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EXISTENCE AND APPROXIMATION OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The paper deals with existence, uniqueness and spline approximation of solutions to boundary value problems for delay integro-differential equations. An iterative approximation scheme based on the use of cubic splines with defect two is presented, and sufficient conditions for its convergence are obtained.

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1. INTRODUCTION

Dynamic processes in many applied problems are described by delay differential and integral equations (Andreeva, Kolmanovsky and Shayhet 1992). An analytical solution of such equation exists only in the simplest cases, so the construction and study of approximate algorithms for solutions of these equations are important.

In the present note we study an approximate method of solving boundary value problems for delay integro-differential equations based on approximation of the solution by cubic splines with defect two.

Existence and uniqueness of a solution of delay boundary value problems in various function spaces were considered by Grim and Schmitt (1968), Kamensky and Myshkis (1972), Biga and Gaber (2007), Athanasiadou (2013). Applying spline functions for solving differential-difference equations was investigated by Nikolova and Bainov (1981), Cherevko and Yakimov (1989), Nastasyeva and Cherevko (1999).

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2. PROBLEM STATEMENT. EXISTENCE OF A SOLUTION

Let us consider the following boundary value problem

$$(1)\qquad y''(x)=f\left(x,y(x),y(x-\tau_0(x)),y'(x),y'(x-\tau_1(x))\right)+ \\ +\int_a^b g\left(x,s,y(s),y(s-\tau_0(s)),y'(s),y'(s-\tau_1(s))\right)ds,$$

$$(2)\qquad y^{(i)}(x)=\varphi^{(i)}(x),\ i=0,1,\ x\in[a^*;a],\ y(b)=\gamma,$$

where

$$a^*=\min\left\{\inf_{x\in[a;b]}(x-\tau_0(x)),\ \inf_{x\in[a;b]}(x-\tau_1(x))\right\},\ \gamma\in R,\ \tau_0(x)\geq 0,\ \tau_1(x)\geq 0.$$

Let $f(x,u_0,u_1,v_0,v_1)$, $g(x,s,u_0,u_1,v_0,v_1)$ be continuous functions in $G=[a,b]\times G_1^2\times G_2^2$ and $Q=[a,b]\times G$, where $G_1=\{u\ \in R:\ |u|<P_1\}$, $G_2=\{v\ \in R:\ |v|\leq P_2\}$, P_1,P_2 are positive constants, $\varphi(t)\in C^1[a^*;a]$, delays $\tau_0(x)$ and $\tau_1(x)$ are continuous functions on $[a,b]$, and additionally, $\tau_1(x)$ is such that the set $E=\{x_i\in[a,b]:\ x_i-\tau_1(x_i)=a,\ i=\overline{1,k}\}$ is finite.

We introduce the notations:

$$P=\sup\left\{|f(x,u,u_1,v,v_1)|+\left|\int_a^b g(x,s,u,u_1,v,v_1)ds\right|:\right. \\ \left.|u_i|<P_1,|v_i|<P_2,\ i=0,1,\ x,s\in[a,b]\right\},$$

$$J=[a^*;a],I=[a,b],\ I_1=[a,x_1],\ I_2=[x_1,x_2],\ \ldots,$$

$$I_k=[x_{k-1},x_k],\ I_{k+1}=[x_k,b],$$

$$B(J\cup I)=\left\{y(x):y(x)\in\left(C(J\cup I)\cap(C^1(J)\cup C^1(I))\cap\left(\bigcup_{j=1}^{k+1}C^2(I_j)\right)\right),\right. \\ \left.|y(x)|\leq P_1,\ |y'(x)|\leq P_2\right\}.$$

A function $y=y(x)$ from the space $B(J\cup I)$ is called a solution of the problem (1)-(2) if it satisfies the equation (1) on $[a;b]$ (with the possible exception of the set E) and boundary conditions (2).

From the definition of the space $B(J\cup I)$ we conclude that the solution of the problem (1)-(2) is continuously differentiable for each $x\in[a,b]$ where $y'(a)$ is the right derivative.

Let us introduce a norm in the space $B(J\cup I)$:

$$\|y\|_B=\max\left\{\tfrac{8}{(b-a)^2}\max_{x\in J\cup I}|y(x)|,\tfrac{2}{b-a}\max\left\{\max_{x\in J}|y'(x)|,\max_{x\in I}|y'(x)|\right\}\right\}.$$

The space $B(J\cup I)$ with this norm is a Banach space.

