6))
d5l6 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l6))
```

```
d6l6 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l6))
d7l6 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l6))
d8l6 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l6))

d1l7 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l7))
d2l7 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l7))
d3l7 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l7))
d4l7 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l7))
d5l7 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l7))
d6l7 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l7))
d7l7 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l7))
d8l7 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l7))

d1l8 = 0.9962/((ped[x] /. x -> xi1) (xi1 - l8))
d2l8 = 0.9962/((ped[x] /. x -> xi2) (xi2 - l8))
d3l8 = 0.9962/((ped[x] /. x -> xi3) (xi3 - l8))
d4l8 = 0.9962/((ped[x] /. x -> xi4) (xi4 - l8))
d5l8 = 0.9962/((ped[x] /. x -> xi5) (xi5 - l8))
d6l8 = 0.9962/((ped[x] /. x -> xi6) (xi6 - l8))
d7l8 = 0.9962/((ped[x] /. x -> xi7) (xi7 - l8))
d8l8 = 0.9962/((ped[x] /. x -> xi8) (xi8 - l8))

Abs[d1l1] + Abs[d1l2] + Abs[d1l3] + Abs[d1l4] + Abs[d1l5] +
 Abs[d1l6] + Abs[d1l7] + Abs[d1l8] + Abs[d2l1] + Abs[d2l2] +
 Abs[d2l3] + Abs[d2l4] + Abs[d2l5] + Abs[d2l6] + Abs[d2l7] +
 Abs[d2l8] + Abs[d3l1] + Abs[d3l2] + Abs[d3l3] + Abs[d3l4] +
 Abs[d3l5] + Abs[d3l6] + Abs[d3l7] + Abs[d3l8] + Abs[d4l1] +
 Abs[d4l2] + Abs[d4l3] + Abs[d4l4] + Abs[d4l5] + Abs[d4l6] +
 Abs[d4l7] + Abs[d4l8] + Abs[d5l1] + Abs[d5l2] + Abs[d5l3] +
 Abs[d5l4] + Abs[d5l5] + Abs[d5l6] + Abs[d5l7] + Abs[d5l8] +
 Abs[d6l1] + Abs[d6l2] + Abs[d6l3] + Abs[d6l4] + Abs[d6l5] +
 Abs[d6l6] + Abs[d6l7] + Abs[d6l8] + Abs[d7l1] + Abs[d7l2] +
 Abs[d7l3] + Abs[d7l4] + Abs[d7l5] + Abs[d7l6] + Abs[d7l7] +
 Abs[d7l8] + Abs[d8l1] + Abs[d8l2] + Abs[d8l3] + Abs[d8l4] +
 Abs[d8l5] + Abs[d8l6] + Abs[d8l7] + Abs[d8l8]

46.6599
```

A.4 Degree 10 polynomial

```
xi = Solve[(z - 1)(z - E^(I Pi/5))(z - E^(I (2 Pi/5 - Pi/70)))
  (z + 1)(z - E^(I 4 Pi/5))(z - E^(I (3 Pi/5 - Pi/70)))
  (z - E^(I 6 Pi/5))(z - E^(I 9 Pi/8))(z - E^(I (7 Pi/5 - Pi/70)))
  (z - E^(I (8 Pi/5 - Pi/70))) == w, z] /. w -> 0.9979

xi1 = N[Part[xi, 1, 1, 2]]
xi2 = N[Part[xi, 2, 1, 2]]
xi3 = N[Part[xi, 3, 1, 2]]
xi4 = N[Part[xi, 4, 1, 2]]
xi5 = N[Part[xi, 5, 1, 2]]
```

```
xi6 = N[Part[xi, 6, 1, 2]]
xi7 = N[Part[xi, 7, 1, 2]]
xi8 = N[Part[xi, 8, 1, 2]]
xi9 = N[Part[xi, 9, 1, 2]]
xi10 = N[Part[xi, 10, 1, 2]]

l = NSolve[(z - 1)(z - E^(I Pi/5))(z - E^(I (2 Pi/5 - Pi/70)))
(z + 1) (z - E^(I 4 Pi/5)) (z - E^(I (3 Pi/5 - Pi/70)))
(z - E^(I 6 Pi/5))(z - E^(I 9 Pi/8))(z - E^(I (7 Pi/5 - Pi/70)))
(z - E^(I (8 Pi/5 - Pi/70))) == 0, z];

l1 = N[Part[l, 1, 1, 2]]
l2 = N[Part[l, 2, 1, 2]]
l3 = N[Part[l, 3, 1, 2]]
l4 = N[Part[l, 4, 1, 2]]
l5 = N[Part[l, 5, 1, 2]]
l6 = N[Part[l, 6, 1, 2]]
l7 = N[Part[l, 7, 1, 2]]
l8 = N[Part[l, 8, 1, 2]]
l9 = N[Part[l, 9, 1, 2]]
l10 = N[Part[l, 10, 1, 2]]

ped[z_] :=
 D[(z - 1)(z - E^(I Pi/5))(z - E^(I (2 Pi/5 - Pi/70)))(z + 1)
   (z - E^(I 4 Pi/5)) (z - E^(I (3 Pi/5 - Pi/70)))
   (z - E^(I 6 Pi/5))(z - E^(I 9 Pi/8))(z -E^(I (7 Pi/5 - Pi/70)))
   (z - E^(I (8 Pi/5 - Pi/70))), z]

d1l1 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l1))
d4l1 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l1))
d5l1 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l1))
d6l1 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l1))
d7l1 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l1))
d8l1 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l1))
d9l1 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l1))
d10l1 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l1))

d1l2 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l2))
d5l2 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l2))
d6l2 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l2))
d7l2 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l2))
d8l2 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l2))
d9l2 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l2))
d10l2 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l2))
```

```
d1l3 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l3))
d5l3 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l3))
d6l3 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l3))
d7l3 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l3))
d8l3 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l3))
d9l3 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l3))
d10l3 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l3))

d1l4 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l4))
d5l4 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l4))
d6l4 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l4))
d7l4 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l4))
d8l4 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l4))
d9l4 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l4))
d10l4 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l4))

d1l5 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l5))
d2l5 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l5))
d3l5 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l5))
d4l5 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l5))
d5l5 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l5))
d6l5 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l5))
d7l5 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l5))
d8l5 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l5))
d9l5 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l5))
d10l5 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l5))

d1l6 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l6))
d2l6 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l6))
d3l6 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l6))
d4l6 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l6))
d5l6 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l6))
d6l6 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l6))
d7l6 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l6))
d8l6 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l6))
d9l6 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l6))
d10l6 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l6))

d1l7 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l7))
d2l7 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l7))
d3l7 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l7))
d4l7 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l7))
d5l7 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l7))
d6l7 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l7))
```

```
d7l7 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l7))
d8l7 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l7))
d9l7 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l7))
d10l7 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l7))

d1l8 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l8))
d2l8 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l8))
d3l8 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l8))
d4l8 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l8))
d5l8 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l8))
d6l8 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l8))
d7l8 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l8))
d8l8 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l8))
d9l8 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l8))
d10l8 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l8))

d1l9 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l9))
d2l9 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l9))
d3l9 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l9))
d4l9 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l9))
d5l9 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l9))
d6l9 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l9))
d7l9 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l9))
d8l9 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l9))
d9l9 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l9))
d10l9 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l9))

d1l10 = 0.9979/((ped[x] /. x -> xi1) (xi1 - l10))
d2l10 = 0.9979/((ped[x] /. x -> xi2) (xi2 - l10))
d3l10 = 0.9979/((ped[x] /. x -> xi3) (xi3 - l10))
d4l10 = 0.9979/((ped[x] /. x -> xi4) (xi4 - l10))
d5l10 = 0.9979/((ped[x] /. x -> xi5) (xi5 - l10))
d6l10 = 0.9979/((ped[x] /. x -> xi6) (xi6 - l10))
d7l10 = 0.9979/((ped[x] /. x -> xi7) (xi7 - l10))
d8l10 = 0.9979/((ped[x] /. x -> xi8) (xi8 - l10))
d9l10 = 0.9979/((ped[x] /. x -> xi9) (xi9 - l10))
d10l10 = 0.9979/((ped[x] /. x -> xi10) (xi10 - l10))

Abs[d1l1] + Abs[d1l2] + Abs[d1l3] + Abs[d1l4] + Abs[d1l5] +
 Abs[d1l6] + Abs[d1l7] + Abs[d1l8] + Abs[d1l9] + Abs[d1l10] +
 Abs[d2l1] + Abs[d2l2] + Abs[d2l3] + Abs[d2l4] + Abs[d2l5] +
 Abs[d2l6] + Abs[d2l7] + Abs[d2l8] + Abs[d2l9] + Abs[d2l10] +
 Abs[d3l1] + Abs[d3l2] + Abs[d3l3] + Abs[d3l4] + Abs[d3l5] +
 Abs[d3l6] + Abs[d3l7] + Abs[d3l8] + Abs[d3l9] + Abs[d3l10] +
 Abs[d4l1] + Abs[d4l2] + Abs[d4l3] + Abs[d4l4] + Abs[d4l5] +
 Abs[d4l6] + Abs[d4l7] + Abs[d4l8] + Abs[d4l9] + Abs[d4l10] +
 Abs[d5l1] + Abs[d5l2] + Abs[d5l3] + Abs[d5l4] + Abs[d5l5] +
 Abs[d5l6] + Abs[d5l7] + Abs[d5l8] + Abs[d5l9] + Abs[d5l10] +
 Abs[d6l1] + Abs[d6l2] + Abs[d6l3] + Abs[d6l4] + Abs[d6l5] +
 Abs[d6l6] + Abs[d6l7] + Abs[d6l8] + Abs[d6l9] + Abs[d6l10] +
```

```
 Abs[d7l1] + Abs[d7l2] + Abs[d7l3] + Abs[d7l4] + Abs[d7l5] +
 Abs[d7l6] + Abs[d7l7] + Abs[d7l8] + Abs[d7l9] + Abs[d7l10] +
 Abs[d8l1] + Abs[d8l2] + Abs[d8l3] + Abs[d8l4] + Abs[d8l5] +
 Abs[d8l6] + Abs[d8l7] + Abs[d8l8] + Abs[d8l9] + Abs[d8l10] +
 Abs[d9l1] + Abs[d9l2] + Abs[d9l3] + Abs[d9l4] + Abs[d9l5] +
 Abs[d9l6] + Abs[d9l7] + Abs[d9l8] + Abs[d9l9] + Abs[d9l10] +
 Abs[d10l1] + Abs[d10l2] + Abs[d10l3] + Abs[d10l4] +
 Abs[d10l5] + Abs[d10l6] + Abs[d10l7] + Abs[d10l8] +
 Abs[d10l9] + Abs[d10l10]

16.3586
```

A.5 Degree 12 polynomial

```
xi =  Solve[(z - E^(I Pi/12)) (z - E^(I Pi/4))
  (z - E^(I (5 Pi/12 - Pi/70)))(z - E^(I (7 Pi/12 - Pi/70)))
  (z - E^(I (3 Pi/4)))(z - E^(I 11 Pi/12))(z - E^(I 13 Pi/12))
  (z - E^(I 5 Pi/4))(z - E^(I (17 Pi/12 - Pi/70)))
  (z - E^(I (19 Pi/12 - Pi/70)))(z - E^(I 7 Pi/4))
  (z - E^(I 23 Pi/12)) == w, z] /. w -> 0.9978

xi1 = N[Part[xi, 1, 1, 2]]
xi2 = N[Part[xi, 2, 1, 2]]
xi3 = N[Part[xi, 3, 1, 2]]
xi4 = N[Part[xi, 4, 1, 2]]
xi5 = N[Part[xi, 5, 1, 2]]
xi6 = N[Part[xi, 6, 1, 2]]
xi7 = N[Part[xi, 7, 1, 2]]
xi8 = N[Part[xi, 8, 1, 2]]
xi9 = N[Part[xi, 9, 1, 2]]
xi10 = N[Part[xi, 10, 1, 2]]
xi11 = N[Part[xi, 11, 1, 2]]
xi12 = N[Part[xi, 12, 1, 2]]

l = NSolve[
    (z - E^(I Pi/12))(z - E^(I Pi/4))(z - E^(I (5 Pi/12 - Pi/70)))
    (z - E^(I (7 Pi/12 - Pi/70))) (z - E^(I (3 Pi/4)))
    (z - E^(I 11 Pi/12))(z - E^(I 13 Pi/12))(z - E^(I 5 Pi/4))
    (z - E^(I (17 Pi/12 - Pi/70)))(z - E^(I (19 Pi/12 - Pi/70)))
    (z - E^(I 7 Pi/4)) (z - E^(I 23 Pi/12)) == 0, z]

l1 = N[Part[l, 1, 1, 2]]
l2 = N[Part[l, 2, 1, 2]]
l3 = N[Part[l, 3, 1, 2]]
l4 = N[Part[l, 4, 1, 2]]
l5 = N[Part[l, 5, 1, 2]]
l6 = N[Part[l, 6, 1, 2]]
l7 = N[Part[l, 7, 1, 2]]
l8 = N[Part[l, 8, 1, 2]]
l9 = N[Part[l, 9, 1, 2]]
```

