JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 44 (2015) no. 4, pp. 166–179
ictp.acad.ro/jnaat

COMMUTATIVITY AND SPECTRAL PROPERTIES OF GENUINE BASKAKOV-DURRMEYER TYPE OPERATORS AND THEIR kTH ORDER KANTOROVICH MODIFICATION*

MARGARETA HEILMANN†

Abstract. In this paper we present an overview of commutativity results and different methods for the proofs for Baskakov-Durrmeyer type operators and associated differential operators. We discuss the spectral properties and generalize all results to kth order Kantorovich modifications and corresponding Durrmeyer type variants of Bleimann, Butzer and Hahn operators and Meyer-König and Zeller operators.

MSC 2010. 41A36, 41A28.
Keywords. Positive linear operators, Durrmeyer type operators, Kantorovich type modification, commutativity, differential operators, spectral properties.

1. INTRODUCTION AND DEFINITION OF THE OPERATORS

In 1957 Baskakov [5] introduced a general method to construct a class of positive linear operators depending on a real parameter c including the classical Bernstein, Szász-Mirakjan and Baskakov operators as special cases. The so-called Bernstein-Durrmeyer operators were introduced by Durrmeyer in [17] and independently developed by Lupaş [31]. Afterwards this construction was carried over to many other classical operators; for instance see [32, 35] for the Szász-Mirakjan and Baskakov operators, [20, 22] in the general setting for so-called Baskakov-Durrmeyer type operators, [33, 10, 11] for the Jacobi weighted Bernstein-Durrmeyer operators, [14, 15] for the non-weighted and Jacobi weighted multivariate Bernstein-Durrmeyer operators defined on a simplex. These operators have a lot of nice properties; they commute, they commute with certain differential operators, they are self-adjoint but they only reproduce constants. Let us mention also [2, 24] for general Durrmeyer-type modifications of Meyer-König and Zeller operators and [3] for Durrmeyer variants of the Bleimann, Butzer and Hahn operators. The Durrmeyer modification of the Bleimann, Butzer and Hahn operators are closely connected to the Bernstein-Durrmeyer operators and the Meyer-König and Zeller operators to the Baskakov-Durrmeyer operators. Due to this relation we can carry over

†Faculty of Mathematics and Natural Sciences, University of Wuppertal, Gaußstr. 20, Wuppertal, Germany, e-mail: heilmann@math.uni-wuppertal.de.

several results which will be discussed in a separate section at the end of this paper.

The consideration of so-called genuine Baskakov-Durrmeyer type operators leads to a class of operators reproducing linear functions and interpolating at (finite) endpoints of the corresponding interval. These operators are related to the Baskakov-Durrmeyer type operators in the same way as the Baskakov type operators to their corresponding Kantorovich variants, i. e., $D^1 \circ B_n \circ I_1 = B_n^{(1)}$ with the notation below.

In what follows for $c \in \mathbb{R}$ we use the notations

$$a^{c,\overline{j}} := \prod_{l=0}^{j-1}(a+cl),\ a^{c,\underline{j}} := \prod_{l=0}^{j-1}(a-cl),\ j \in \mathbb{N}; \quad a^{c,\overline{0}} = a^{c,\underline{0}} := 1$$

which can be considered as a generalization of rising and falling factorials. Note that $a^{-c,\overline{j}} = a^{c,\underline{j}}$ and $a^{c,\overline{j}} = a^{-c,\underline{j}}$. This notation enables us to state the results for the different operators in a unified form.

In the following definitions of the operators we omit the parameter c in the notations in order to reduce the necessary sub- and superscripts.

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore let $j \in \mathbb{N}_0$, $x \in I_c$ with $I_c = [0,\infty)$ for $c \geq 0$ and $I_c = [0,-1/c]$ for $c < 0$. Then the basis functions are given by

$$p_{n,j}(x) = \begin{cases} \dfrac{n^j}{j!} x^j e^{-nx} & , c = 0, \\ \dfrac{n^{c,\overline{j}}}{j!} x^j (1+cx)^{-\left(\frac{n}{c}+j\right)} & , c \neq 0. \end{cases}$$

Note that $p_{n,j}(x) \equiv 0$ for $j > -n/c$ if $c < 0$ and

$$p'_{n,j}(x) = n[p_{n+c,j-1}(x) - p_{n+c,j}(x)] \tag{1}$$

with the convention $p_{n,l}(x) = 0$, if $l < 0$.

For $c < 0$ we consider the space $L_1(I_c)$ and denote by $L_1^0(I_c)$ the set of all functions $f \in L_1(I_c)$ with finite limits $f(0) = \lim_{x\to 0^+} f(x)$ and $f(-1/c) = \lim_{x\to -1/c^-} f(x)$ at the endpoints of the interval. For $c \geq 0$, $\alpha \geq 0$ we denote by $W_\alpha(I_c)$ the space of all locally integrable functions on I_c, satisfying for $t \geq 0$ the growth condition

$$|f(t)| \leq Me^{\alpha t} \text{ if } c = 0 \text{ and } |f(t)| \leq M(1+ct)^{\frac{\alpha}{c}} \text{ if } c > 0$$

for some positive constant M. $W_\alpha^0(I_c)$ consists of all functions $f \in W_\alpha(I_c)$ with finite limit $f(0) = \lim_{x\to 0^+} f(x)$. Furthermore $\mathcal{P}_l$ denotes the set of all polynomials of degree at most l.

Now we can define the genuine Baskakov-Durrmeyer type operators.