The boundary value problem (1)-(2) is equivalent to the following integral equation (Grim and Schmitt 1968; Kamensky and Myshkis 1972)

(3)

$$y(x) = \int_{a^*}^{b} \left[f\left(s, y(s), y(s-\tau_0(s)), y'(s), y'(s-\tau_1(s))\right) + \int_a^b g\left(s,\xi, y(\xi), y(\xi-\tau_0(\xi)), y'(\xi), y'(\xi-\tau_1(\xi))\right) d\xi \right] \bar{G}(x,s)\, ds + l(x), \qquad x \in J \cup I,$$

where

$$\bar{G}(x,s) = \begin{cases} G(x,s), & x,s \in I, \\ 0, & \text{otherwise}, \end{cases} \qquad l(x) = \begin{cases} \varphi(x), & x \in J, \\ \frac{\gamma-\varphi(a)}{b-a}(x-a) + \varphi(a), & x \in I, \end{cases}$$

and $G(x,s)$ is the Green's function of the following boundary value problem

$$y''(x) = 0, \quad x \in I, y(a) = y(b) = 0.$$

We define an operator T in the space $B(J \cup I)$ in the following way

$$(Ty)(x) = \int_{a^*}^{b} \left[f\left(s, y(s), y(s-\tau_0(s)), y'(s), y'(s-\tau_1(s))\right) + \int_a^b g\left(s,\xi, y(\xi), y(\xi-\tau_0(\xi)), y'(\xi), y'(\xi-\tau_1(\xi))\right) d\xi \right] \bar{G}(x,s)\, ds + l(x), \quad x \in J \cup I.$$

Hence,

(4)

$$(Ty)'(x) = \int_{a^*}^{b} \left[f\left(s, y(s), y(s-\tau_0(s)), y'(s), y'(s-\tau_1(s))\right) + \int_a^b g\left(s,\xi, y(\xi), y(\xi-\tau_0(\xi)), y'(\xi), y'(\xi-\tau_1(\xi))\right) d\xi \right] \bar{G}'_x(x,s)\, ds + \frac{\gamma-\varphi(a)}{b-a}, \quad x \in J \cup I.$$

Theorem 1. *Let the following conditions hold:*

1) $\max\left\{ \max\limits_{x \in J} |\varphi(x)|, \ \frac{(b-a)^2}{8} P + \max(|\varphi(a)|, |\gamma|) \right\} \le P_1,$
2) $\max\left\{ \max\limits_{x \in J} |\varphi'(x)|, \ \frac{b-a}{2} P + \left| \frac{\gamma-\varphi(a)}{b-a} \right| \right\} \le P_2,$
3) *the functions* $f(x,u_0,u_1,v_0,v_1)$, $g(x,s,u_0,u_1,v_0,v_1)$ *satisfy the Lipschitz condition for variables* $u_i, v_i, i = \overline{0,1}$ *with constants* L_j^1, L_j^2, $j = \overline{1,4}$ *in* G *and* Q,

4) $\frac{(b-a)^2}{8}\sum\limits_{j=1}^{2}\left(L_j^1+(b-a)\,L_j^2\right)+\frac{b-a}{2}\sum\limits_{j=3}^{4}\left(L_j^1+(b-a)\,L_j^2\right)<1.$

Then there exists a unique solution of the problem (1)-(2) *in* $B\,(J\cup I)$.

Proof. Based on Green's function (Hartman 2002)

$$G\,(t,s)=\begin{cases}\frac{(s-a)(t-b)}{b-a}, & a\le s\le t\le b,\\ \frac{(t-a)(s-b)}{b-a}, & a\le t\le s\le b,\end{cases}$$

we obtain the following estimates

$$\int_a^b |G\,(t,s)|\,ds\le\frac{(b-a)^2}{8},\qquad \int_a^b G_t'\,(t,s)\,ds\le\frac{b-a}{2}. \tag{5}$$

When the conditions 1)-2) and the inequalities (5) are true the operator T maps the space $B\,(J\cup I)$ on itself.