```
l10 = N[Part[l, 10, 1, 2]]
l11 = N[Part[l, 11, 1, 2]]
l12 = N[Part[l, 12, 1, 2]]

ped[z_] :=
  D[(z - E^(I Pi/12))(z - E^(I Pi/4))(z - E^(I (5 Pi/12 - Pi/70)))
    (z - E^(I (7 Pi/12 - Pi/70))) (z - E^(I (3 Pi/4)))
    (z - E^(I 11 Pi/12)) (z - E^(I 13 Pi/12)) (z - E^(I 5 Pi/4))
    (z - E^(I (17 Pi/12 - Pi/70))) (z - E^(I (19 Pi/12 - Pi/70)))
    (z - E^(I 7 Pi/4)) (z - E^(I 23 Pi/12)), z];

d1l1 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l1))
d4l1 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l1))
d5l1 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l1))
d6l1 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l1))
d7l1 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l1))
d8l1 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l1))
d9l1 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l1))
d10l1 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l1))
d11l1 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l1))
d12l1 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l1))

d1l2 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l2))
d5l2 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l2))
d6l2 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l2))
d7l2 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l2))
d8l2 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l2))
d9l2 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l2))
d10l2 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l2))
d11l2 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l2))
d12l2 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l2))

d1l3 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l3))
d5l3 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l3))
d6l3 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l3))
d7l3 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l3))
d8l3 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l3))
d9l3 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l3))
d10l3 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l3))
d11l3 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l3))
d12l3 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l3))
```

```
d1l4 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l4))
d5l4 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l4))
d6l4 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l4))
d7l4 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l4))
d8l4 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l4))
d9l4 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l4))
d10l4 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l4))
d11l4 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l4))
d12l4 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l4))

d1l5 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l5))
d2l5 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l5))
d3l5 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l5))
d4l5 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l5))
d5l5 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l5))
d6l5 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l5))
d7l5 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l5))
d8l5 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l5))
d9l5 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l5))
d10l5 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l5))
d11l5 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l5))
d12l5 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l5))

d1l6 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l6))
d2l6 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l6))
d3l6 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l6))
d4l6 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l6))
d5l6 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l6))
d6l6 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l6))
d7l6 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l6))
d8l6 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l6))
d9l6 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l6))
d10l6 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l6))
d11l6 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l6))
d12l6 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l6))

d1l7 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l7))
d2l7 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l7))
d3l7 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l7))
d4l7 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l7))
d5l7 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l7))
d6l7 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l7))
d7l7 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l7))
d8l7 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l7))
d9l7 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l7))
d10l7 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l7))
d11l7 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l7))
```

```
d12l7 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l7))

d1l8 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l8))
d2l8 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l8))
d3l8 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l8))
d4l8 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l8))
d5l8 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l8))
d6l8 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l8))
d7l8 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l8))
d8l8 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l8))
d9l8 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l8))
d10l8 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l8))
d11l8 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l8))
d12l8 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l8))

d1l9 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l9))
d2l9 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l9))
d3l9 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l9))
d4l9 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l9))
d5l9 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l9))
d6l9 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l9))
d7l9 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l9))
d8l9 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l9))
d9l9 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l9))
d10l9 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l9))
d11l9 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l9))
d12l9 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l9))

d1l10 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l10))
d2l10 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l10))
d3l10 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l10))
d4l10 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l10))
d5l10 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l10))
d6l10 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l10))
d7l10 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l10))
d8l10 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l10))
d9l10 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l10))
d10l10 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l10))
d11l10 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l10))
d12l10 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l10))

d1l11 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l11))
d2l11 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l11))
d3l11 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l11))
d4l11 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l11))
d5l11 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l11))
d6l11 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l11))
d7l11 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l11))
d8l11 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l11))
d9l11 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l11))
```

```
d10l11 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l11))
d11l11 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l11))
d12l11 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l11))

d1l12 = 0.9978/((ped[x] /. x -> xi1) (xi1 - l12))
d2l12 = 0.9978/((ped[x] /. x -> xi2) (xi2 - l12))
d3l12 = 0.9978/((ped[x] /. x -> xi3) (xi3 - l12))
d4l12 = 0.9978/((ped[x] /. x -> xi4) (xi4 - l12))
d5l12 = 0.9978/((ped[x] /. x -> xi5) (xi5 - l12))
d6l12 = 0.9978/((ped[x] /. x -> xi6) (xi6 - l12))
d7l12 = 0.9978/((ped[x] /. x -> xi7) (xi7 - l12))
d8l12 = 0.9978/((ped[x] /. x -> xi8) (xi8 - l12))
d9l12 = 0.9978/((ped[x] /. x -> xi9) (xi9 - l12))
d10l12 = 0.9978/((ped[x] /. x -> xi10) (xi10 - l12))
d11l12 = 0.9978/((ped[x] /. x -> xi11) (xi11 - l12))
d12l12 = 0.9978/((ped[x] /. x -> xi12) (xi12 - l12))

Abs[d1l1] + Abs[d1l2] + Abs[d1l3] + Abs[d1l4] + Abs[d1l5] +
Abs[d1l6] + Abs[d1l7] + Abs[d1l8] + Abs[d1l9] + Abs[d1l10]+
Abs[d1l11] + Abs[d1l12] + Abs[d2l1] + Abs[d2l2] + Abs[d2l3]+
Abs[d2l4] + Abs[d2l5] + Abs[d2l6] + Abs[d2l7] + Abs[d2l8] +
Abs[d2l9] + Abs[d2l10] + Abs[d2l11] + Abs[d2l12] +Abs[d3l1]+
Abs[d3l2] + Abs[d3l3] + Abs[d3l4] + Abs[d3l5] + Abs[d3l6] +
Abs[d3l7] + Abs[d3l8] + Abs[d3l9] + Abs[d3l10] + Abs[d3l11]+
Abs[d3l12] + Abs[d4l1] + Abs[d4l2] + Abs[d4l3] + Abs[d4l4] +
Abs[d4l5] + Abs[d4l6] + Abs[d4l7] + Abs[d4l8] + Abs[d4l9] +
Abs[d4l10] + Abs[d4l11] + Abs[d4l12] + Abs[d5l1] +Abs[d5l2]+
Abs[d5l3] + Abs[d5l4] + Abs[d5l5] + Abs[d5l6] + Abs[d5l7] +
Abs[d5l8] + Abs[d5l9] + Abs[d5l10] + Abs[d5l11] +Abs[d5l12]+
Abs[d6l1] + Abs[d6l2] + Abs[d6l3] + Abs[d6l4] + Abs[d6l5] +
Abs[d6l6] + Abs[d6l7] + Abs[d6l8] + Abs[d6l9] + Abs[d6l10] +
Abs[d6l11] + Abs[d6l12] + Abs[d7l1] + Abs[d7l2] + Abs[d7l3]+
Abs[d7l4] + Abs[d7l5] + Abs[d7l6] + Abs[d7l7] + Abs[d7l8] +
Abs[d7l9] + Abs[d7l10] + Abs[d7l11] + Abs[d7l12] +Abs[d8l1]+
Abs[d8l2] + Abs[d8l3] + Abs[d8l4] + Abs[d8l5] + Abs[d8l6] +
Abs[d8l7] + Abs[d8l8] + Abs[d8l9] + Abs[d8l10] + Abs[d8l11]+
Abs[d8l12] + Abs[d9l1] + Abs[d9l2] + Abs[d9l3] + Abs[d9l4] +
Abs[d9l5] + Abs[d9l6] + Abs[d9l7] + Abs[d9l8] + Abs[d9l9] +
Abs[d9l10] + Abs[d9l11] + Abs[d9l12] + Abs[d10l1] +
Abs[d10l2] + Abs[d10l3] + Abs[d10l4] + Abs[d10l5] +
Abs[d10l6] + Abs[d10l7] + Abs[d10l8] + Abs[d10l9] +
Abs[d10l10] + Abs[d10l11] + Abs[d10l12] + Abs[d11l1] +
Abs[d11l2] + Abs[d11l3] + Abs[d11l4] + Abs[d11l5] +
Abs[d11l6] + Abs[d11l7] + Abs[d11l8] + Abs[d11l9] +
Abs[d11l10] + Abs[d11l11] + Abs[d11l12] + Abs[d12l1] +
Abs[d12l2] + Abs[d12l3] + Abs[d12l4] + Abs[d12l5] +
Abs[d12l6] + Abs[d12l7] + Abs[d12l8] + Abs[d12l9] +
Abs[d12l10] + Abs[d12l11] + Abs[d12l12]

83.4547
```

A.6 Degree 14 polynomial

```
xi = Solve[(z - 1) (z + 1) (z - E^(I Pi/7)) (z - E^(2 I Pi/7))
     (z - E^(I (3 Pi/7 - Pi/70))) (z - E^(I (4 Pi/7 - Pi/70)))
     (z - E^(I (5 Pi/7))) (z - E^(I 6 Pi/7)) (z - E^(I 8 Pi/7))
     (z - E^(I 9 Pi/7)) (z - E^(I (10 Pi/7 - Pi/70)))
     (z - E^(I (11 Pi/7 - Pi/70))) (z - E^(I 12 Pi/7))
     (z - E^(I 13 Pi/7)) == w, z] /. w -> 0.9955

xi1 = N[Part[xi, 1, 1, 2]]
xi2 = N[Part[xi, 2, 1, 2]]
xi3 = N[Part[xi, 3, 1, 2]]
xi4 = N[Part[xi, 4, 1, 2]]
xi5 = N[Part[xi, 5, 1, 2]]
xi6 = N[Part[xi, 6, 1, 2]]
xi7 = N[Part[xi, 7, 1, 2]]
xi8 = N[Part[xi, 8, 1, 2]]
xi9 = N[Part[xi, 9, 1, 2]]
xi10 = N[Part[xi, 10, 1, 2]]
xi11 = N[Part[xi, 11, 1, 2]]
xi12 = N[Part[xi, 12, 1, 2]]
xi13 = N[Part[xi, 13, 1, 2]]
xi14 = N[Part[xi, 14, 1, 2]]

l = NSolve[(z - 1) (z + 1) (z - E^(I Pi/7)) (z - E^(2 I Pi/7))
(z - E^(I (3 Pi/7 - Pi/70))) (z - E^(I (4 Pi/7 - Pi/70)))
(z - E^(I (5 Pi/7))) (z - E^(I 6 Pi/7)) (z - E^(I 8 Pi/7))
(z - E^(I 9 Pi/7)) (z - E^(I (10 Pi/7 - Pi/70)))
(z - E^(I (11 Pi/7 - Pi/70))) (z - E^(I 12 Pi/7))
(z - E^(I 13 Pi/7)) == 0, z]

l1 = N[Part[l, 1, 1, 2]]; l2 = N[Part[l, 2, 1, 2]];
l3 = N[Part[l, 3, 1, 2]]; l4 = N[Part[l, 4, 1, 2]];
l5 = N[Part[l, 5, 1, 2]]; l6 = N[Part[l, 6, 1, 2]];
l7 = N[Part[l, 7, 1, 2]]; l8 = N[Part[l, 8, 1, 2]];
l9 = N[Part[l, 9, 1, 2]]; l10 = N[Part[l, 10, 1, 2]];
l11 = N[Part[l, 11, 1, 2]]; l12 = N[Part[l, 12, 1, 2]];
l13 = N[Part[l, 13, 1, 2]]; l14 = N[Part[l, 14, 1, 2]];

ped[z_] :=
  D[(z - 1) (z + 1) (z - E^(I Pi/7)) (z - E^(2 I Pi/7))
    (z - E^(I (3 Pi/7 - Pi/70))) (z - E^(I (4 Pi/7 - Pi/70)))
    (z - E^(I (5 Pi/7))) (z - E^(I 6 Pi/7)) (z - E^(I 8 Pi/7))
    (z - E^(I 9 Pi/7)) (z - E^(I (10 Pi/7 - Pi/70)))
    (z - E^(I (11 Pi/7 - Pi/70))) (z - E^(I 12 Pi/7))
    (z - E^(I 13 Pi/7)), z];

d1l1 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l1))
d2l1 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l1))
d3l1 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l1))
```

```
d4l1 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l1))
d5l1 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l1))
d6l1 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l1))
d7l1 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l1))
d8l1 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l1))
d9l1 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l1))
d10l1 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l1))
d11l1 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l1))
d12l1 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l1))
d13l1 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l1))
d14l1 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l1))

d1l2 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l2))
d2l2 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l2))
d3l2 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l2))
d4l2 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l2))
d5l2 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l2))
d6l2 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l2))
d7l2 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l2))
d8l2 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l2))
d9l2 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l2))
d10l2 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l2))
d11l2 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l2))
d12l2 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l2))
d13l2 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l2))
d14l2 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l2))

d1l3 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l3))
d2l3 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l3))
d3l3 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l3))
d4l3 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l3))
d5l3 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l3))
d6l3 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l3))
d7l3 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l3))
d8l3 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l3))
d9l3 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l3))
d10l3 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l3))
d11l3 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l3))
d12l3 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l3))
d13l3 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l3))
d14l3 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l3))

d1l4 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l4))
d2l4 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l4))
d3l4 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l4))
d4l4 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l4))
d5l4 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l4))
d6l4 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l4))
d7l4 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l4))
d8l4 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l4))
```