Definition 1. *For $c<0$, $n\in\mathbb{R}^+$, $-n/c\in\mathbb{N}$, $f\in L_1^0(I_c)$ define*

$$\begin{aligned}(B_nf)(x) &= f(0)p_{n,0}(x)+f\left(-\tfrac{1}{c}\right)p_{n,-\frac{n}{c}}(x)\\ &\quad +(n+c)\sum_{j=1}^{-\frac{n}{c}-1}p_{n,j}(x)\int_0^{-\frac{1}{c}}p_{n+2c,j-1}(t)f(t)dt,\ x\in\left[0,-\tfrac{1}{c}\right].\end{aligned}$$

For $c\geq 0$, $\alpha\geq 0$, $n\in\mathbb{R}^+$, $n>\alpha-c$, $f\in W_\alpha^0(I_c)$ define

$$\begin{aligned}(B_nf)(x) &= f(0)p_{n,0}(x)\\ &\quad +(n+c)\sum_{j=1}^{\infty}p_{n,j}(x)\int_0^{\infty}p_{n+2c,j-1}(t)f(t)dt,\ x\in[0,\infty).\end{aligned}$$

Setting $c=-1$ leads to the genuine Bernstein-Durrmeyer operators first defined in [12] and independently in [18], $c=0$ to the Phillips operators [34], $c>0$ was investigated in [36].

Similar as in [27, 28, 6] we also consider the kth order Kantorovich modification of the operators B_n, i.e.,

$$B_n^{(k)} := D^k\circ B_n\circ I_k,\ k\in\mathbb{N}_0, \tag{2}$$

where D^k denotes the kth order ordinary differential operator and

$$I_kf=f,\ \text{if } k=0,\ \text{and } (I_kf)(x)=\int_0^x\frac{(x-t)^{k-1}}{(k-1)!}f(t)dt,\ \text{if } k\in\mathbb{N}.$$

For $k=0$ we omit the superscript (k) as indicated by the definition above.

This general definition contains many known operators as special cases. For $k=1$ we get the Baskakov-Durrmeyer type operators $B_n^{(1)}$ (see [17] for $c=-1$, [32] for $c=0$ and [22, (1.3)] for $c\geq 0$, named M_{n+c} there) and for $k\geq 2$ the auxiliary operators $B_n^{(k)}$ considered in [23, (3.5)] (named $M_{n+c,k-1}$ there).

For $k\in\mathbb{N}$, $f\in L_1(I_c)$ for $c<0$ and $f\in W_\alpha(I_c)$ for $c\geq 0$, we have the explicit representation [23, (3.5)]

$$(B_n^{(k)}f)(x)=\frac{n^{c,\overline{k}}}{n^{c,\underline{k-1}}}\sum_{j=0}^{\infty}p_{n+ck,j}(x)\int_{I_c}p_{n-c(k-2),j+k-1}(t)f(t)dt,$$

where the upper limit of the sum is $-\frac{n}{c}-k$ in case $c<0$, as $p_{n+ck,j}(x)\equiv 0$ for $j>-\frac{n}{c}-k$.

In this paper we summarize known results, give an overview of different methods for the proofs and establish general results for the kth order Kantorovich modification concerning the commutativity properties and results for the eigenfunctions of the operators and appropriate differential operators. The proofs are mainly based on the fact that for a suitable function g, $s\in\mathbb{N}_0$, $l\in\mathbb{N}$

$$I_lD^sg=\begin{cases} D^{s-l}g-q_{l-1} & ,\quad s\geq l,\\ I_{l-s}g-q_{l-1} & ,\quad s\leq l,\end{cases} \tag{3}$$

where

$$q_{l-1}(x) = \sum_{i=\max\{0,l-s\}}^{l-1} \frac{g^{(i+s-l)}(0)}{i!} x^i \in \mathcal{P}_{l-1}.$$

Furthermore we need that for each $k \in \mathbb{N}_0$

$$p \in \mathcal{P}_l \implies B_n^{(k)} p \in \mathcal{P}_l \tag{4}$$

(see [27, 28, Theorem 1, Theorem 2]).

2. COMMUTATIVITY OF THE OPERATORS

First we summarize known results and give a survey over the different methods of proofs.

In 1981 Derriennic [13, Théorème III.3] proved that the eigenfunctions of the Bernstein-Durrmeyer operators $B_n^{(1)}$, i. e. , $c = -1$, $k = 1$, are the Legendre polynomials

$$Q_0(x) = 1,\, Q_l(x) = \tfrac{\sqrt{2l+1}}{l!} D^l \left[x^l (1-x)^l \right],\, l \in \mathbb{N},$$

with corresponding eigenvalues

$$\lambda_{n,l} = \begin{cases} \frac{n!(n-1)!}{(n-l-1)!(n+l)!} & , \quad l \le n-1, \\ 0 & , \quad l \ge n, \end{cases}$$

and deduced the representation of the operators in terms of these eigenfunctions, i. e.,

$$(B_n^{(1)} f)(x) = \sum_{l=0}^{n-1} \lambda_{n,l} Q_l(x) \int_0^1 Q_l(t) f(t) dt,\ f \in L_1[0,1].$$

Ditzian and Ivanov [16] remarked that from this result it follows immediately that the operators commute:

$$B_m^{(1)} B_n^{(1)} f = B_n^{(1)} B_m^{(1)} f = \sum_{l=0}^{\min\{m-1,n-1\}} \lambda_{m,l} \lambda_{n,l} Q_l \int_0^1 Q_l(t) f(t) dt.$$

So, the proof of the commutativity is quite elegant in case $c = -1$. The general case $c < 0$, $k = 1$ can be proved in the same way by using the corresponding eigenfunctions and eigenvalues given in Theorem 9.