Let $y_1,y_2\in B\,(J\cup I)$. Considering the condition 3) and the estimates (5), we get:

$$\begin{aligned}
&|(Ty_1)\,(t)-(Ty_2)\,(t)|\le\\
&\le\int_{a^*}^b\Bigg[\left(L_1^1+L_2^1\right)\max_{t\,\in\,J\cup I}|y_1\,(t)-y_2\,(t)|\\
&\quad+\left(L_3^1+L_4^1\right)\max\left\{\max_{t\,\in\,I}\left|y_1'\,(t)-y_2'\,(t)\right|,\ \max_{t\,\in\,J}\left|y_1'\,(t)-y_2'\,(t)\right|\right\}\\
&\quad+(b-a)\left(L_1^2+L_2^2\right)\max_{t\,\in\,J\cup I}|y_1\,(t)-y_2\,(t)|\\
&\quad+(b-a)\left(L_3^2+L_4^2\right)\max\left\{\max_{t\,\in\,I}\left|y_1'\,(t)-y_2'\,(t)\right|,\max_{t\,\in\,J}\left|y_1'\,(t)-y_2'\,(t)\right|\right\}\Bigg]\bar{G}\,(t,s)\,ds\\
&\le\frac{(b-a)^2}{8}\Bigg[\frac{(b-a)^2}{8}\left(L_1^1+L_2^1+(b-a)\left(L_1^2+L_2^2\right)\right)\\
&\quad+\frac{b-a}{2}\left(L_3^1+L_4^1+(b-a)\left(L_3^2+L_4^2\right)\right)\Bigg]\|y_1-y_2\|_B,\\
&\left|(Ty_1)'\,(t)-(Ty_2)'\,(t)\right|\le\\
&\le\frac{b-a}{2}\Bigg[\frac{(b-a)^2}{8}\left(L_1^1+L_2^1+(b-a)\left(L_1^2+L_2^2\right)\right)\\
&\quad+\frac{b-a}{2}\left(L_3^1+L_4^1+(b-a)\left(L_3^2+L_4^2\right)\right)\Bigg]\|y_1-y_2\|_B.
\end{aligned}$$

Based on the obtained estimates and on the definition of the norm in the space $B\,(J \cup I)$ we have:

(6)

$$\begin{aligned}&\left\|(Ty_1)\,(t) - (Ty_2)\,(t)\right\|_B \le \\ &\le \left[\tfrac{(b-a)^2}{8}\sum_{i=1}^{2}\left(L_i^1 + (b-a)\,L_i^2\right) + \tfrac{b-a}{2}\sum_{i=3}^{4}\left(L_i^1 + (b-a)\,L_i^2\right)\right]\left\|y_1 - y_2\right\|_B.\end{aligned}$$

The inequality (6) and the condition 4) imply that the operator T is a contraction in $B\,(J \cup I)$ and it has a single fixed point in this space, therefore the boundary value problem (1)-(2) has a unique solution $y\,(t) \in B\,(J \cup I)$. The proof is complete. □

3. CUBIC SPLINES WITH DEFECT TWO

Let us consider an irregular grid $\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$ on the segment $[a;b]$, $E \subset \Delta$. We denote by $S\,(y,x)$ an interpolating cubic spline with defect two on Δ which belongs to the space $B\,(J \cup I)$.

We can obtain a formula of $S\,(y,x)$ (Nikolova and Bainov 1981; Nastasyeva and Cherevko 1999; Dorosh and Cherevko 2014):

$$\begin{aligned}S\,(y,x) =& M_{j-1}^{+}\tfrac{(x_j-x)^3}{6h_j} + M_j^{-}\tfrac{(x-x_{j-1})^3}{6h_j} + \left(y_{j-1} - \tfrac{M_{j-1}^{+}h_j^2}{6}\right)\tfrac{x_j-x}{h_j} \\ &+ \left(y_j - \tfrac{M_j^{-}h_j^2}{6}\right)\tfrac{x-x_{j-1}}{h_j},\ x \in [x_{j-1};x_j]\,,\ h_j = x_j - x_{j-1},\ j = \overline{1,n},\end{aligned} \tag{7}$$

where $M_j^{+} = S''\,(y, x_j+0)\,,\ j = \overline{0,n-1},\ M_j^{-} = S''\,(y,x_j-0)\,,\ j = \overline{1,n}$ satisfy the following system of equations

$$\begin{aligned}&h_{j+1}y_{j-1} - (h_j + h_{j+1})\,y_j + h_j y_{j+1} = \\ &= \tfrac{h_j h_{j+1}}{6}\left(h_j M_{j-1}^{+} + 2h_j M_j^{-} + 2h_{j+1}M_j^{+} + h_{j+1}M_{j+1}^{-}\right),\ j = \overline{1,n-1}, \\ &y_0 = \varphi\,(a)\,, y_n = \gamma.\end{aligned} \tag{8}$$