```
d9l4 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l4))
d10l4 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l4))
d11l4 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l4))
d12l4 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l4))
d13l4 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l4))
d14l4 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l4))

d1l5 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l5))
d2l5 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l5))
d3l5 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l5))
d4l5 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l5))
d5l5 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l5))
d6l5 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l5))
d7l5 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l5))
d8l5 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l5))
d9l5 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l5))
d10l5 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l5))
d11l5 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l5))
d12l5 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l5))
d13l5 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l5))
d14l5 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l5))

d1l6 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l6))
d2l6 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l6))
d3l6 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l6))
d4l6 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l6))
d5l6 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l6))
d6l6 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l6))
d7l6 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l6))
d8l6 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l6))
d9l6 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l6))
d10l6 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l6))
d11l6 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l6))
d12l6 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l6))
d13l6 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l6))
d14l6 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l6))

d1l7 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l7))
d2l7 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l7))
d3l7 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l7))
d4l7 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l7))
d5l7 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l7))
d6l7 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l7))
d7l7 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l7))
d8l7 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l7))
d9l7 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l7))
d10l7 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l7))
d11l7 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l7))
d12l7 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l7))
d13l7 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l7))
```

```
d14l7 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l7))

d1l8 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l8))
d2l8 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l8))
d3l8 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l8))
d4l8 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l8))
d5l8 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l8))
d6l8 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l8))
d7l8 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l8))
d8l8 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l8))
d9l8 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l8))
d10l8 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l8))
d11l8 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l8))
d12l8 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l8))
d13l8 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l8))
d14l8 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l8))

d1l9 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l9))
d2l9 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l9))
d3l9 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l9))
d4l9 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l9))
d5l9 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l9))
d6l9 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l9))
d7l9 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l9))
d8l9 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l9))
d9l9 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l9))
d10l9 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l9))
d11l9 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l9))
d12l9 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l9))
d13l9 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l9))
d14l9 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l9))

d1l10 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l10))
d2l10 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l10))
d3l10 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l10))
d4l10 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l10))
d5l10 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l10))
d6l10 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l10))
d7l10 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l10))
d8l10 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l10))
d9l10 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l10))
d10l10 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l10))
d11l10 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l10))
d12l10 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l10))
d13l10 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l10))
d14l10 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l10))

d1l11 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l11))
d2l11 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l11))
d3l11 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l11))
```

```
d4l11 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l11))
d5l11 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l11))
d6l11 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l11))
d7l11 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l11))
d8l11 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l11))
d9l11 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l11))
d10l11 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l11))
d11l11 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l11))
d12l11 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l11))
d13l11 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l11))
d14l11 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l11))

d1l12 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l12))
d2l12 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l12))
d3l12 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l12))
d4l12 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l12))
d5l12 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l12))
d6l12 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l12))
d7l12 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l12))
d8l12 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l12))
d9l12 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l12))
d10l12 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l12))
d11l12 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l12))
d12l12 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l12))
d13l12 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l12))
d14l12 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l12))

d1l13 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l13))
d2l13 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l13))
d3l13 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l13))
d4l13 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l13))
d5l13 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l13))
d6l13 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l13))
d7l13 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l13))
d8l13 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l13))
d9l13 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l13))
d10l13 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l13))
d11l13 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l13))
d12l13 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l13))
d13l13 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l13))
d14l13 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l13))

d1l14 = 0.9955/((ped[x] /. x -> xi1) (xi1 - l14))
d2l14 = 0.9955/((ped[x] /. x -> xi2) (xi2 - l14))
d3l14 = 0.9955/((ped[x] /. x -> xi3) (xi3 - l14))
d4l14 = 0.9955/((ped[x] /. x -> xi4) (xi4 - l14))
d5l14 = 0.9955/((ped[x] /. x -> xi5) (xi5 - l14))
d6l14 = 0.9955/((ped[x] /. x -> xi6) (xi6 - l14))
d7l14 = 0.9955/((ped[x] /. x -> xi7) (xi7 - l14))
d8l14 = 0.9955/((ped[x] /. x -> xi8) (xi8 - l14))
```

```
d9l14 = 0.9955/((ped[x] /. x -> xi9) (xi9 - l14))
d10l14 = 0.9955/((ped[x] /. x -> xi10) (xi10 - l14))
d11l14 = 0.9955/((ped[x] /. x -> xi11) (xi11 - l14))
d12l14 = 0.9955/((ped[x] /. x -> xi12) (xi12 - l14))
d13l14 = 0.9955/((ped[x] /. x -> xi13) (xi13 - l14))
d14l14 = 0.9955/((ped[x] /. x -> xi14) (xi14 - l14))

Abs[d1l1] + Abs[d1l2] + Abs[d1l3] + Abs[d1l4] + Abs[d1l5] +
 Abs[d1l6] + Abs[d1l7] + Abs[d1l8] + Abs[d1l9] + Abs[d1l10] +
 Abs[d1l11] + Abs[d1l12] + Abs[d1l13] + Abs[d1l14] +
 Abs[d2l1] + Abs[d2l2] + Abs[d2l3] + Abs[d2l4] + Abs[d2l5] +
 Abs[d2l6] + Abs[d2l7] + Abs[d2l8] + Abs[d2l9] + Abs[d2l10] +
 Abs[d2l11] + Abs[d2l12] + Abs[d2l13] + Abs[d2l14] +
 Abs[d3l1] + Abs[d3l2] + Abs[d3l3] + Abs[d3l4] + Abs[d3l5] +
 Abs[d3l6] + Abs[d3l7] + Abs[d3l8] + Abs[d3l9] + Abs[d3l10] +
 Abs[d3l11] + Abs[d3l12] + Abs[d3l13] + Abs[d3l14] +
 Abs[d4l1] + Abs[d4l2] + Abs[d4l3] + Abs[d4l4] + Abs[d4l5] +
 Abs[d4l6] + Abs[d4l7] + Abs[d4l8] + Abs[d4l9] + Abs[d4l10] +
 Abs[d4l11] + Abs[d4l12] + Abs[d4l13] + Abs[d4l14] +
 Abs[d5l1] + Abs[d5l2] + Abs[d5l3] + Abs[d5l4] + Abs[d5l5] +
 Abs[d5l6] + Abs[d5l7] + Abs[d5l8] + Abs[d5l9] + Abs[d5l10] +
 Abs[d5l11] + Abs[d5l12] + Abs[d5l13] + Abs[d5l14] +
 Abs[d6l1] + Abs[d6l2] + Abs[d6l3] + Abs[d6l4] + Abs[d6l5] +
 Abs[d6l6] + Abs[d6l7] + Abs[d6l8] + Abs[d6l9] + Abs[d6l10] +
 Abs[d6l11] + Abs[d6l12] + Abs[d6l13] + Abs[d6l14] +
 Abs[d7l1] + Abs[d7l2] + Abs[d7l3] + Abs[d7l4] + Abs[d7l5] +
 Abs[d7l6] + Abs[d7l7] + Abs[d7l8] + Abs[d7l9] + Abs[d7l10] +
 Abs[d7l11] + Abs[d7l12] + Abs[d7l13] + Abs[d7l14] +
 Abs[d8l1] + Abs[d8l2] + Abs[d8l3] + Abs[d8l4] + Abs[d8l5] +
 Abs[d8l6] + Abs[d8l7] + Abs[d8l8] + Abs[d8l9] + Abs[d8l10] +
 Abs[d8l11] + Abs[d8l12] + Abs[d8l13] + Abs[d8l14] +
 Abs[d9l1] + Abs[d9l2] + Abs[d9l3] + Abs[d9l4] + Abs[d9l5] +
 Abs[d9l6] + Abs[d9l7] + Abs[d9l8] + Abs[d9l9] + Abs[d9l10] +
 Abs[d9l11] + Abs[d9l12] + Abs[d9l13] + Abs[d9l14] +
 Abs[d10l1] + Abs[d10l2] + Abs[d10l3] + Abs[d10l4] +
 Abs[d10l5] + Abs[d10l6] + Abs[d10l7] + Abs[d10l8] +
 Abs[d10l9] + Abs[d10l10] + Abs[d10l11] + Abs[d10l12] +
 Abs[d10l13] + Abs[d10l14] + Abs[d11l1] + Abs[d11l2] +
 Abs[d11l3] + Abs[d11l4] + Abs[d11l5] + Abs[d11l6] +
 Abs[d11l7] + Abs[d11l8] + Abs[d11l9] + Abs[d11l10] +
 Abs[d11l11] + Abs[d11l12] + Abs[d11l13] + Abs[d11l14] +
 Abs[d12l1] + Abs[d12l2] + Abs[d12l3] + Abs[d12l4] +
 Abs[d12l5] + Abs[d12l6] + Abs[d12l7] + Abs[d12l8] +
 Abs[d12l9] + Abs[d12l10] + Abs[d12l11] + Abs[d12l12] +
 Abs[d12l13] + Abs[d12l14] + Abs[d13l1] + Abs[d13l2] +
 Abs[d13l3] + Abs[d13l4] + Abs[d13l5] + Abs[d13l6] +
 Abs[d13l7] + Abs[d13l8] + Abs[d13l9] + Abs[d13l10] +
 Abs[d13l11] + Abs[d13l12] + Abs[d13l13] + Abs[d13l14] +
 Abs[d14l1] + Abs[d14l2] + Abs[d14l3] + Abs[d14l4] +
 Abs[d14l5] + Abs[d14l6] + Abs[d14l7] + Abs[d14l8] +
```

```
 Abs[d14l9] + Abs[d14l10] + Abs[d14l11] + Abs[d14l12] +
 Abs[d14l13] + Abs[d14l14]

16.7046
```

B Matlab program - computes s from Remark 2.10

```
function pituus = minlength(piste,polyn,rho,loota)
  %UNTITLED piste=[x;y] (column vector) is the critical point
  % and polyn=[a_n, .. , a_1,a_0] (row vector) for the REAL
  % polynomial an*z^n + ... + a1*z+a0, rho is the level
  % (p(z)=rho gives the lemniscate)
  % loota=the size of the drawing area. MAKE BIG ENOUGH FOR THE
  % LEMNISCATE TO FIT IN, oterwise the Matlab's contour command
  % returns only partof the lemniscate drawn and you get wrong
  % (and strange) answers

% get the level curve data
C = kontour(polyn,[rho],[-loota,-loota,loota,loota]);
[~,m]=size(C);
% (copypaste from lemnlength:)
katkot=[];
for ii=2:m
  if (C(2,ii)>10)
      katkot=[katkot, ii];
  end
end
% build M matrix that contains the start and end points
% for the pieces of the lemniscate in the data
valienlkm=length(katkot)+1;
M=zeros(2,valienlkm); M(1,1)=2; M(2,valienlkm)=m;
for ii=1:(valienlkm-1)
   M(2,ii)=katkot(ii)-1; M(1,ii+1)=katkot(ii)+1;
end
% go through M and calculate all the angles
pituudet=sqrt(2)*loota*ones(1,m);
for ii=1:valienlkm
   for jj=M(1,ii):(M(2,ii))
      pituudet(jj)=norm([C(1,jj)-piste(1,1),C(2,jj)-piste(2,1)]);
   end
end
pituus=min(pituudet);
end
```

The programme above uses the following programme:

```
function C = kontour(p,tasot,ruutu)
 % UNTITLED Draws the lemniscates of the polynomial p.
 % The picture coordinates are in ruutu vector.
 % ruutu=[xmin,ymin,xmax,ymax]
```