For $c = 0$ we have the eigenfunction $e_0 = 1$, for $c > 0$ certain polynomial eigenfunctions (see [25, Remark 2.2, Corollary 2.5]). So, the method for $c = -1$ is not applicable to the non-compact interval $[0, \infty)$ in case $c \ge 0$.

In [20, 21] the author proved the commutativity for $c \ge 0$, $k = 1$, $f \in L_p[0, \infty)$, $1 \le p \le \infty$ with a completely different method. Here we give an outline of the main steps of the proof. Note that the proof is also valid for $f \in W_\alpha(I_c)$.

- First the integral representations

$$
\begin{aligned}
(B_n^{(1)} B_m^{(1)} f)(x) &= \int_0^\infty f(y) G_{n,m}(x,y)dy, \\
(B_m^{(1)} B_n^{(1)} f)(x) &= \int_0^\infty f(y) G_{m,n}(x,y)dy
\end{aligned}
$$

  for all $x \in [0,\infty)$ were derived with the kernel functions

$$
\begin{aligned}
G_{n,m}(x,y) &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} p_{n+c,j}(x) p_{m+c,l}(y) \tbinom{j+l}{j} \tfrac{n^{c,\overline{j+1}} m^{c,\overline{l+1}}}{(n+m+c)^{c,\overline{j+l+1}}}, \\
G_{m,n}(x,y) &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} p_{n+c,j}(y) p_{m+c,l}(x) \tbinom{j+l}{j} \tfrac{n^{c,\overline{j+1}} m^{c,\overline{l+1}}}{(n+m+c)^{c,\overline{j+l+1}}}.
\end{aligned}
$$

- Next, the kernel functions were considered as functions of two complex variables and it was shown that they are holomorphic in a certain region.
- The equality of the kernel functions was proved in an open neighborhood of $(0,0)$ by considering the Taylor series at $(0,0)$.
- Finally, by using the identity theorem for analytic functions, the equality of the kernel functions was established for all $x, y \in [0,\infty)$.

In 2005 Abel and Ivan [4] presented a nice alternative proof for the commutativity in case $c = 0$. They proved that for every $f \in W_\alpha(I_c)$, $n, m > \alpha$ with $\frac{nm}{n+m} > \alpha$

$$
B_n^{(1)} B_m^{(1)} f = B_{\frac{nm}{n+m}}^{(1)} f \tag{5}
$$

from which the commutativity follows as a corollary.

In 2011 Tachev and the author [29] proved an analogue for the case $c = 0$, $k = 0$, i. e., for every $f \in W_\alpha^0(I_c)$, $n, m > \alpha$ with $\frac{nm}{n+m} > \alpha$

$$
B_n B_m f = B_{\frac{nm}{n+m}} f. \tag{6}
$$

Now we generalize (5) and (6), respectively, to $k \geq 2$.

Theorem 2. *Let $c = 0$, $k \geq 2$, $f \in W_\alpha(I_c)$, $\alpha \geq 0$, $n, m > \alpha$ with $\frac{nm}{n+m} > \alpha$. Then*

$$
B_n^{(k)} B_m^{(k)} f = B_{\frac{nm}{n+m}}^{(k)} f \tag{7}
$$

Proof. Using the definition of $B_n^{(k)}$ and applying (3) for $g = B_m^{(1)} I_{k-1} f$ we derive

$$
\begin{aligned}
B_n^{(k)} B_m^{(k)} f &= D^{k-1} B_n^{(1)} I_{k-1} D^{k-1} B_m^{(1)} I_{k-1} f \\
&= D^{k-1} B_n^{(1)} B_m^{(1)} I_{k-1} f - D^{k-1} B_n^{(1)} q_{k-2}.
\end{aligned}
$$

As $B_n^{(1)} q_{k-2} \in \mathcal{P}_{k-2}$ by (4) the last term on the right hand side vanishes. Together with (5) this leads to

$$B_n^{(k)} B_m^{(k)} f = D^{k-1} B^{(1)}_{\frac{nm}{n+m}} I_{k-1} f = B^{(k)}_{\frac{nm}{n+m}} f.$$

□

From Theorem 2 together with (5) and (6) we now get the commutativity of the operators $B_n^{(k)}$ for each $k \in \mathbb{N}_0$ in case $c = 0$.

Now we consider $c \neq 0$. Since identities as given in (5), (6) and (7), respectively, are not true for $c \neq 0$, the method by Abel and Ivan is not applicable in this case. For $k = 0$ we need the following result.

Lemma 3. *For $c < 0$ let $n \in \mathbb{R}^+$, $-n/c \in \mathbb{N}$, $f \in L_1^0(I_c)$ such that $D^1 f \in L_1(I_c)$. For $c > 0$, $\alpha \geq 0$ let $n \in \mathbb{R}^+$, $n > \alpha - c$, $f \in W_\alpha^0(I_c)$ such that $D^1 f \in W_\alpha(I_c)$. Then*

$$B_n f = f(0) + I_1 B_n^{(1)} D^1 f.$$

Proof. We only prove the case $c < 0$ as the case $c > 0$ is completely analogue. Using integration by parts and (1) we have

$$\int_0^{-1/c} p_{n+c,j}(t) f'(t) dt = -(n+c) \int_0^{-1/c} [p_{n+2c,j-1}(t) - p_{n+2c,j}(t)] f(t) dt + \begin{cases} f\left(-\frac{1}{c}\right) & , \quad j = -\frac{n}{c} - 1, \\ -f(0) & , \quad j = 0, \\ 0 & , \quad 1 \leq j \leq -\frac{n}{c} - 2. \end{cases}$$