We shall present the equations (8) in a matrix form

$$Ay = BM + d, \tag{9}$$

where

$$A = \begin{pmatrix} -(h_1+h_2) & h_1 & 0 & 0 & \cdots & 0 \\ h_3 & -(h_2+h_3) & h_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & h_n & -(h_{n-1}+h_n) \end{pmatrix}$$

is an $(n-1)\times(n-1)$ matrix, $d = (-h_2 y_0,\ 0,\ \ldots,\ 0,\ -h_{n-1}y_n)^T$, B is a right side of the relations (8) coefficient matrix with dimensions$(n-1)\times 2n$,

$$M = \left(M_0^{+},\ M_1^{-}, M_1^{+},\ M_2^{-},\ M_2^{+},\ \ldots,\ M_{n-1}^{-},\ M_{n-1}^{+},\ M_n^{-}\right)^T.$$

LEMMA. *The following correlations are true:*

(10) 1) $\det(A) = (-1)^{n-1} h_2 h_3 \dots h_{n-1}(b-a)$,

(11) 2)$\|A^{-1}\| \le \frac{K^2}{8h^3}(b-a)$,

(12) 3)$\max\limits_{1\le i<n-2} \sum\limits_{j=1}^{n-1} \left(a_{i+1,j}^{-1} - a_{i,j}^{-1}\right) \le \frac{K^2(b-a)}{2h^2}$,

(13) 4)$\|B\| \le H^3$,

where $h = \min\limits_i h_i$, $H = \max\limits_i h_i$, $K = \frac{H}{h}$ *and* a_{ij}^{-1} *are elements of a matrix*A^{-1}.

The proof of the lemma statements is easy to obtain by applying the principle of mathematical induction and using the structure of the matrices A, B.

4. COMPUTATIONAL SCHEME

A) Choose a cubic spline $S\left(y^{(0)}, x\right)$ randomly so that the boundary conditions (2) are enforced, for instance, $S\left(y^{(0)}, x\right) = \frac{\gamma-\varphi(a)}{b-a}(x-a)+\varphi(a)$.

B) Using the original equation (1) and the spline $S\left(y^{(k)}, x\right)$, find

$$(14)\quad M_j^{+(k+1)} = f\Big(x_j, S\left(y^{(k)}, x_j+0\right), S\left(y^{(k)}, x_j-\tau_0(x_j)+0\right), S'\left(y^{(k)}, x_j+0\right), S'\left(y^{(k)}, x_j-\tau_1(x_j)+0\right)\Big)+ + \int_a^b g\Big(x_j, s, S\left(y^{(k)}, s\right), S\left(y^{(k)}, s-\tau_0(s)\right), S'\left(y^{(k)}, s\right), S'\left(y^{(k)}, s-\tau_1(s)\right)\Big)ds,\ j=\overline{0, n-1},$$

$$(15)\quad M_j^{-(k+1)} = f\Big(x_j, S\left(y^{(k)}, x_j-0\right), S\left(y^{(k)}, x_j-\tau_0(x_j)-0\right), S'\left(y^{(k)}, x_j-0\right), S'\left(y^{(k)}, x_j-\tau_1(x_j)-0\right)\Big)+ + \int_a^b g\Big(x_j, s, S\left(y^{(k)}, s\right), S\left(y^{(k)}, s-\tau_0(s)\right), S'\left(y^{(k)}, s\right), S'\left(y^{(k)}, s-\tau_1(s)\right)\Big)ds,\ j=\overline{1, n}.$$

In (14), (15) put $S^{(p)}\left(y^{(k)}, t\right) = \varphi^{(p)}(t)$, $p = 0, 1$ if $t < a$.

C) Compute $\left\{y_j^{k+1}\right\}$, $j = \overline{0, n}$ from the equations (8).

D) Build a cubic spline $S\left(y^{(k+1)},x\right)$ according to (7), using the values of $\left\{y_j^{k+1}\right\}$, $M_j^{+(k+1)}$, $M_j^{-(k+1)}$. This spline will be the next approximation.