```
 % tasot=[l1,l2,..,ln] (the levels to be drawn)
 % p=[an,...,a1,a0] (the polynomial an*z^n + ... + a1*z+a0)

juuret=roots(p);
xx=linspace(ruutu(1),ruutu(3),501);
yy=linspace(ruutu(2),ruutu(4),501);
[X,Y]=meshgrid(xx,yy);
Z=abs(polyval(p,X+1i*Y));
if length(tasot)==1
  [C,~]=contour(X,Y,Z,[tasot(1),tasot(1)]);
else
  [C,~]=contour(X,Y,Z,tasot);
end
hold on, plot(real(juuret),imag(juuret),'.k')
end
```

C Expansions for the Riesz projection

C.1 Degree 2

Let $p(z) = z^2 - 1$ with solutions $\lambda_1 = 1$, and $\lambda_2 = -1$. Denoting $\delta_1(z) = (1+z)/2$ and $\delta_2(z) = (1-z)/2$ we obtain

$$f_1(z^2-1) = \frac{1}{2}[f(z)+f(-z)] + \frac{f(z)-f(-z)}{2z} \tag{C.1}$$

$$f_2(z^2-1) = \frac{1}{2}[f(z)+f(-z)] - \frac{f(z)-f(-z)}{2z}. \tag{C.2}$$

Consider the Riesz projection which is obtained by assuming φ to be identically 1 near 1 and -1 near -1. We have, for $|w| < 1$, where $w = z^2 - 1$

$$(w+1)^{1/2} = 1 + \frac{1}{2}w - \frac{1}{8}w^2 + \frac{1}{16}w^3 + \cdots$$

and

$$(w+1)^{-1/2} = 1 - \frac{1}{2}w + \frac{3}{8}w^2 - \frac{5}{16}w^3 + \cdots .$$

Let us compute the two-centric representation first around the point 1. There with $z = (w+1)^{1/2}$

$$\begin{aligned} \delta_1(z) &= 1 + \frac{1}{4}w - \frac{1}{16}w^2 + \frac{1}{32}w^3 + \cdots , \\ \delta_2(z) &= -\frac{1}{4}w + \frac{1}{16}w^2 - \frac{1}{32}w^3 + \cdots . \end{aligned}$$

From (C.1) and (C.2) we obtain

$$\begin{aligned} f_1(w) &= 1 - \frac{1}{2}w + \frac{3}{8}w^2 - \frac{5}{16}w^3 + \cdots , \\ f_2(w) &= -1 + \frac{1}{2}w - \frac{3}{8}w^2 + \frac{5}{16}w^3 + \cdots . \end{aligned}$$

This gives

$$\begin{aligned}
\delta_1(z)f_1(w) &= 1-\frac{1}{4}w+\frac{3}{16}w^2-\frac{5}{32}w^3+\cdots, \\
\delta_2(z)f_2(w) &= \frac{1}{4}w-\frac{3}{16}w^2+\frac{5}{32}w^3+\cdots
\end{aligned}$$

so their sum is identically 1. Near -1 we have

$$\begin{aligned}
\delta_1(z) &= -\frac{1}{4}w+\frac{1}{16}w^2-\frac{1}{32}w^3+\cdots, \\
\delta_2(z) &= 1+\frac{1}{4}w-\frac{1}{16}w^2+\frac{1}{32}w^3+\cdots,
\end{aligned}$$

which gives

$$\begin{aligned}
\delta_1(z)f_1(w) &= -\frac{1}{4}w+\frac{3}{16}w^2-\frac{5}{32}w^3+\cdots, \\
\delta_2(z)f_2(w) &= -1+\frac{1}{4}w-\frac{3}{16}w^2+\frac{5}{32}w^3+\cdots.
\end{aligned}$$

So, near -1 their sum is identically -1.

C.2 Degree 4

Let $p(z) = z^4+1$ with roots

$$\lambda_1 = (1+i)/\sqrt{2},\ \lambda_2 = (-1+i)/\sqrt{2},\ \lambda_3 = (-1-i)/\sqrt{2},\ \lambda_4 = (1-i)/\sqrt{2}.$$

Denoting

$$\begin{aligned}
\delta_1(z) &= \frac{(-1-i)z^3-i\sqrt{2}z^2+(1-i)z+\sqrt{2}}{4\sqrt{2}} \\
&= (-\frac{1}{8}-\frac{i}{8})((1+i)+\sqrt{2}z)(z^2+i) \\
\delta_2(z) &= \frac{(1-i)z^3+i\sqrt{2}z^2-(1+i)z+\sqrt{2}}{4\sqrt{2}} \\
&= (\frac{1}{8}+\frac{i}{8})((1+i)-\sqrt{2}z)(z^2-i)
\end{aligned}$$

$$\begin{aligned}
\delta_3(z) &= \frac{(1+i)z^3-i\sqrt{2}z^2-(1-i)z+\sqrt{2}}{4\sqrt{2}} \\
&= (-\frac{1}{8}-\frac{i}{8})((1+i)-\sqrt{2}z)(z^2+i) \\
\delta_4(z) &= \frac{(-1+i)z^3+i\sqrt{2}z^2+(1+i)z+\sqrt{2}}{4\sqrt{2}} \\
&= (\frac{1}{8}+\frac{i}{8})((1+i)+\sqrt{2}z)(z^2-i)
\end{aligned}$$

we obtain

$$
\begin{aligned}
f_1(z^4+1) &= \frac{1}{4z^3}[(\frac{-1+i}{\sqrt{2}} + iz + \frac{1+i}{\sqrt{2}}z^2 + z^3)f(z) \\
&+ (\frac{-1-i}{\sqrt{2}} - iz + \frac{1-i}{\sqrt{2}}z^2 + z^3)f(iz) \\
&+ (\frac{1-i}{\sqrt{2}} + iz - \frac{1+i}{\sqrt{2}}z^2 + z^3)f(-z) \\
&+ (\frac{1+i}{\sqrt{2}} - iz - \frac{1-i}{\sqrt{2}}z^2 + z^3)f(-iz)] \qquad \text{(C.3)} \\
f_2(z^4+1) &= \frac{1}{4z^3}[(\frac{1+i}{\sqrt{2}} - iz - \frac{1-i}{\sqrt{2}}z^2 + z^3)f(z) \\
&+ (\frac{-1+i}{\sqrt{2}} + iz + \frac{1+i}{\sqrt{2}}z^2 + z^3)f(iz) \\
&+ (\frac{-1-i}{\sqrt{2}} - iz + \frac{1-i}{\sqrt{2}}z^2 + z^3)f(-z) \\
&+ (\frac{1-i}{\sqrt{2}} + iz - \frac{1+i}{\sqrt{2}}z^2 + z^3)f(-iz)] \qquad \text{(C.4)} \\
f_3(z^4+1) &= \frac{1}{4z^3}[(\frac{1-i}{\sqrt{2}} + iz - \frac{1+i}{\sqrt{2}}z^2 + z^3)f(z) \\
&+ (\frac{1+i}{\sqrt{2}} - iz - \frac{1-i}{\sqrt{2}}z^2 + z^3)f(iz) \\
&+ (\frac{-1+i}{\sqrt{2}} + iz + \frac{1+i}{\sqrt{2}}z^2 + z^3)f(-z) \\
&+ (\frac{-1-i}{\sqrt{2}} - iz + \frac{1-i}{\sqrt{2}}z^2 + z^3)f(-iz)] \qquad \text{(C.5)} \\
f_4(z^4+1) &= \frac{1}{4z^3}[(\frac{-1-i}{\sqrt{2}} - iz + \frac{1-i}{\sqrt{2}}z^2 + z^3)f(z) \\
&+ (\frac{1-i}{\sqrt{2}} + iz - \frac{1+i}{\sqrt{2}}z^2 + z^3)f(iz) \\
&+ (\frac{1+i}{\sqrt{2}} - iz - \frac{1-i}{\sqrt{2}}z^2 + z^3)f(-z) \\
&+ (\frac{-1+i}{\sqrt{2}} + iz + \frac{1+i}{\sqrt{2}}z^2 + z^3)f(-iz)] \qquad \text{(C.6)}
\end{aligned}
$$

Consider the Riesz spectral projection which is obtained by assuming φ to be identically 1 near λ_1 and λ_4, and -1 near λ_2 and λ_3. We have, for $|w| < 1$, where $w = z^4+1$

$$(w-1)^{1/4} = \frac{1+i}{\sqrt{2}} - \frac{1}{4}\frac{1+i}{\sqrt{2}}w - \frac{3}{32}\frac{1+i}{\sqrt{2}}w^2 - \frac{7}{128}\frac{1+i}{\sqrt{2}}w^3 + \cdots .$$

Let us compute the four-centric representation first around λ_1. There with $z = (w-1)^{1/4}$ we have

$$
\begin{aligned}
\delta_1 &= 1 - \frac{3}{8}w - \frac{5}{64}w^2 - \frac{5}{128}w^3 + \cdots \\
\delta_2 &= (\frac{1}{8} - \frac{i}{8})w + \frac{1}{32}w^2 + (\frac{1}{64} + \frac{i}{256})w^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\delta_3 &= \frac{1}{8}w + \frac{1}{64}w^2 + \frac{1}{128}w^3 + \cdots \\
\delta_4 &= (\frac{1}{8} + \frac{i}{8})w + \frac{1}{32}w^2 + (\frac{1}{64} - \frac{i}{256})w^3 + \cdots .
\end{aligned}
$$

From (C.3), (C.4), (C.5) and (C.6) we obtain

$$
\begin{aligned}
f_1(w) &= 1 + (\frac{1}{2} - \frac{i}{4})w + (\frac{13}{32} - \frac{i}{4})w^2 + (\frac{23}{64} - \frac{31i}{128})w^3 + \cdots , \\
f_2(w) &= -1 + (-\frac{1}{2} - \frac{i}{4})w + (-\frac{13}{32} - \frac{i}{4})w^2 + (-\frac{23}{64} - \frac{31i}{128})w^3 + \cdots , \\
f_3(w) &= -1 + (-\frac{1}{2} + \frac{i}{4})w + (-\frac{13}{32} + \frac{i}{4})w^2 + (-\frac{23}{64} + \frac{31i}{128})w^3 + \cdots , \\
f_4(w) &= 1 + (\frac{1}{2} + \frac{i}{4})w + (\frac{13}{32} + \frac{i}{4})w^2 + (\frac{23}{64} + \frac{31i}{128})w^3 + \cdots .
\end{aligned}
$$

This gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= 1 + (\frac{1}{8} - \frac{i}{4})w + (\frac{9}{64} - \frac{5i}{32})w^2 + (\frac{33}{256} - \frac{33i}{256})w^3 + \cdots , \\
\delta_2(z)f_2(w) &= -(\frac{1}{8} - \frac{i}{8})w - (\frac{1}{8} - \frac{i}{32})w^2 - (\frac{29}{256} - \frac{i}{128})w^3 + \cdots , \\