Thus, again using (1), we derive

$$\begin{aligned} (8) \quad & (B_n^{(1)} D^1 f)(x) \\ &= n\left[f\left(-\tfrac{1}{c}\right) p_{n+c,-\frac{n}{c}-1}(x) - f(0) p_{n+c,0}(x)\right] \\ &\quad -n \sum_{j=0}^{-\frac{n}{c}-1} p_{n+c,j}(x)(n+c) \int_0^{-1/c} (p_{n+2c,j-1}(t) - p_{n+2c,j}(t)) f(t) dt \\ &= n\left[f\left(-\frac{1}{c}\right) p_{n+c,-\frac{n}{c}-1}(x) - f(0) p_{n+c,0}(x)\right] \\ &\quad +(n+c) \sum_{j=1}^{-\frac{n}{c}-1} p'_{n,j}(x) \int_0^{-1/c} p_{n+2c,j-1}(t) f(t) dt. \end{aligned}$$

As

$$\begin{aligned} \int_0^x p_{n+c,-\frac{n}{c}-1}(u) du &= \tfrac{1}{n}(-cx)^{-\frac{n}{c}} = \tfrac{1}{n} p_{n,-\frac{n}{c}}(x), \\ \int_0^x p_{n+c,0}(u) du &= \tfrac{1}{n}\left[1 - (1+cx)^{-\frac{n}{c}}\right] = \tfrac{1}{n}(1 - p_{n,0}(x)), \end{aligned}$$

we get by applying I_1 on both sides of (8)

$$\begin{aligned}(I_1 B_n^{(1)} D^1 f)(x) &= f\left(-\tfrac{1}{c}\right) p_{n,-\frac{n}{c}}(x) - f(0)(1 - p_{n,0}(x)) \\ &\quad + (n+c) \sum_{j=1}^{-\frac{n}{c}-1} p_{n,j}(x) \int_0^{-1/c} p_{n+2c,j-1}(t) f(t) dt \\ &= -f(0) + (B_n f)(x).\end{aligned}$$

□

THEOREM 4. *With the same assumptions as in Lemma 3 we have*

$$B_n B_m f = B_m B_n f.$$

Proof. With Lemma 3 and the interpolation property of the genuine operators, i. e., $(B_n f)(0) = (B_m f)(0) = f(0)$, we get

$$\begin{aligned}B_n B_m f &= f(0) + I_1 B_n^{(1)} D^1 I_1 B_m^{(1)} D^1 f \\ &= f(0) + I_1 B_n^{(1)} B_m^{(1)} D^1 f \\ &= f(0) + I_1 B_m^{(1)} B_n^{(1)} D^1 f \\ &= B_m B_n f.\end{aligned}$$

□

Next we consider the case $k \geq 2$.

THEOREM 5. *Let $k \in \mathbb{N}$, $k \geq 2$. For $c < 0$ let $n \in \mathbb{R}^+$, $-n/c \in \mathbb{N}$, $f \in L_1(I_c)$. For $c > 0$, $\alpha \geq 0$ let $n \in \mathbb{R}^+$, $n > \alpha - c$, $f \in W_\alpha(I_c)$. Then*

$$B_n^{(k)} B_m^{(k)} f = B_m^{(k)} B_n^{(k)} f.$$

Proof. With similar arguments as in the proof of Theorem 4 we get

$$\begin{aligned}B_n^{(k)} B_m^{(k)} f &= D^{k-1} B_n^{(1)} I_{k-1} D^{k-1} B_m^{(1)} I_{k-1} f \\ &= D^{k-1} B_n^{(1)} B_m^{(1)} I_{k-1} f - D^{k-1} B_n^{(1)} q_{k-2} \\ &= D^{k-1} B_m^{(1)} B_n^{(1)} I_{k-1} f \\ &= B_m^{(k)} B_n^{(k)} f.\end{aligned}$$

□

3. ADAPTED DIFFERENTIAL OPERATORS

The operators $B_n^{(k)}$ are strongly connected to appropriate differential operators. This was used for example for the construction of quasi-interpolants (see, *e.g.*, [9, 1, 30, 36]).

In the following we use the notation $\varphi(x) = \sqrt{x(1+cx)}$.

Definition 6. *For $r \in \mathbb{N}$ we define*

$$\widetilde{D}^{2r,(k)} = \begin{cases} D^{r-1+k}\varphi^{2r}D^{r+1-k} & , \quad k \leq r+1, \\ D^{r-1+k}\varphi^{2r}I_{k-r-1} & , \quad k \geq r+1. \end{cases}.$$

Formally we denote $\widetilde{D}^{0,(k)} = Id$.

The following recursion formula for the differential operators was proved for the special cases $c = -1$, $k = 1$ also in the multivariate setting in [8, (4.4)], for $c \geq 0$, $k = 1$ in [7, Lemma 4], for $c = -1$, $k = 0$ in [30, Lemma 3] and for $c \geq 0$, $k = 0$ in [36, Lemma 2.3].

Theorem 7. *For $r \in \mathbb{N}_0$ we have*

$$\widetilde{D}^{2r+2,(k)} = \widetilde{D}^{2r,(k)}\left[\widetilde{D}^{2,(k)} - cr(r+1)Id\right].$$

Proof. In view of the already known results we only have to consider $k \geq 2$. We distinguish between the cases $2 \leq k \leq r+1$ and $k \geq r+2$.