Let us denote

$$\lambda_1 = L_1^1 + L_2^1 + (b-a)\left(L_1^2 + L_2^2\right),\ \lambda_2 = L_3^1 + L_4^1 + (b-a)\left(L_3^2 + L_4^2\right),$$

$$u = \tfrac{K^5}{8}(b-a)^2 + \tfrac{H^2}{8},\ v = \tfrac{K^5}{2}(b-a) + \tfrac{2H}{3}. \tag{16}$$

Theorem 2. *Assume that the conditions of Theorem* 1 *hold. If the following inequality is true*

$$\theta = u\lambda_1 + v\lambda_2 < 1, \tag{17}$$

then there exists $H^* > 0$ *such that for each* $0 < H < H^*$ *the sequence of splines* $\left\{S\left(y^{(k)},x\right)\right\}$, $k = 0,1\ldots$ *converges uniformly on* $[a;b]$.

Proof. The equation (10) implies that it is possible to construct an iterative spline sequence $S\left(y^{(k)},x\right)$, $k = 0,1,\ldots$ using the scheme A)-D). We shall demonstrate that the series

$$S^{(p)}\left(y^{(0)},x\right) + \sum_{i=1}^{\infty}\left[S^{(p)}\left(y^{(i)},x\right) - S^{(p)}\left(y^{(i-1)},x\right)\right],\ p = 0,1$$

are uniformly convergent on $[a;b]$ and thus the sequences $S^{(p)}\left(y^{(k)},x\right)$, $k = 0,1,\ldots,\ p = 0,1$ are also uniformly convergent.

Let us define scalar functions $y(x)$, $M(x)$ on $[a;b]$ and denote the following vectors

$$\bar{y} = \left(y(x_1),\ldots,y(x_{n-1})\right)^T,$$

$$\bar{M} = \Big(M(x_0+0), M(x_1-0), M(x_1+0),\ \ldots, M(x_{n-1}-0),$$

$$M(x_{n-1}+0), M(x_n-0)\Big)^T.$$

We shall write the iterative algorithm A)-D) in a matrix form

$$\bar{y}^{(k+1)} = A^{-1}B\bar{M}^{k+1} + A^{-1}d, \tag{18}$$

where the vector $\bar{M}$ components are defined according to (14)-(15) and the constant vector d depends only on the boundary conditions (2).

From (18) we obtain the estimate

$$\begin{aligned}\left\|y^{(k+1)} - y^{(k)}\right\| &= \left\|A^{-1}BM^{k+1} - A^{-1}BM^k\right\| \\ &\le \left\|A^{-1}\right\|\|B\|\left\|\bar{M}^{k+1} - \bar{M}^k\right\|.\end{aligned} \tag{19}$$

From (14)-(15) and the properties of the functions f and g we obtain the following inequalities

(20)
$$\left\|M_j^{+(k+1)} - M_j^{+(k)}\right\| \le \lambda_1 \max_{x\in[a;b]} \left|S\left(y^{(k)},x\right) - S\left(y^{(k-1)},x\right)\right| + \lambda_2 \max_{x\in[a;b]}\left|S'\left(y^{(k)},x\right) - S'\left(y^{(k-1)},x\right)\right|, j = 0,1,\dots,n-1,$$
$$\left\|M_j^{-(k+1)} - M_j^{-(k)}\right\| \le \lambda_1 \max_{x\in[a;b]} \left|S\left(y^{(k)},x\right) - S\left(y^{(k-1)},x\right)\right| + \lambda_2 \max_{x\in[a;b]}\left|S'\left(y^{(k)},x\right) - S'\left(y^{(k-1)},x\right)\right|, j = 1,2,\dots,n.$$

Therefore, taking into account the above mentioned lemma, (19) can be written in the following way

$$\left\|y^{(k+1)} - y^{(k)}\right\| \le \tfrac{K^5}{8}(b-a)\left[\lambda_1\left\|S\left(y^{(k)},x\right) - S\left(y^{(k-1)},x\right)\right\| + \lambda_2\left\|S'\left(y^{(k)},x\right) - S'\left(y^{(k-1)},x\right)\right\|\right]. \tag{21}$$