\delta_3(z)f_3(w) &= -\frac{1}{8}w - (\frac{5}{64} - \frac{i}{32})w^2 - (\frac{17}{256} - \frac{9i}{256})w^3 + \cdots , \\
\delta_4(z)f_4(w) &= (\frac{1}{8} + \frac{i}{8})w + (\frac{1}{16} + \frac{3i}{32})w^2 + (\frac{13}{256} + \frac{11i}{128})w^3 + \cdots
\end{aligned}
$$

so their sum is identically 1. Near λ_2 we have

$$
\delta_1 = \delta_4(z), \quad \delta_2 = \delta_1(z) \quad \delta_3 = \delta_2(z) \quad \delta_4 = \delta_3(z)
$$

which gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= (\frac{1}{8} + \frac{i}{8})w + (\frac{1}{8} + \frac{i}{32})w^2 + (\frac{29}{256} + \frac{i}{128})w^3 + \cdots , \\
\delta_2(z)f_2(w) &= -1 - (\frac{1}{8} + \frac{i}{4})w - (\frac{9}{64} + \frac{5i}{32})w^2 - (\frac{33}{256} + \frac{33i}{256})w^3 + \cdots , \\
\delta_3(z)f_3(w) &= -(\frac{1}{8} - \frac{i}{8})w - (\frac{1}{16} - \frac{3i}{32})w^2 - (\frac{13}{256} - \frac{11i}{128})w^3 + \cdots , \\
\delta_4(z)f_4(w) &= \frac{1}{8}w + (\frac{5}{64} + \frac{i}{32})w^2 + (\frac{17}{256} + \frac{9i}{256})w^3 + \cdots .
\end{aligned}
$$

So, near λ_2 their sum is -1. Near λ_3 we have

$$
\delta_1 = \delta_3(z), \quad \delta_2 = \delta_4(z) \quad \delta_3 = \delta_1(z) \quad \delta_4 = \delta_2(z)
$$

which gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= \frac{1}{8}w + (\frac{5}{64} - \frac{i}{32})w^2 + (\frac{17}{256} - \frac{9i}{256})w^3 + \cdots , \\
\delta_2(z)f_2(w) &= -(\frac{1}{8} + \frac{i}{8})w - (\frac{1}{16} + \frac{3i}{32})w^2 - (\frac{13}{256} + \frac{11i}{128})w^3 + \cdots , \\
\delta_3(z)f_3(w) &= -1 - (\frac{1}{8} - \frac{i}{4})w - (\frac{9}{64} - \frac{5i}{32})w^2 - (\frac{33}{256} - \frac{33i}{256})w^3 + \cdots , \\
\delta_4(z)f_4(w) &= (\frac{1}{8} - \frac{i}{8})w + (\frac{1}{8} - \frac{i}{32})w^2 + (\frac{29}{256} - \frac{i}{128})w^3 + \cdots
\end{aligned}
$$

so their sum is identically -1 Near λ_4 we have

$$\delta_1 = \delta_2(z), \quad \delta_2 = \delta_3(z) \quad \delta_3 = \delta_4(z) \quad \delta_4 = \delta_1(z)$$

which gives

$$\begin{aligned}
\delta_1(z)f_1(w) &= (\frac{1}{8} - \frac{i}{8})w + (\frac{1}{16} - \frac{3i}{32})w^2 + (\frac{13}{256} - \frac{11i}{128})w^3 + \cdots, \\
\delta_2(z)f_2(w) &= -\frac{1}{8}w - (\frac{5}{16} + \frac{i}{32})w^2 - (\frac{17}{256} + \frac{9i}{256})w^3 + \cdots, \\
\delta_3(z)f_3(w) &= -(\frac{1}{8} + \frac{i}{8})w - (\frac{1}{8} + \frac{i}{32})w^2 - (\frac{29}{256} + \frac{i}{128})w^3 + \cdots, \\
\delta_4(z)f_4(w) &= 1 + (\frac{1}{8} + \frac{i}{4})w + (\frac{9}{64} + \frac{5i}{32})w^2 + (\frac{33}{256} + \frac{33i}{256})w^3 + \cdots.
\end{aligned}$$

So, near λ_4 their sum is 1.

C.3 Degree 4 with perturbed roots

Now let's perturb the roots of $p(z) = z^4+1$ with ε. Therefore, our polynomial becomes

$$p_\varepsilon(z) = z^4 - 2z^2 \sin(2\varepsilon) + 1$$

with roots

$$\lambda_1 = e^{i(\pi/4-\varepsilon)}, \quad \lambda_2 = -e^{-i(\pi/4-\varepsilon)}, \quad \lambda_3 = -e^{i(\pi/4-\varepsilon)}, \quad \lambda_4 = e^{-i(\pi/4-\varepsilon)},$$

and derivative $p'_\varepsilon(z) = 4z^3 - 4z\sin(2\varepsilon)$.

Denoting

$$\begin{aligned}
\delta_1(z) &= (\frac{1}{8} - \frac{i}{8})((1+i) + \sqrt{2}e^{i\varepsilon}z)(e^{2i\varepsilon} - iz^2)\sec(2\varepsilon) \\
\delta_2(z) &= \frac{e^{-3i\varepsilon}}{4\sqrt{2}}(\sqrt{2}e^{i\varepsilon} - (1+i)z)(1 + ie^{2i\varepsilon}z^2)\sec(2\varepsilon) \\
\delta_3(z) &= (\frac{1}{8} + \frac{i}{8})((-1-i) + \sqrt{2}e^{i\varepsilon}z)(ie^{2i\varepsilon} + z^2)\sec(2\varepsilon) \\
\delta_4(z) &= \frac{e^{-3i\varepsilon}}{4\sqrt{2}}(\sqrt{2}e^{i\varepsilon} + (1+i)z)(1 + ie^{2i\varepsilon}z^2)\sec(2\varepsilon)
\end{aligned}$$

we get

$$f_1(p_\varepsilon(z)) = \frac{i}{\sqrt{2}z^3}e^{-3i\varepsilon}(1 + e^{2i\varepsilon}z^2) \tag{C.7}$$

$$f_2(p_\varepsilon(z)) = \frac{1}{\sqrt{2}z^3}e^{i\varepsilon}(e^{2i\varepsilon} - z^2) \tag{C.8}$$

$$f_3(p_\varepsilon(z)) = \frac{-i}{\sqrt{2}z^3}e^{-3i\varepsilon}(1 + e^{2i\varepsilon}z^2) \tag{C.9}$$

$$f_4(p_\varepsilon(z)) = \frac{-1}{\sqrt{2}z^3}e^{i\varepsilon}(e^{2i\varepsilon} - z^2). \tag{C.10}$$

Considering the Riesz projection which is obtained by assuming φ to be identically 1 near the roots on the right hand side of the imaginary axis and -1 near the others, we have, for $|w| < 1$, where $w = p_\varepsilon(z)$,

$$
\begin{aligned}
z &= (-1)^{1/4}\left(1 - i\varepsilon - \frac{1}{2}\varepsilon^2 + \frac{i}{6}\varepsilon^3 + \dots\right) \\
&\quad -\frac{1}{4}(-1)^{1/4}\left(1 + i\varepsilon + \frac{3}{2}\varepsilon^2 + \frac{11i}{6}\varepsilon^3 + \dots\right) w \\
&\quad +\frac{1}{32}(-1)^{3/4}\left(1 + 3i\varepsilon - \frac{1}{2}\varepsilon^2 + \frac{15i}{5}\varepsilon^3 + \dots\right) w^2 + \dots
\end{aligned}
$$

Let us compute the four-centric representation first around λ_1. There with $w = z^4 - 2z^2\sin(2\varepsilon) + 1$

$$
\begin{aligned}
\delta_1(z) &= 1 - \frac{1}{8}\left(3 + 2i\varepsilon + 8\varepsilon^2 + \frac{8i}{3}\varepsilon^3 + \dots\right) w + \frac{1}{32}\left(\left(2 + \frac{3i}{2}\right) - (4-4i)\varepsilon\right. \\
&\quad \left. + (8+2i)\,\varepsilon^2 - \left(\frac{40}{3} - \frac{64i}{3}\right)\varepsilon^3 + \dots\right) w^2 + \dots \\
\delta_2(z) &= \frac{1}{8}\left((1-i) - 2\varepsilon + (4-2i)\,\varepsilon^2 - \frac{20}{3}\varepsilon^3 + \dots\right) w \\
&\quad -\frac{1}{32}\left((1-i) - (1-i)\varepsilon + (5-7i)\varepsilon^2 - \left(\frac{16}{3} - \frac{16i}{3}\right)\varepsilon^3 + \dots\right) w^2 + \dots \\
\delta_3(z) &= \frac{1}{8}\left(1 + 2i\varepsilon + \frac{8i}{3}\varepsilon^3 + \dots\right) w - \frac{1}{32}\left(\left(1 + \frac{i}{2}\right) - (2-2i)\,\varepsilon\right. \\
&\quad \left. + (4-2i)\,\varepsilon^2 - \left(\frac{8}{3} - \frac{32i}{3}\right)\varepsilon^3 + \dots\right) w^2 + \dots \\
\delta_4(z) &= \frac{1}{8}\left((1-i) + 2\varepsilon + (4+2i)\,\varepsilon^2 + \frac{20}{3}\varepsilon^3 + \dots\right) w \\
&\quad -\frac{1}{32}\left(2i - (1-i)\varepsilon - (1-11i)\varepsilon^2 - \left(\frac{16}{3} - \frac{16i}{3}\right)\varepsilon^3 + \dots\right) w^2 + \dots.
\end{aligned}
$$

From (C.7), (C.8), (C.9) and (C.10) we obtain

$$
\begin{aligned}
f_1(w) &= 1 + \left(\left(\frac{1}{2} - \frac{i}{4}\right) + \left(\frac{1}{2} + i\right)\varepsilon + \dots\right) w \\