$\boldsymbol{2 \leq k \leq r+1}$:

$$\begin{aligned}\widetilde{D}^{2r,(k)}\widetilde{D}^{2,(k)} &= D^{r+k-1}\varphi^{2r}D^{r-k+1}D^{k}\varphi^{2}I_{k-2} \\ &= D^{r+k-1}\varphi^{2r}D^{r+1}\varphi^{2}I_{k-2}.\end{aligned}$$

By using Leibniz' formula we derive

$$\begin{aligned}&D^{r+1}\varphi^{2}I_{k-2} \\ =\ &\sum_{l=0}^{r+1}\binom{r+1}{l}\left(D^{l}\varphi^{2}\right)\left(D^{r+1-l}I_{k-2}\right) \\ =\ &\varphi^{2}D^{r+1}I_{k-2} + (r+1)\left(D\varphi^{2}\right)\left(D^{r}I_{k-2}\right) + cr(r+1)\left(D^{r-1}I_{k-2}\right) \\ =\ &\varphi^{2}D^{r+3-k} + (r+1)\left(D\varphi^{2}\right)D^{r+2-k} + cr(r+1)D^{r+1-k}.\end{aligned}$$

Thus,

$$\begin{aligned}(9)\qquad &\widetilde{D}^{2r,(k)}\widetilde{D}^{2,(k)} \\ &= D^{r+k-1}\varphi^{2r+2}D^{r+3-k} + (r+1)D^{r+k-1}\varphi^{2r}\left(D\varphi^{2}\right)D^{r+2-k} \\ &\quad + cr(r+1)D^{r+k-1}\varphi^{2r}D^{r+1-k}.\end{aligned}$$

Furthermore,

$$\begin{aligned}(10)\quad \widetilde{D}^{2r+2,(k)} &= D^{r+k-1}D\varphi^{2r+2}D^{r+2-k} \\ &= D^{r+k-1}\left[(r+1)\varphi^{2r}(D\varphi^{2})D^{r+2-k} + \varphi^{2r+2}D^{r+3-k}\right].\end{aligned}$$

The proposition now follows from (9) and (10).

$\boldsymbol{k \geq r+2}$: By using (3) for $l = s = k-r-1$ with $g = D^{r+1}\varphi^{2}I_{k-2}$ we derive

$$\begin{aligned}\widetilde{D}^{2r,(k)}\widetilde{D}^{2,(k)} &= D^{r+k-1}\varphi^{2r}I_{k-r-1}D^{k-r-1}D^{r+1}\varphi^{2}I_{k-2} \\ &= D^{r+k-1}\varphi^{2r}D^{r+1}\varphi^{2}I_{k-2}.\end{aligned}$$

Again by Leibniz' formula we get

$$\begin{aligned} & D^{r+1}\varphi^2 I_{k-2} = \\ = \; & \varphi^2 D^{r+1} I_{k-2} + (r+1)\left(D\varphi^2\right)\left(D^r I_{k-2}\right) + cr(r+1)\left(D^{r-1} I_{k-2}\right) \\ = \; & \varphi^2 D I_{k-2-r} + (r+1)\left(D\varphi^2\right) I_{k-2-r} + cr(r+1) I_{k-r-1}. \end{aligned}$$

Thus,

$$(11)\quad \begin{aligned} & \widetilde{D}^{2r,(k)}\widetilde{D}^{2,(k)} = \\ = \; & D^{r+k-1}\varphi^{2r+2} D I_{k-2-r} + (r+1) D^{r+k-1}\varphi^{2r}\left(D\varphi^2\right) I_{k-2-r} \\ & + cr(r+1) D^{r+k-1}\varphi^{2r} I_{k-r-1}. \end{aligned}$$

Furthermore

$$(12)\quad \begin{aligned} \widetilde{D}^{2r+2,(k)} &= D^{r+k-1} D\varphi^{2r+2} I_{k-r-2} \\ &= D^{r+k-1}\left[(r+1)\varphi^{2r}(D\varphi^2) I_{k-r-2} + \varphi^{2r+2} D I_{k-r-2}\right]. \end{aligned}$$

The proposition now follows from (11) and (12). □

From Theorem 7 the following product formula can be easily established by induction (see [8, (4.5)] for $k = 1$ also in the multivariate setting, [30, Lemma 4] for $c = -1$, $k = 0$ and [36, Lemma 2.4] for $c \in \mathbb{R}$, $k = 0$).

$$(13)\quad \begin{aligned} \widetilde{D}^{2r,(k)} &= \prod_{j=0}^{r-1}\left[\widetilde{D}^{2,(k)} - j(j+1)cId\right] \\ &= \widetilde{D}^{2,(k)} \circ \left(\widetilde{D}^{2,(k)} - 2cId\right) \circ \cdots \circ \left(\widetilde{D}^{2,(k)} - (r-1)rcId\right). \end{aligned}$$

For the special case $c = 0$ this means

$$\widetilde{D}^{2r,(k)} = \left(\widetilde{D}^{2,(k)}\right)^r.$$

The commutativity of the differential operators now follows as a corollary.