Let $x \in [x_{j-1}; x_j]$. Considering (7), we have

$$\begin{aligned}\left|S\left(y^{(k+1)},x\right) - S\left(y^{(k)},x\right)\right| \le& \left|\tfrac{x_j-x}{6h_j}\right|\left((x_j-x)^2 - h_j^2\right) + \\ &+ \tfrac{x-x_{j-1}}{6h_j}\left((x-x_{j-1})^2 - h_j^2\right)\left\|\bar{M}^{k+1} - \bar{M}^k\right\| + \\ &+ \left|y_{j-1}^{k+1} - y_j^k\right|\left|\tfrac{x_j-x}{h_j}\right| + \left|y_j^{k+1} - y_j^k\right|\left|\tfrac{x-x_{j-1}}{h_j}\right|.\end{aligned} \tag{22}$$

It is easy to show that

$$\max_{x\in[x_{j-1};x_j]}\left|\tfrac{x_j-x}{6h_j}\left(h_j^2 - (x_j-x)^2\right) + \tfrac{x-x_{j-1}}{6h_j}\left(h_j^2 - (x-x_{j-1})^2\right)\right| \le \tfrac{H^2}{8}. \tag{23}$$

Using (20), (21), (23), from (22) we obtain

$$\begin{aligned}\left\|S\left(y^{(k+1)},x\right) - S\left(y^{(k)},x\right)\right\| &\le \tfrac{H^2}{8}\left\|\bar{M}^{k+1} - \bar{M}^k\right\| + \left\|y^{(k+1)} - y^{(k)}\right\| \\ &\le \left(\tfrac{K^5}{8}(b-a)^2 + \tfrac{H^2}{8}\right)\left(\lambda_1\left\|S\left(y^{(k)},x\right) - S\left(y^{(k-1)},x\right)\right\| + \right. \\ &\left. + \lambda_2\left\|S'\left(y^{(k)},x\right) - S'\left(y^{(k-1)},x\right)\right\|\right).\end{aligned} \tag{24}$$

According to the spline (7), we get

(25)
$$\begin{aligned}&\left|S'\left(y^{(k+1)},x\right)-S'\left(y^{(k)},x\right)\right|\le\\&\le\left|\frac{h_j}{6}-\frac{(x_j-x)^2}{2h_j}\right|\left|M_{j-1}^{+(k+1)}-M_{j-1}^{+(k)}\right|\\&\quad+\left|\frac{(x-x_{j-1})^2}{2h_j}-\frac{h_j}{6}\right|\left|M_j^{-(k+1)}-M_j^{-(k)}\right|+\frac{1}{h_j}\left|y_j^{k+1}-y_{j-1}^{k+1}-\left(y_j^k-y_{j-1}^k\right)\right|.\end{aligned}$$

One can show that

$$\max_{x\in[x_{j-1};x_j]}\left(\left|\frac{h_j}{6}-\frac{(x_j-x)^2}{2h_j}\right|+\left|\frac{(x-x_{j-1})^2}{2h_j}-\frac{h_j}{6}\right|\right)\le\frac{2H}{3},\tag{26}$$

$$\max_{1<j<n}\left|y_j^{k+1}-y_{j-1}^{k+1}-\left(y_j^k-y_{j-1}^k\right)\right|\le\frac{K^4}{2}(b-a)H\left\|\bar{M}^{k+1}-\bar{M}^k\right\|.\tag{27}$$

Due to (26)-(27), the inequality (25) implies that

$$\begin{aligned}&\left\|S'\left(y^{(k+1)},x\right)-S'\left(y^{(k)},x\right)\right\|\le\\&\le\left(\frac{K^5}{2}(b-a)+\frac{2}{3}H\right)\left(\lambda_1\left\|S\left(y^{(k)},x\right)-S\left(y^{(k-1)},x\right)\right\|\right.\\&\quad\left.+\lambda_2\left\|S'\left(y^{(k)},x\right)-S'\left(y^{(k-1)},x\right)\right\|\right).\end{aligned}\tag{28}$$