&+ \left(\left(\frac{3}{16} - \frac{7i}{32}\right) + \left(\frac{7}{8} - \frac{3i}{4}\right)\varepsilon - \left(\frac{3}{4} - \frac{7i}{8}\right)\varepsilon^2 + \dots\right) w^2 + \dots \\
f_2(w) &= \left(-1 + (2-4i)\varepsilon + (10+8i)\varepsilon^2 - \dots\right) \\
&- \left(\left(\frac{1}{2} + \frac{i}{4}\right) - \left(\frac{5}{2} - \frac{7i}{2}\right)\varepsilon - \left(12 - \frac{21i}{2}\right)\varepsilon^2 + \dots\right) w \\
&- \left(\left(\frac{1}{4} + \frac{3i}{32}\right) - \left(\frac{9}{8} - \frac{39i}{16}\right)\varepsilon - \left(11 + \frac{93i}{16}\right)\varepsilon^2 + \dots\right) w^2 + \dots \\
f_3(w) &= -1 - \left(\left(\frac{1}{2} - \frac{i}{4}\right) + \left(\frac{1}{2} - i\right)\varepsilon + \dots\right) w \\
&- \left(\left(\frac{3}{16} - \frac{7i}{32}\right) + \left(\frac{7}{8} + \frac{3i}{4}\right)\varepsilon - \left(\frac{3}{4} - \frac{7i}{8}\right)\varepsilon^2 + \dots\right) w^2 + \dots
\end{aligned}
$$

$$
\begin{aligned}
f_4(w) &= \big(1-(2-4i)\varepsilon-(10+8i)\varepsilon^2+\dots\big)\\
&+ \left(\left(\frac{1}{2}+\frac{i}{4}\right)-\left(\frac{5}{2}-\frac{7i}{2}\right)\varepsilon-\left(12+\frac{21i}{2}\right)\varepsilon^2+\dots\right)w\\
&+ \left(\left(\frac{1}{4}+\frac{3i}{32}\right)-\left(\frac{9}{8}-\frac{39i}{16}\right)\varepsilon-\left(11+\frac{93i}{16}\right)\varepsilon^2+\dots\right)w^2+\dots
\end{aligned}
$$

This gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= 1+\frac{w}{8}\big((1-2i)+(4+6i)\varepsilon-8\varepsilon^2+\dots\big)\\
&+ \frac{w^2}{32}\left(\left(2-\frac{5i}{2}\right)+(16+12i)\varepsilon-(24-34i)\varepsilon^2+\dots\right)+\dots\\
\delta_2(z)f_2(w) &= \frac{w}{8}\big(-(1-i)-6i\varepsilon+(10+8i)\varepsilon^2+\dots\big)\\
&- \frac{w^2}{32}\big(2-(1-15i)\varepsilon-(45+11i)\varepsilon^2+\dots\big)+\dots\\
\delta_3(z)f_3(w) &= \frac{w}{8}\left(-1-2i\varepsilon+\dots\right)\\
&- \frac{w^2}{32}\left(\left(1-\frac{3i}{2}\right)+(6+6i)\varepsilon-(12-12i)\varepsilon^2+\dots\right)+\dots\\
\delta_4(z)f_4(w) &= \frac{w}{8}\big((1+i)-(4-2i)\varepsilon-(2+8i)\varepsilon^2+\dots\big)\\
&+ \frac{w^2}{32}\big((1+i)-(11-9i)\varepsilon-(33+39i)\varepsilon^2+\dots\big)+\dots
\end{aligned}
$$

so their sum is identically 1. Near λ_2 we have

$$
\begin{aligned}
\delta_1(z) &= \left(-(1-i)\varepsilon-\left(\frac{4}{3}-\frac{4i}{3}\right)\varepsilon^3+\dots\right)\\
&\quad +\frac{w}{8}\left((1+i)+2\varepsilon+(4+8i)\varepsilon^2+\frac{8}{3}\varepsilon^3+\dots\right)\\
&\quad -\frac{w^2}{32}\left(2-(4-i)\varepsilon-(4-8i)\varepsilon^2-\left(\frac{64}{3}+\frac{8i}{3}\right)\varepsilon^3+\dots\right)+\dots\\
\delta_2(z) &= \big(1-3i\varepsilon-3\varepsilon^2+\dots\big)+\frac{w}{8}\left(-3+4i\varepsilon-8\varepsilon^2+\frac{40i}{3}\varepsilon^3+\dots\right)\\
&\quad +\frac{w^2}{32}\left(\left(2+\frac{3i}{2}\right)-(1-i)\varepsilon+(11+8i)\varepsilon^2-\left(\frac{16}{3}-\frac{16i}{3}\right)\varepsilon^3+\dots\right)+\dots\\
\delta_3(z) &= \left((1+i)\varepsilon+\left(\frac{4}{3}+\frac{4i}{3}\right)\varepsilon^3+\dots\right)\\
&\quad +\frac{w}{8}\left((1-i)-2\varepsilon+(4-8i)\varepsilon^2-\frac{8}{3}\varepsilon^3\dots\right)\\
&\quad -\frac{w^2}{32}\left((1-i)+(2+i)\varepsilon+(8-4i)\varepsilon^2+\left(\frac{32}{3}+\frac{40i}{3}\right)\varepsilon^3+\dots\right)+\dots
\end{aligned}
$$

$$
\begin{aligned}
\delta_4(z) &= \left(i\varepsilon + 3\varepsilon^2 - \frac{8i}{3}\varepsilon^3 + \dots\right) + \frac{w}{8}\left(1 - 4i\varepsilon - \frac{40i}{3}\varepsilon^3 + \dots\right) \\
&\quad - \frac{w^2}{32}\left(\left(1 + \frac{i}{2}\right) + (1-i)\varepsilon + (7+4i)\varepsilon^2 + \left(\frac{16}{3} - \frac{16i}{3}\right)\varepsilon^3 + \dots\right) + ...
\end{aligned}
$$

which gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= \left(-(1-i)\varepsilon - \left(\frac{4}{3} - \frac{4i}{3}\right)\varepsilon^3 + \dots\right) \\
&\quad + \frac{w}{8}\left((1+i) + 6i\varepsilon - (8-4i)\varepsilon^2 + \dots\right) \\
&\quad + \frac{w^2}{32}\left((3-i) + (7+16i)\varepsilon - (28-16i)\varepsilon^2 + \dots\right) + \dots \\
\delta_2(z)f_2(w) &= \left(-1 + (2-i)\varepsilon + (1+2i)\varepsilon^2 + \dots\right) \\
&\quad - \frac{w}{8}\left((1+2i) - (8-8i)\varepsilon - (18+14i)\varepsilon^2 + \dots\right) \\
&\quad - \frac{w^2}{32}\left(\left(4 + \frac{3i}{2}\right) - (12-26i)\varepsilon - (69+38i)\varepsilon^2 + \dots\right) + \dots \\
\delta_3(z)f_3(w) &= \left(-(1+i)\varepsilon - \left(\frac{4}{3} + \frac{4i}{3}\right)\varepsilon^3 + \dots\right) \\
&\quad - \frac{w}{8}\left((1-i) + (4+2i)\varepsilon + 4i\varepsilon^2 + \dots\right) \\
&\quad + \frac{w^2}{32}\left(2i - (13+2i)\varepsilon + (8-28i)\varepsilon^2 + \dots\right) + \dots \\
\delta_4(z)f_4(w) &= \left(i\varepsilon - (1+2i)\varepsilon^2 + \left(2 - \frac{2i}{3}\right)\varepsilon^3 + \dots\right) \\
&\quad + \frac{w}{8}\left(1 - (4-4i)\varepsilon - (10+14i)\varepsilon^2 + \dots\right) \\
&\quad + \frac{w^2}{32}\left(\left(1 + \frac{i}{2}\right) - (6-12i)\varepsilon - (49-26i)\varepsilon^2 + \dots\right) + \dots
\end{aligned}
$$

So, near λ_2 their sum is -1. Near λ_3 we have

$$
\begin{aligned}
\delta_1(z) &= \frac{w}{8}\left(1 + 2i\varepsilon + \frac{8i}{3}\varepsilon^3 + \dots\right) + \frac{w^2}{32}\left(-\left(1 + \frac{i}{2}\right) + (2-2i)\,\varepsilon\right. \\
&\quad \left. - (4-2i)\,\varepsilon^2 + \left(\frac{8}{3} - \frac{32i}{3}\right)\varepsilon^3 + \dots\right) + \dots \\
\delta_2(z) &= \frac{w}{8}\left((1-i) + 2\varepsilon + (4+2i)\,\varepsilon^2 + \frac{20}{3}\varepsilon^3 + \dots\right) \\
&\quad - \frac{w^2}{32}\left(2i - (1-i)\varepsilon - (1-11i)\varepsilon^2 - \left(\frac{16}{3} - \frac{16i}{3}\right)\varepsilon^3 + \dots\right) + \dots \\
\delta_3(z) &= 1 - \frac{w}{8}\left(3 + 2i\varepsilon + 8\varepsilon^2 + \frac{8i}{3}\varepsilon^3 + \dots\right) + \frac{w^2}{32}\left(\left(2 + \frac{3i}{2}\right) - (4-4i)\varepsilon\right. \\
&\quad \left. + (8+2i)\,\varepsilon^2 - \left(\frac{40}{3} - \frac{64i}{3}\right)\varepsilon^3 + \dots\right) + \dots
\end{aligned}
$$

$$
\begin{aligned}
\delta_4(z) &= \frac{w}{8}\left((1-i)-2\varepsilon+(4-2i)\,\varepsilon^2-\frac{20}{3}\varepsilon^3+\dots\right)\\