Corollary 8. *Let* $r, l \in \mathbb{N}$, $k \in \mathbb{N}_0$. *Then*

$$\widetilde{D}^{2r,(k)}\widetilde{D}^{2l,(k)} = \widetilde{D}^{2l(k)}\widetilde{D}^{2r,(k)}.$$

4. SPECTRAL PROPERTIES

Next we generalize results concerning the spectral properties of the operators $B_n^{(k)}$ and the differential operators. For $B_n^{(k)}$ the special case $k = 1, c = -1$ was considered in [13, Théorème III.3], for $c = -1, k = 0$ see [19, Theorem 4], for $k = 1, c = 1$ [24, Corollary 2.5] and for $k = 0, c \neq 0$ [36, Lemma 1.16]. References concerning the differential operators are [8, Theorem 4], [9, (2.1), (2.2)] for $c = -1$, $k = 1$ (also in the Jacobi weighted multivariate setting) and [36, Lemmas 2.2, 2.3, 2.4].

THEOREM 9. *For $c \neq 0$, $l \in \mathbb{N}_0$ and $n > c(l+k-1)$ in case $c > 0$ it holds*

$$B_n^{(k)} g_{l,k} = \lambda_{n,l,k} g_{l,k} \text{ and } \widetilde{D}^{2r,(k)} g_{l,k} = \gamma_{n,l,k} g_{l,k},$$

where

$$g_{0,0}(x) = 1,\ g_{1,0}(x) = x,\ g_{l,k}(x) = D^{l+2(k-1)} \varphi^{2(l+k-1)},\ l + 2(k-1) \geq 0$$

and

$$\lambda_{n,l,k} = \frac{n^{c,\overline{l+k}}}{n^{c,\underline{l+k}}}, \quad \gamma_{r,l,k} = \begin{cases} c^r \dfrac{(l+k+r-1)!}{(l+k-r-1)!} & , \quad l+k-1 \geq r, \\ 0 & , \quad l+k-1 < r. \end{cases}$$

Proof. First we consider $B_n^{(k)}$. We use the known results for $k = 0$. For $k = 1, l = 0$ we have

$$g_{0,1} = 1 \text{ and } B_n^{(1)} g_{0,1} = g_{0,1}$$

as $B_n^{(1)}$ preserves constants.

Now let $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ with $l + k \geq 2$. Then, again using (3) and (4),

$$\begin{aligned} B_n^{(k)} g_{l,k} &= D^k B_n I_k D^{l+2(k-1)} \varphi^{2(l+k-1)} \\ &= D^k B_n D^{l+k-2} \varphi^{2(l+k-1)} \\ &= D^k B_n g_{l+k,0} \\ &= \frac{n^{c,\overline{l+k}}}{n^{c,\underline{l+k}}} g_{l,k}. \end{aligned}$$

Next we treat the differential operators. With

$$\gamma_{r,l,k} = \gamma_{r,l+k,0} \text{ and } g_{l,k} = D^k g_{l+k,0}$$

we derive

$$\begin{aligned} \gamma_{r,l,k} g_{l,k} &= D^k \gamma_{r,l+k,0} g_{l+k,0} \\ &= D^k \widetilde{D}^{2r,(0)} g_{l+k,0} \\ &= D^{k+r-1} \varphi^{2r} D^{r+1} g_{l+k,0} \\ &= \begin{cases} D^{k+r-1} \varphi^{2r} D^{r+1-k} D^k g_{l+k,0}, & k \leq r+1 \\ D^{k+r-1} \varphi^{2r} I_{k-r-1} D^k g_{l+k,0}, & k \geq r+1 \end{cases} \\ &= \widetilde{D}^{2r,k} g_{l,k}. \end{aligned}$$

□

5. COMMUTATIVITY OF THE OPERATORS AND APPROPRIATE DIFFERENTIAL OPERATORS

In [23, Lemma 3.1] the author proved that the operators $B_n^{(1)}$ and the differential operators $\widetilde{D}^{2r,(1)}$ commute for sufficiently smooth functions. The corresponding result for $c = 0$, $k = 0$ was proved in [29, Theorem 3.2, Remark 3.1] and was generalized for $c \in \mathbb{R}$, $k = 0$ in [36, Satz 2.8].

Theorem 10. *For $k \geq 2$ we have*

$$\widetilde{D}^{2r,(k)} B_n^{(k)} = B_n^{(k)} \widetilde{D}^{2r,(k)}.$$

Proof. With regard to the above mentioned known results we only have to treat the case $k \geq 2$ and prove our proposition by induction.
$\boldsymbol{r=1}$: Using (3) with $l = k-2$ and $g = B_n^{(1)} I_{k-1} f$ if $k \geq 3$ we get

$$\begin{aligned}
\widetilde{D}^{2,(k)} B_n^{(k)} f &= D^{k-1} D\varphi^2 D B_n^{(1)} I_{k-1} f \\
&= D^{k-1} B_n^{(1)} D\varphi^2 D I_{k-1} f \\
&= D^{k-1} B_n^{(1)} D\varphi^2 I_{k-2} f \\
&= D^{k-1} B_n^{(1)} I_{k-1} D^k \varphi^2 I_{k-2} f \\
&= B_n^{(k)} \widetilde{D}^{2,(k)}.
\end{aligned}$$

The conclusion $\boldsymbol{r \Rightarrow r+1}$ follows easily from (13). □

6. RELATED DURRMEYER TYPE OPERATORS

In this section we consider $c \neq 0$. Let

$$\begin{aligned}
\sigma : I_c \longrightarrow I_{-c} \qquad & \sigma(x) = \frac{x}{1+cx} \\
\psi : I_c \longrightarrow I_{-c} \qquad & \psi(x) = \frac{x}{1-cx}.
\end{aligned}$$

The consideration of

$$\left(\overline{B}_n^{(k)} f(t)\right)(x) := \left(B_n^{(k)} f(\sigma(t))\right)(\psi(x))$$

leads to kth order Kantorovich modifications of Durrmeyer type variants of Bleimann, Butzer and Hahn operators (BBH-D operators) for $c < 0$ and Meyer-König and Zeller operators (MKZ-D operators) for $c > 0$.
With the notation

$$\overline{p}_{n,j}(x) := p_{n,j}(\psi(x)) = \frac{n^{c,\overline{j}}}{j!} x^j (1-cx)^{\frac{n}{c}}$$

they are explicitly given by the following formulas.