After iterating (24), (28) and considering the notations (16)-(17) we obtain

(29)
$$\begin{aligned}&\left\|S\left(y^{(k+1)},x\right)-S\left(y^{(k)},x\right)\right\|\le\\&\le u\theta^{k-1}\left(\lambda_1\left\|S\left(y^{(1)},x\right)-S\left(y^{(0)},x\right)\right\|+\lambda_2\left\|S'\left(y^{(1)},x\right)-S'\left(y^{(0)},x\right)\right\|\right),\\&\left\|S'\left(y^{(k+1)},x\right)-S'\left(y^{(k)},x\right)\right\|\le\\&\le v\theta^{k-1}\left(\lambda_1\left\|S\left(y^{(1)},x\right)-S\left(y^{(0)},x\right)\right\|+\lambda_2\left\|S'\left(y^{(1)},x\right)-S'\left(y^{(0)},x\right)\right\|\right).\end{aligned}$$

The correlations (29) with the condition (17) ensure the convergence of the sequences $\left\{S^{(p)}\left(y^{(k)},x\right)\right\}$, $k=0,1,\ldots$, $p=0,1$. Theorem 2 is proved. □

Let us denote $\lim\limits_{k\to\infty}S^{(p)}\left(y^{(k)},x\right)=S^{(p)}(\tilde{y},x)$, $p=0,1,\ldots$ Note that the parameters $\widetilde{M}_j^+$, $\widetilde{M}_j^-$ of the spline $S(\tilde{y},x)$ satisfy the system (8) and equations (14)-(15).

Let $S(y,x)$ be a cubic spline with defect 2 which interpolates the solution $y(x)$ of the boundary value problem (1)-(2). Thus,

$$\begin{aligned}\left\|S^{(p)}(\tilde{y},x)-y^{(p)}(x)\right\|&\le\left\|S^{(p)}(\tilde{y},x)-S^{(p)}(y,x)\right\|\\&\quad+\left\|S^{(p)}(y,x)-y^{(p)}(x)\right\|,\quad p=0,1.\end{aligned}\tag{30}$$

For the second term on the right side of the inequality (30) it is true (Alberg, Nilson and Walsh 1967) that

$$\left\| S^{(p)}(y,x) - y^{(p)}(x) \right\| \le K_p H^{2-p} \omega\left(y''(x), H\right), \tag{31}$$
$$p = 0,1,2,\ K_0 = \tfrac{5}{2},\ K_1 = K_2 = 5,$$

where $\omega\left(y''(x), H\right) = \max\limits_{1 \le r \le k+1} \omega_r\left(y''(x), H\right)$, $\omega_r\left(y''(x), H\right)$ is a modulus of continuity for $y''(x)$ on $I_r = [x_{r-1}; x_r]$.

We shall denote

$$\max_{x \in [a;b]} \left| S^{(p)}(\tilde{y}, x) - S^{(p)}(y, x) \right| = \alpha_p, \quad p = 0, 1.$$

According to the properties of the functions f, g and estimates (31), we obtain

$$\begin{aligned} &\Big| M_j^{+} - f\Big(x_j, y(x_j), y(x_j - \tau_0(x_j)), y'(x_j), y'(x_j - \tau_1(x_j))\Big) \\ &\quad - \int_a^b g\Big(x_j, s, y(s), y(s - \tau_0(s)), y'(s), y'(s - \tau_1(s))\Big) ds \Big| \\ &\le 5\left(1 + \tfrac{1}{2}\lambda_1 H^2 + \lambda_2 H\right) \omega\left(y''(x), H\right), \end{aligned} \tag{32}$$

$$\begin{aligned} &\Big| M_j^{-} - f\Big(x_j, y(x_j), y(x_j - \tau_0(x_j)), y'(x_j), y'(x_j - \tau_1(x_j))\Big) \\ &\quad - \int_a^b g\Big(x_j, s, y(s), y(s - \tau_0(s)), y'(s), y'(s - \tau_1(s))\Big) ds \Big| \\ &\le 5\left(1 + \tfrac{1}{2}\lambda_1 H^2 + \lambda_2 H\right) \omega\left(y''(x), H\right). \end{aligned} \tag{33}$$

Using the formulas of $S(\tilde{y}, x)$, $S(y, x)$ and the inequalities (32)-(33), one can get the following system of inequalities

$$\begin{aligned} \alpha_0 &\le u\left(\alpha_0\lambda_1 + \alpha_1\lambda_2 + 5\left(1 + \tfrac{1}{2}\lambda_1 H^2 + \lambda_2 H\right)\omega\left(y''(x), H\right)\right), \\ \alpha_1 &\le v\left(\alpha_0\lambda_1 + \alpha_1\lambda_2 + 5\left(1 + \tfrac{1}{2}\lambda_1 H^2 + \lambda_2 H\right)\omega\left(y''(x), H\right)\right). \end{aligned} \tag{34}$$