&\quad -\frac{w^2}{32}\left((1-i)-(1-i)\varepsilon+(5-7i)\varepsilon^2-\left(\frac{16}{3}-\frac{16i}{3}\right)\varepsilon^3+\dots\right)+\dots
\end{aligned}
$$

which gives

$$
\begin{aligned}
\delta_1(z)f_1(w) &= \frac{w}{8}\left(1+2i\varepsilon+\frac{8i}{3}\varepsilon^3+\dots\right)\\
&\quad +\frac{w^2}{32}\left(\left(1-\frac{3i}{2}\right)+(6+6i)\varepsilon-(12-12i)\varepsilon^2+\dots\right)+\dots\\
\delta_2(z)f_2(w) &= -\frac{w}{8}\left((1+i)-(4-2i)\varepsilon-(2+8i)\varepsilon^2+\dots\right)\\
&\quad -\frac{w^2}{32}\left((1+i)-(11-9i)\varepsilon-(33+39i)\varepsilon^2+\dots\right)+\dots\\
\delta_3(z)f_3(w) &= -1-\frac{w}{8}\left((1-2i)+(4+6i)\varepsilon-8\varepsilon^2+\dots\right)\\
&\quad -\frac{w^2}{32}\left(\left(2-\frac{5i}{2}\right)+(16+12i)\varepsilon-(24-34i)\varepsilon^2+\dots\right)+\dots\\
\delta_4(z)f_4(w) &= \frac{w}{8}\left((1-i)+6i\varepsilon-(10+8i)\varepsilon^2+\dots\right)\\
&\quad +\frac{w^2}{32}\left(2-(1-15i)\varepsilon-(45+11i)\varepsilon^2+\dots\right)+\dots
\end{aligned}
$$

so their sum is 1. Near λ_4 we have

$$
\begin{aligned}
\delta_1(z) &= \left((1+i)\varepsilon+\left(\frac{4}{3}+\frac{4i}{3}\right)\varepsilon^3+\dots\right)\\
&\quad +\frac{w}{8}\left((1-i)-2\varepsilon+(4-8i)\varepsilon^2-\frac{8}{3}\varepsilon^3\dots\right)\\
&\quad -\frac{w^2}{32}\left((1-i)+(2+i)\varepsilon+(8-4i)\varepsilon^2+\left(\frac{32}{3}+\frac{40i}{3}\right)\varepsilon^3+\dots\right)+\dots\\
\delta_2(z) &= \left(i\varepsilon+3\varepsilon^2-\frac{8i}{3}\varepsilon^3+\dots\right)+\frac{w}{8}\left(1-4i\varepsilon-\frac{40i}{3}\varepsilon^3+\dots\right)\\
&\quad -\frac{w^2}{32}\left(\left(1+\frac{i}{2}\right)+(1-i)\varepsilon+(7+4i)\varepsilon^2+\left(\frac{16}{3}-\frac{16i}{3}\right)\varepsilon^3+\dots\right)+...\\
\delta_3(z) &= \left(-(1-i)\varepsilon-\left(\frac{4}{3}-\frac{4i}{3}\right)\varepsilon^3+\dots\right)\\
&\quad +\frac{w}{8}\left((1+i)+2\varepsilon+(4+8i)\varepsilon^2+\frac{8}{3}\varepsilon^3+\dots\right)\\
&\quad -\frac{w^2}{32}\left(2-(4-i)\varepsilon-(4-8i)\varepsilon^2-\left(\frac{64}{3}+\frac{8i}{3}\right)\varepsilon^3+\dots\right)+\dots\\
\delta_4(z) &= \left(1-3i\varepsilon-3\varepsilon^2+\dots\right)-\frac{w}{8}\left(3-4i\varepsilon+8\varepsilon^2-\frac{40i}{3}\varepsilon^3+\dots\right)\\
&\quad +\frac{w^2}{32}\left(\left(2+\frac{3i}{2}\right)-(1-i)\varepsilon+(11+8i)\varepsilon^2-\left(\frac{16}{3}-\frac{16i}{3}\right)\varepsilon^3+\dots\right)+\dots
\end{aligned}
$$

This gives

$$\begin{aligned}
\delta_1(z)f_1(w) &= \left((1+i)\varepsilon + \left(\frac{4}{3}+\frac{4i}{3}\right)\varepsilon^3 + \dots\right) \\
&\quad +\frac{w}{8}\left((1-i)+(4+2i)\varepsilon+4i\varepsilon^2+\dots\right) \\
&\quad -\frac{w^2}{32}\left(2i-(13+2i)\varepsilon+(8-28i)\varepsilon^2+\dots\right)+\dots \\
\delta_2(z)f_2(w) &= -\left(i\varepsilon-(1+2i)\varepsilon^2+\left(2-\frac{2i}{3}\right)\varepsilon^3+\dots\right) \\
&\quad -\frac{w}{8}\left(1-(4-4i)\varepsilon-(10+14i)\varepsilon^2+\dots\right) \\
&\quad -\frac{w^2}{32}\left(\left(1+\frac{i}{2}\right)-(6-12i)\varepsilon-(49-26i)\varepsilon^2+\dots\right)+\dots \\
\delta_3(z)f_3(w) &= \left((1-i)\varepsilon+\left(\frac{4}{3}-\frac{4i}{3}\right)\varepsilon^3+\dots\right) \\
&\quad -\frac{w}{8}\left((1+i)+6i\varepsilon-(8-4i)\varepsilon^2+\dots\right) \\
&\quad -\frac{w^2}{32}\left((3-i)+(7+16i)\varepsilon-(28-16i)\varepsilon^2+\dots\right)+\dots \\
\delta_4(z)f_4(w) &= \left(1-(2-i)\varepsilon-(1+2i)\varepsilon^2-\left(\frac{2}{3}+\frac{2i}{3}\right)\varepsilon^3+\dots\right) \\
&\quad +\frac{w}{8}\left((1+2i)-(8-8i)\varepsilon-(18+14i)\varepsilon^2+\dots\right) \\
&\quad +\frac{w^2}{32}\left(\left(4+\frac{3i}{2}\right)-(12-26i)\varepsilon-(69+38i)\varepsilon^2+\dots\right)+\dots
\end{aligned}$$

so, near λ_4 their sum is 1.

C.4 Degree 2 to power n

As a final step we will compute the expansions for the Riesz projection for the polynomial $p^n = (z^2-1)^n$. As before we let $w = z^2-1$ so for $|w| < 1$ we have

$$z = (w+1)^{1/2} = 1 + \frac{1}{2}w - \frac{1}{8}w^2 + \frac{1}{16}w^3 + \cdots .$$

At the begining of this section we have computed the expansion for $p = z^2-1$, when taking $\varphi \equiv 1$ near $\lambda_1 = 1$ and $\varphi \equiv -1$ near $\lambda_2 = -1$, (λ_j, $j = 1, 2$, being the roots of p). Thus we have, $\delta_1(z) = (1+z)/2,\ \delta_2(z) = (1-z)/2$ and

$$\begin{aligned}
f_1(w) &= 1 - \frac{1}{2}w + \frac{3}{8}w^2 - \frac{5}{16}w^3 + \dots, \\
f_2(w) &= -1 + \frac{1}{2}w - \frac{3}{8}w^2 + \frac{5}{16}w^3 + \dots
\end{aligned}$$

from which we note that $f_2(w) = -f_1(w)$.

From section 2 we know that for $\alpha = 2\pi/n$ and a given function $g(w)$

$$w^k g_k(w^n) = \frac{1}{n}\{g(w) + e^{-ik\alpha}g(e^{i\alpha}w) + \cdots + e^{-i(n-1)k\alpha}g(e^{i(n-1)\alpha}w)\} \quad \text{(C.11)}$$

and from Theorem 2.1 we have

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z)[f_{j0}(p(z)^n) + \cdots + p(z)^{n-1} f_{jn-1}(p(z)^n)] \tag{C.12}$$

where d is the degree of p. Denoting

$$\begin{aligned} F_1(w) &= f_{10}(w^n) + \cdots + w^{n-1} f_{1n-1}(w^n), \\ F_2(w) &= f_{20}(w^n) + \cdots + w^{n-1} f_{2n-1}(w^n), \end{aligned}$$

since $p(z) = w$, we have $\varphi(z) = \delta_1(z)F_1(w) + \delta_2(z)F_2(w)$.

Since $f_2(w) = -f_1(w)$ we note that also $F_2(w) = -F_1(w)$. Thus it is enough to compute $F_1(w)$. We start by replacing $g(w)$ from (C.11) with $f_1(w)$ and we get

$$\begin{aligned} f_{10}(w^n) &= \frac{1}{n}\{f_1(w) + f_1(e^{i\alpha}w) + f_1(e^{2i\alpha}) + \ldots \\ &\quad + f_1(e^{i(n-1)\alpha}w)\} \\ f_{11}(w^n) &= \frac{1}{nw}\{f_1(w) + e^{-i\alpha} f_1(e^{i\alpha}w) + e^{-2i\alpha} f_1(e^{2i\alpha}w) + \ldots \\ &\quad + e^{-i(n-1)\alpha} f_1(e^{i(n-1)\alpha}w)\} \\ f_{12}(w^n) &= \frac{1}{nw^2}\{f_1(w) + e^{-2i\alpha} f_1(e^{i\alpha}w) + e^{-4i\alpha} f_1(e^{2i\alpha}w) + \ldots \\ &\quad + e^{-2i(n-1)\alpha} f_1(e^{i(n-1)\alpha}w)\} \\ &\ldots \\ f_{1n-1}(w^n) &= \frac{1}{nw^{n-1}}\{f_1(w) + e^{-i(n-1)\alpha} f_1(e^{i\alpha}w) + e^{-2i(n-1)\alpha} f_1(e^{2i\alpha}w) + \ldots \\ &\quad + e^{-i(n-1)^2\alpha} f_1(e^{i(n-1)\alpha}w)\}. \end{aligned}$$

Then