For $c < 0$, $n \in \mathbb{R}^+$, $-n/c \in \mathbb{N}$, $(1-c\cdot)^{-2} f(\cdot) \in L_1[0,\infty)$ with finite limits $f(0) = \lim_{x\to 0^+} f(x)$ and $f_\infty = \lim_{x\to\infty} f(x)$

$$\begin{aligned}
(\overline{B}_n f)(x) &= f(0)\overline{p}_{n,0}(x) + f_\infty \overline{p}_{n,-\frac{n}{c}}(x) \\
&\quad + (n+c) \sum_{j=1}^{-\frac{n}{c}-1} \overline{p}_{n,j}(x) \int_{I_{-c}} \overline{p}_{n+2c,j-1}(t) f(t) (1-ct)^{-2} dt,
\end{aligned}$$

$x \in [0,\infty)$, we have a genuine variant of BBH-D operators.
For $c > 0$, $\alpha \geq 0$, $n \in \mathbb{R}^+$, $n > \alpha - c$, f locally integrable on $[0, \frac{1}{c})$

satisfying $|f(t)| \leq M(1-ct)^{-\frac{\alpha}{c}}$, $t \in [0, \frac{1}{c})$, and possessing a finite limit $f(0) = \lim_{x\to 0^+} f(x)$

$$(\overline{B}_n f)(x) = f(0)\overline{p}_{n,0} + (n+c)\sum_{j=1}^{\infty} \overline{p}_{n,j}(x) \int_{I_{-c}} \overline{p}_{n+2c,j-1}(t) f(t)(1-ct)^{-2} dt,$$

$x \in [0, \frac{1}{c})$, defines a genuine variant of MKZ-D operators.
For the kth order Kantorovich modification we derive for f as above without the conditions for the limits with $c \neq 0$, $k \in \mathbb{N}$:

$$(\overline{B}_n^{(k)} f)(x) = \frac{n^{c,\overline{k}}}{n^{c,\underline{k-1}}} \sum_{j=0}^{\infty} \overline{p}_{n+ck,j}(x) \int_{I_{-c}} \overline{p}_{n-c(k-2),j+k-1}(t) f(t)(1-ct)^{-2} dt$$

where the upper limit of the sum is $-\frac{n}{c} - k$ for $c < 0$.

From the results in Section 2 we deduce that the operators $\overline{B}_n^{(k)}$ are commutative. For the special case $k = 1, c = -1$ see [3, Theorem 2.1] and for $k = 1, c = 1$ [26, Theorem 1]. Furthermore they commute with the differential operators

$$\overline{D}^{2r,(k)} = \begin{cases} \widehat{D}^{r-1+k}\overline{\varphi}^{2r}\widehat{D}^{r+1-k} & , \quad k \leq r+1, \\ \widehat{D}^{r-1+k}\overline{\varphi}^{2r}\widehat{I}_{k-r-1} & , \quad k \geq r+1, \end{cases}$$

where, with $\overline{\varphi}(x) = \frac{\sqrt{x}}{1-cx}$,

$$(\widehat{D}f)(x) = (1-cx)^{-2} f'(x),\ \widehat{D}^m f = \widehat{D}^{m-1}(\widehat{D}f)$$

and

$$(\widehat{I}f)(x) = \int_0^x \frac{f(t)}{(1-ct)^2} dt,\ \widehat{I}_m f = \widehat{I}_{m-1}(\widehat{I}f).$$

From Section 4 we get the eigenfunctions

$$\overline{g}_{0,0}(x) = 1,\ \overline{g}_{1,0}(x) = \frac{x}{1-cx},\ \overline{g}_{l,k}(x) = \widehat{D}^{l+2(k-1)}\overline{\varphi}^{2(l+k-1)},\ l + 2(k-1) \geq 0$$

for the operators $\overline{B}_n^{(k)}$ and the differential operators $\overline{D}^{2r,(k)}$.

REFERENCES

[1] U. Abel, *An identity for a general class of approximation operators*, J. Approx. Theory, **142**, pp. 20–35, 2006.

[2] U. Abel, V. Gupta and M. Ivan, *The complete asymptotic expansion for a general Durrmeyer variant of the Meyer-König and Zeller operators* Math. Comput. Modelling **40**, no. 7-8, pp. 867–875, 2004.

[3] U. Abel and M. Ivan, *Durrmeyer variants of the Bleimann, Butzer and Hahn operators*, Mathematical analysis and approximation theory, Burg, Sibiu, pp. 1–8, 2002.

[4] U. Abel and M. Ivan, *Enhanced asymptotic approximation and approximation of truncated functions by linear operators*, Constructive Theory of Functions, Proceedings of the International Conference on Constructive Theory of Functions, Varna, June 2 - June 6, 2005, B. D. Bojanov (Ed.), Prof. Marin Drinov Academic Publishing House, pp. 1–10, 2006.