Solving the system (34), we find estimates for the first terms on the right side of (30):

$$\begin{aligned} \alpha_0 &\le \frac{5\left(1 + \frac{1}{2}\lambda_1 H^2 + \lambda_2 H\right)u}{1-\theta} \omega\left(y''(x), H\right), \\ \alpha_1 &\le \frac{5\left(1 + \frac{1}{2}\lambda_1 H^2 + \lambda_2 H\right)v}{1-\theta} \omega\left(y''(x), H\right). \end{aligned}$$

Now the inequalities (30) can be written in the following form

$$\left\| S^{(p)}(\tilde{y}, x) - y^{(p)}(x) \right\| \le K_p \omega\left(y''(x), H\right), \quad p = 0, 1, \tag{35}$$

where $K_0 = \sup\limits_{H \le H^*} \left(\frac{u\mu}{1-\theta} + \frac{5H^2}{2}\right)$, $K_1 = \sup\limits_{H \le H^*} \left(\frac{v\mu}{1-\theta} + 5H\right)$.

We can summarize the aforementioned arguments concerning accuracy of approximating the solution of the boundary value problem (1)-(2) based on the spline sequence as the following theorem.

THEOREM 3. *Let the solution of the boundary value problem* (1)-(2) *exist, be unique and belong to the space* $B\,[a^*;b]$. *If the condition* (17) *holds, then there exists* $H^* > 0$ *such that for any* $H < H^*$ *the spline sequence* $\left\{S\left(y^{(k)},x\right)\right\}$ *is approximating the solution of the boundary value problem* (1)-(2) *and the correlations* (35) *are true.*

5. EXAMPLE

Let us consider the usage of this calculation scheme for finding an approximate solution of the following boundary value problem

$$y''\,(x) = -\alpha y'\left(x - \tfrac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} y(t - \tfrac{\pi}{2})dt + \cos x, \quad 0 \le x \le \tfrac{\pi}{2},$$
$$y\,(x) = \sin(x) + 1, \quad -\tfrac{\pi}{2} \le x < 0,$$
$$y\,(0) = 1,\ y(\tfrac{\pi}{2}) = 2 + \alpha.$$

In this example $L_1^1 = L_2^1 = L_3^1 = 0$, $L_4^1 = \alpha$, $L_1^2 = L_3^2 = L_4^2 = 0$, $L_2^2 = 1$, so $\lambda_1 = \frac{\pi}{2}$, $\lambda_2 = \alpha$, $h = H = \frac{\pi}{40}$, $K = 1$, $u = \frac{\pi^2}{32} + \frac{H^2}{8}$, $v = \frac{\pi}{4} + \frac{2}{3}H$, $\theta = \left(\frac{\pi^2}{32} + \frac{H^2}{8}\right)\frac{\pi}{2} + \left(\frac{\pi}{4} + \frac{2}{3}H\right)\alpha$. If we put $\alpha = \frac{1}{4}$, then $\theta \approx 0.695 < 1$ and therefore the conditions of the Theorems 1 and 2 are satisfied. The precise solution $y_p\,(x)$ of this boundary value problem, which was found using the step method, is

$$y_p\,(x) = \alpha \sin x - \cos x + \left(\tfrac{\pi}{2} - 1\right)\tfrac{x^2}{2} + \tfrac{\pi}{4}\left(1 - \tfrac{\pi}{2}\right)x + 2.$$

x	$y_a(x)$	$y_p(x)$	Δ	δ
0	1	1	0	0%
$\frac{\pi}{8}$	1.03971	1.03976	0.00005	0.01%
$\frac{\pi}{4}$	1.2935	1.2936	0.00010	0.01%
$\frac{3\pi}{8}$	1.7162	1.7163	0.00010	0.01%
$\frac{\pi}{2}$	2.25	2.25	0	0%

Table 1. Precise and approximate solutions.

The results of the calculation are given in Table 1, where $y_p(x)$ is the precise solution, $y_a(x)$ is the approximate solution obtained with $h = \frac{\pi}{40}$ after 2 iterations, Δ is the absolute error and δ is the relative error.

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