$$\begin{aligned} F_1(w) &= f_{10}(w^n) + w f_{11}(w^n) + w^2 f_{12}(w^n) + \cdots + w^{n-1} f_{1n-1}(w^n) \\ &= \frac{1}{n}\left(n f_1(w) + f_1(e^{i\alpha}w) \sum_{k=0}^{n-1} e^{-ik\alpha} + f_1(e^{2i\alpha}w) \sum_{k=0}^{n-1} e^{-2ki\alpha} \right. \\ &\quad \left. + f_1(e^{3i\alpha}w) \sum_{k=0}^{n-1} e^{-3ki\alpha} + \cdots + f_1(e^{i(n-1)\alpha}w) \sum_{k=0}^{n-1} e^{-ik(n-1)\alpha} \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(f_1(e^{ij\alpha}w) \sum_{k=0}^{n-1} e^{-ijk\alpha} \right) \end{aligned} \tag{C.13}$$

Now we must check that near λ_1, the quantity $\varphi(z) = \delta_1(z)F_1(w) + \delta_2(z)F_2(w)$ is identically 1 and near the other root is -1. Since $f_1(w) = z^{-1} = ((w+1)^{1/2})^{-1}$ it follows that $f_1(e^{ij\alpha}w) = (e^{ij\alpha}w + 1)^{-1/2}$ which has the expansion

$$f_1(e^{ij\alpha}w) = 1 - \frac{1}{2}e^{ij\alpha}w + \frac{3}{8}e^{2ij\alpha}w^2 - \frac{5}{16}e^{3ij\alpha}w^3 + \ldots$$

Replacing this in (C.13) we get

$$
\begin{aligned}
F_1(w) &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\left(1-\frac{1}{2}e^{ij\alpha}w+\frac{3}{8}e^{2ij\alpha}w^2-\frac{5}{16}e^{3ij\alpha}w^3+\dots\right)\sum_{k=0}^{n-1}e^{-ijk\alpha}\right)\\
&= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}e^{-ijk\alpha}-\frac{1}{2n}w\sum_{j=0}^{n-1}\left(e^{ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right)\\
&\quad +\frac{3}{8n}w^2\sum_{j=0}^{n-1}\left(e^{2ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right)-\frac{5}{16n}w^3\sum_{j=0}^{n-1}\left(e^{3ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right)+\dots\\
&= 1-\frac{1}{2}w+\frac{3}{8}w^2-\frac{5}{16}w^3+\dots
\end{aligned}
$$

since

$\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}e^{-ijk\alpha} = n,\ \sum_{j=0}^{n-1}\left(e^{ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right) = n,\ \sum_{j=0}^{n-1}\left(e^{2ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right) = n,$ $\sum_{j=0}^{n-1}\left(e^{3ij\alpha}\sum_{k=0}^{n-1}e^{-ijk\alpha}\right) = n$ and so on. Similarly we get $F_2(w) = f_2(w)$. Hence, near λ_1 we have

$$
\begin{aligned}
\delta_1(z)F_1(w) &= 1-\frac{1}{4}w+\frac{3}{16}w^2-\frac{5}{32}w^3+\dots,\\
\delta_2(z)F_2(w) &= \frac{1}{4}w-\frac{3}{16}w^2+\frac{5}{32}w^3+\dots
\end{aligned}
$$

so their sum is identically 1. Near -1 we have

$$
\begin{aligned}
\delta_1(z)F_1(w) &= -\frac{1}{4}w+\frac{3}{16}w^2-\frac{5}{32}w^3+\dots,\\
\delta_2(z)F_2(w) &= -1+\frac{1}{4}w-\frac{3}{16}w^2+\frac{5}{32}w^3+\dots.
\end{aligned}
$$

So, their sum is -1.

D Separating polynomials

Below, one can see how monic polynomials of degrees $6, 8, 10, 12$ and 14 get separated and how they look like when appling the perturbations.

For $p(z) = z^6 - 1$ we perturb only the four complex roots and we leave unchanged the other two real roots,

$$
\begin{aligned}
p(z) &= z^6-1 = (z^2-1)(z^4+z^2+1)\\
&= (z^2-1)(z-e^{i\theta})(z-e^{2i\theta})(z+e^{i\theta})(z+e^{2i\theta})\\
&= (z^2-1)(z^2-e^{2i\theta})(z^2-e^{4i\theta}) \qquad \text{(D.1)}
\end{aligned}
$$

where $\theta = \pi/3$. Let $\theta_\varepsilon = \theta - \varepsilon$, then $l : |p_\varepsilon(z)| = 1 - \eta$. Letting $\varepsilon = \pi/70$, we have

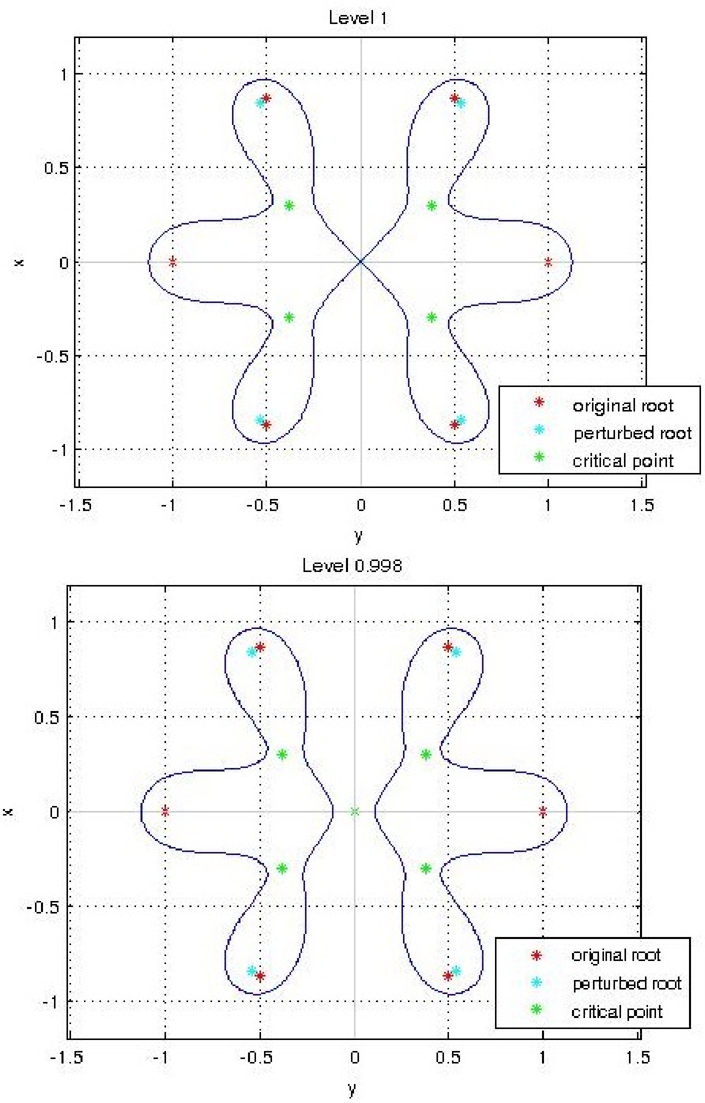


Figure 1: Separation of degree 6 polynomial

For $p(z) = z^8 - 1$ we first need to apply a rotation with $\pi/8$ so that no root lays on the imaginary axis. Thus our polynomial becomes $p(z) = z^8 + 1$ with roots $e^{i\pi/8}$, $e^{3i\pi/8}$, $e^{5i\pi/8}$, $e^{7i\pi/8}$, $e^{9i\pi/8}$, $e^{11i\pi/8}$, $e^{13i\pi/8}$ and $e^{15i\pi/8}$. We perturbed only the four roots closest to the imaginary axis with ε.

Therefore the perturbed polynomial is

$$\begin{aligned} p_\varepsilon(z) &= (z - e^{i\pi/8})(z - e^{i(3\pi/8-\varepsilon)})(z - e^{i(5\pi/8+\varepsilon)})(z - e^{7i\pi/8}) \\ &\quad (z - e^{9i\pi/8})(z - e^{i(11\pi/8-\varepsilon)})(z - e^{i(13\pi/8+\varepsilon)})(z - e^{15i\pi/8}) \end{aligned}$$

and for this one we compute the lemniscate and we plot it.

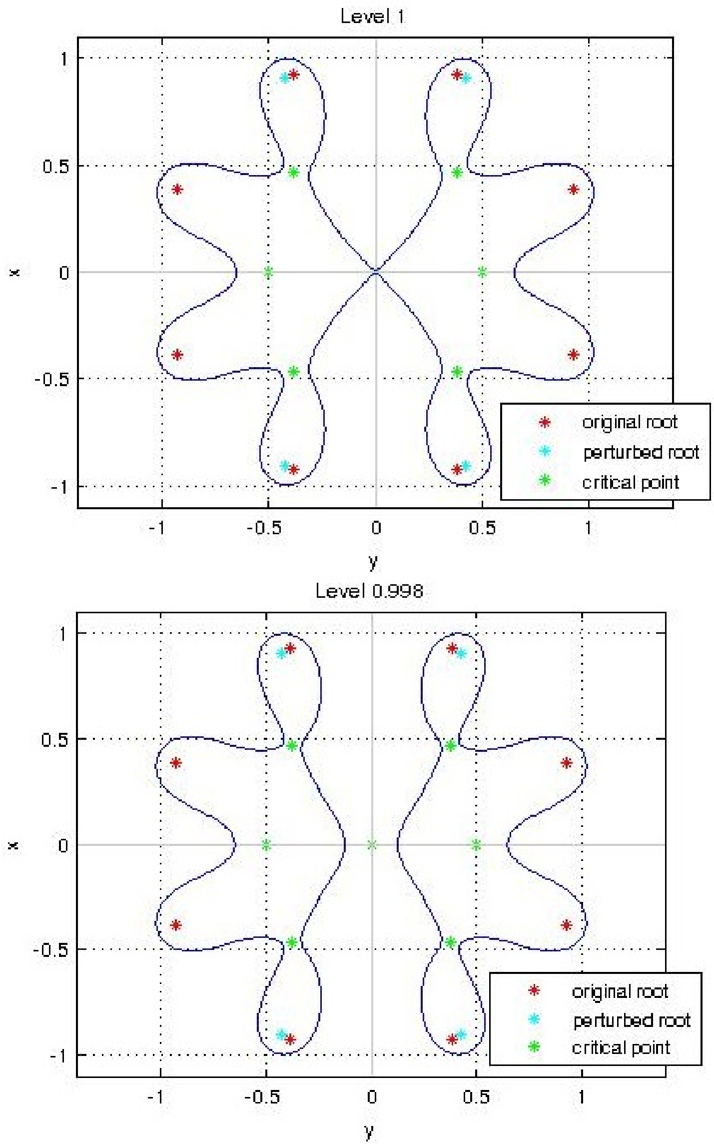


Figure 2: Separation of degree 8 polynomial

For $p(z) = z^{10} - 1$ with roots $1, -1, e^{i\pi/5}, e^{2i\pi/5}, e^{3i\pi/5}, e^{4i\pi/5}, e^{6i\pi/5}, e^{7i\pi/5}, e^{8i\pi/5}$ and $e^{9i\pi/5}$ we perturb the four roots that are closest to $i\mathbb{R}$ and we get

$$\begin{aligned} p_\varepsilon(z) \quad = \quad & (z^2-1)(z-e^{i\pi/5})(z-e^{i(2\pi/5-\varepsilon)})(z-e^{i(3\pi/5+\varepsilon)})(z-e^{4i\pi/5}) \\ & (z-e^{6i\pi/5})(z-e^{i(7\pi/5-\varepsilon)})(z-e^{i(8\pi/5+\varepsilon)})(z-e^{9i\pi/5}). \end{aligned}$$

For $p_\varepsilon(z)$ we compute the lemniscate and we plot it.

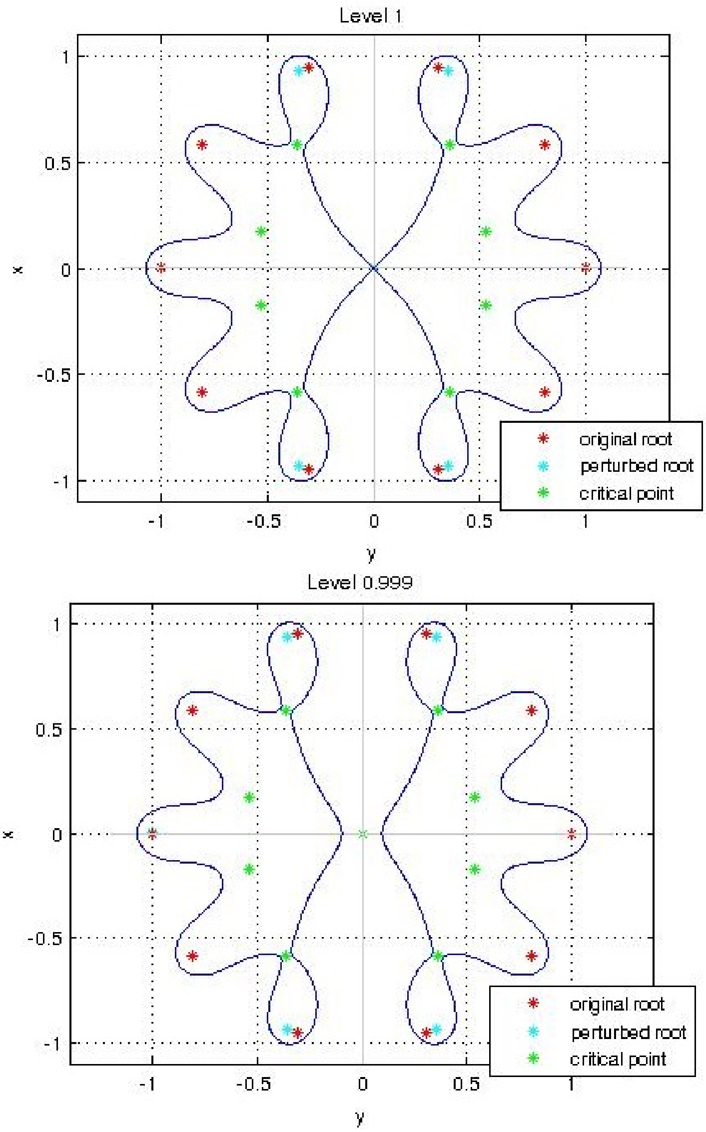


Figure 3: Separation of degree 10 polynomial

For $p(z) = z^{12} - 1$ we first apply a rotation with $\pi/12$ and we get a new polynomial $p(z) = z^{12} + 1$ to which we apply the perturbation with ε. In this case the picture will look like this:

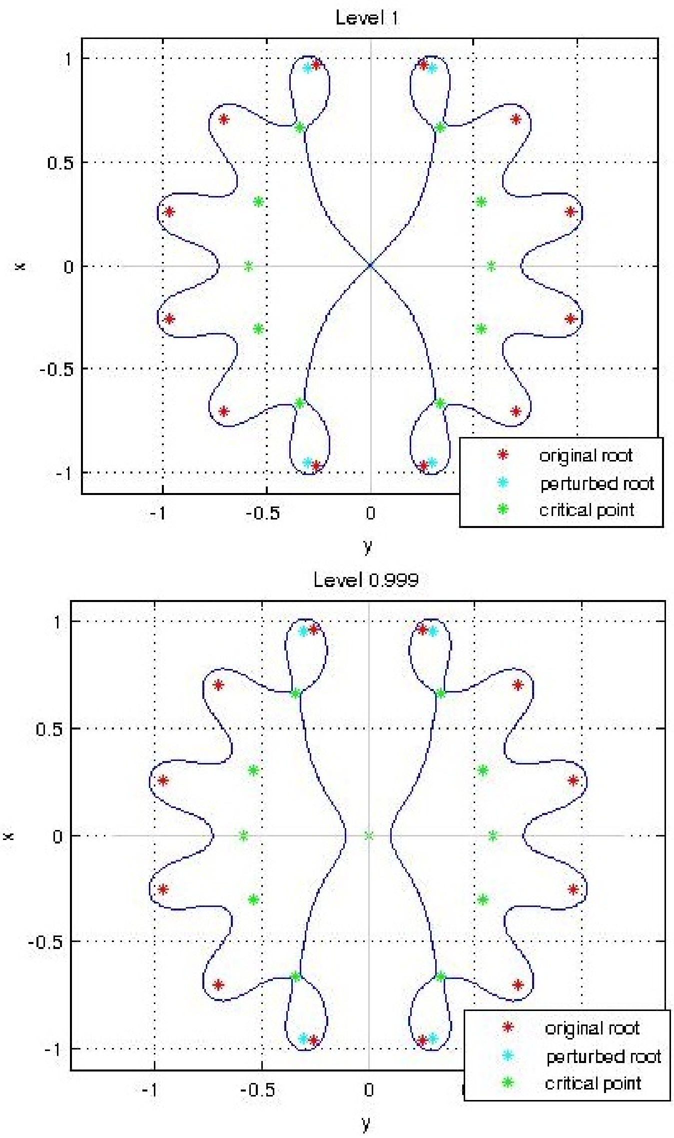


Figure 4: Separation of degree 12 polynomial

And the last case that we discuss here is $p(z) = z^{14} - 1$. The roots of this polynomial don't need any rotation, so we just change the 4 roots that are closest to $i\mathbb{R}$ with ε and then we compute and plot the lemniscate:

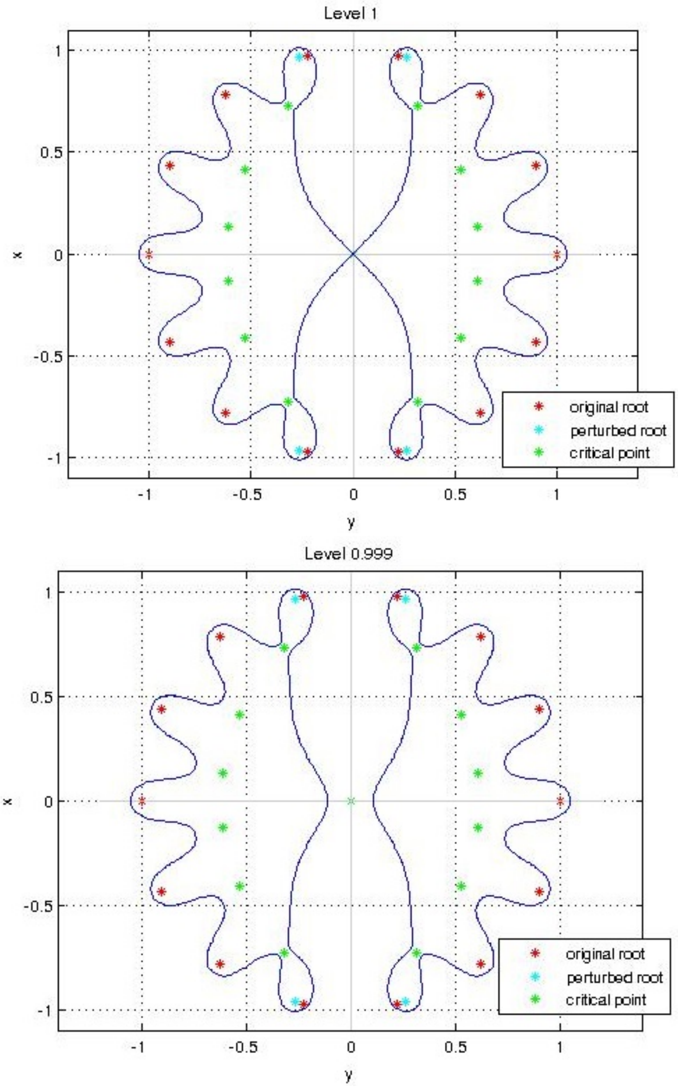
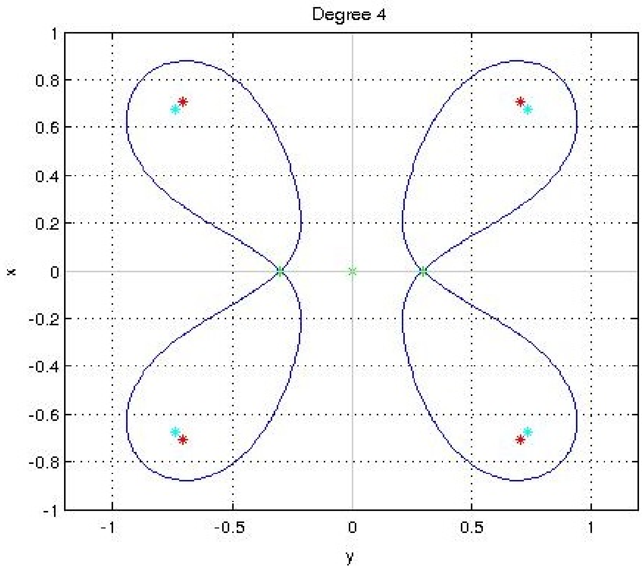
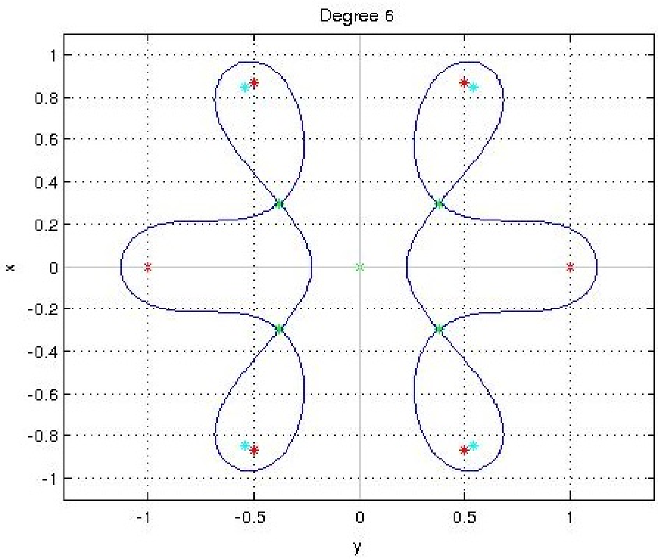


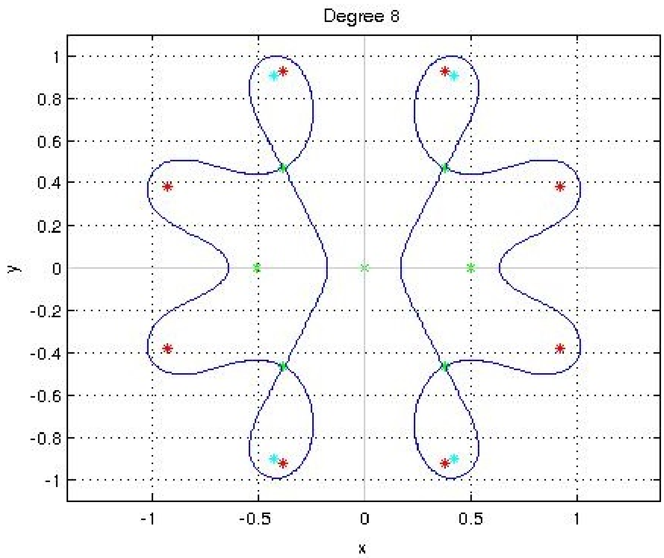
Figure 5: Separation of degree 14 polynomial

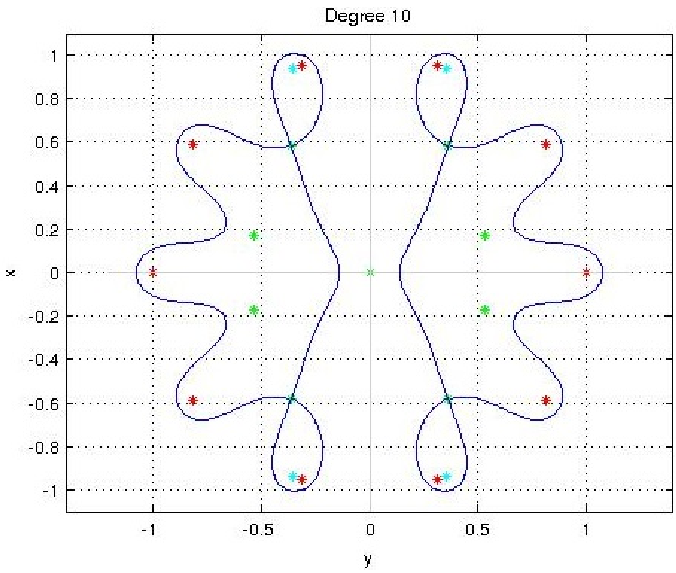
E Pictures with lowest level of the lemniscate that holds the separation

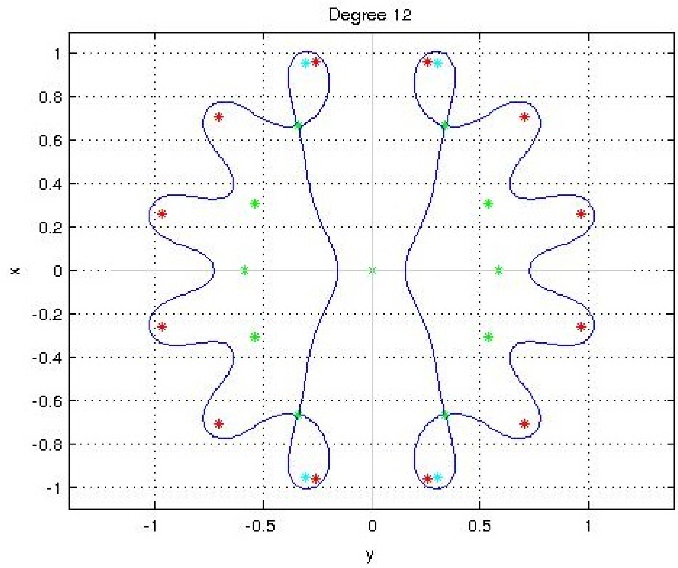
These pictures are what we have riched when perturbing the roots with $\varepsilon = \pi/70$ and decreasing the level to its lowest value that ensures the separation into two parts, each on one side of the imaginary axis :

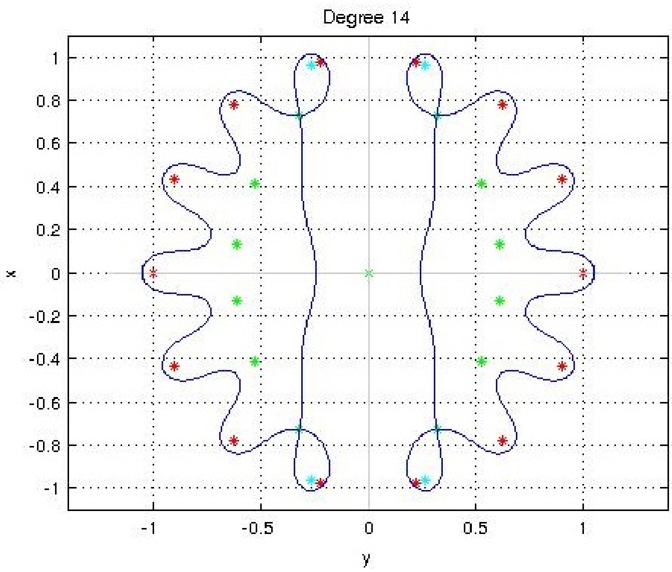

Degree 4
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x
-1
-0.5
0
0.5
1
y


Degree 6
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x
-1
-0.5
0
0.5
1
y


Degree 8
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Degree 10
x
y
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


Degree 12
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x
-1
-0.5
0
0.5
1
y


Degree 14
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x
-1
-0.5
0
0.5
1
y