[5] V. A. Baskakov, *An instance of a sequence of positive linear operators in the space of continuous functions*, Doklady Akademii Nauk SSSR, **113:2**, pp. 249–251, 1957.

[6] K. Baumann, M. Heilmann and I. Raşa, *Further results for k-th order Kantorovich modification of linking Baskakov type operators*, Results Math., 2015.

[7] E. Berdysheva, *Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions* J. Approx. Theory, **149**, no. 2, pp. 131–150, 2007.

[8] E. Berdysheva, K. Jetter and J. Stöckler, *New polynomial preserving operators on simplices: direct results*, J. Approx. Theory **131**, no. 1, pp. 59–73, 2004.

[9] E. Berdysheva, K. Jetter and J. Stöckler, *Bernstein-Durrmeyer type quasi-interpolants on intervals*, Approximation Theory: a Volume dedicated to Borislav Bojanov, Prof. M. Drinov Acad. Publ. House, Sofia, pp. 32–42,2004.

[10] H. Berens and Y. Xu, *On Bernstein-Durrmeyer polynomials with Jacobi weights*, Approximation theory and functional analysis (College Station, TX, 1990), Academic Press, Boston, MA, pp. 25–46,1991.

[11] H. Berens and Y. Xu, *On Bernstein-Durrmeyer polynomials with Jacobi-weights: the cases $p = 1$ and $p = \infty$*, Approximation Interpolation and Summability (Ramat Aviv, 1990/Ramat Gan, 1990), Israel Math. Conf. Proc., 4, Bar-Ilan Univ., Ramat Gan, pp. 51–62,1991.

[12] W. Chen, *On the modified Durrmeyer-Bernstein operator*, (handwritten, Chinese, 3 pages), Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China (1987).

[13] M.-M. Derriennic, *Sur l'approximation de fonctions intégrables sur $[0, 1]$ par des polynômes de Bernstein modifiés*, J. Approx. Theory **31**, no. 4, pp. 325–343, 1981.

[14] M.-M. Derriennic, *On multivariate approximation by Bernstein-type polynomials*, J. Approx. Theory **45**, no. 2, pp. 155-166, 1985.

[15] Z. Ditzian, *Multidimensional Jacobi-type Bernstein-Durrmeyer operators*, Acta Sci. Math. (Szeged) **60**, no. 1-2, pp. 225–243, 1995.

[16] Z. Ditzian and K. Ivanov, *Bernstein-type operators and their derivatives*, J. Approx. Theory **56** , no. 1, pp. 72–90, 1989.

[17] J. L. Durrmeyer, *Une formule d'inversion de la transformée de Laplace: applications à la théorie des moments*, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[18] T. N. T. Goodman and A. Sharma, *A modified Bernstein-Schoenberg operator*, Constructive Theory of Functions (Varna, 1987), Publ. House Bulgar. Acad. Sci., Sofia, pp. 166–173,1988.

[19] T. N. T. Goodman and A. Sharma, *A Bernstein type operator on the simplex*, Math. Balkanica (N.S.) **5** , no. 2, pp. 129–145, 1991.

[20] M. Heilmann, *Approximation auf $[0, \infty)$ durch das Verfahren der Operatoren vom Baskakov-Durrmeyer Typ*, Dissertation, Universität Dortmund, 1987.

[21] M. Heilmann, *Commutativity of operators from Baskakov-Durrmeyer type* Constructive theory of functions (Varna, 1987), Publ. House Bulgar. Acad. Sci., Sofia, pp. 197–206, 1988.

[22] M. Heilmann, *Direct and converse results for operators of Baskakov-Durrmeyer type*, Approx. Theory Appl. **5** , no. 1, pp. 105–127, 1989.

[23] M. Heilmann, *Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren*, Habilitationschrift Universität Dortmund, 1992.

[24] M. Heilmann, *Eigenfunctions of Durrmeyer-type modifications of Meyer-König and Zeller operators*, J. Approx. Theory **125** , no. 1, pp. 63–73, 2003.

[25] M. Heilmann, *Rodriguez-type representation for the eigenfunctions of Durrmeyer-type operators*, Results Math. **44**, no. 1-2, pp. 97–105, 2003.
[26] M. Heilmann, *Commutativity of Durrmeyer-type modifications of Meyer-König and Zeller and Baskakov-operators*, Constructive Theory of Functions, DARBA, Sofia, pp. 295–301, 2003.
[27] M. Heilmann and I. Raşa, *k-th order Kantorovich type modification of the operators* U_n^ρ, J. Appl. Funct. Anal. **9**, no. 3-4, pp. 320–334, 2014.
[28] M. Heilmann and I. Raşa, *k-th order Kantorovich modification of linking Baskakov type operators*, Recent Trends in Mathematical Analysis and its Applications, Rorkee, India, December 2014, (ed. P. N. Agrawal et al.), Springer Proceedings in Mathematics & Statistics, Vol. 143, pp. 229-242, 2015.
[29] M. Heilmann and G. Tachev, *Commutativity, direct and strong converse results for Phillips operators*, East J. Approx. **17**, no. 3, pp. 299–317, 2011.
[30] M. Heilmann and M. Wagner, *The genuine Bernstein-Durrmeyer operators and quasi-interpolants*, Results Math. **62** , nos. 3-4, pp. 319–335, 2012.
[31] A. Lupaş, *Die Folge der Betaoperatoren*, Dissertation, Universität Stuttgart 1972.
[32] S. M. Mazhar and V. Totik, *Approximation by modified Szász operators*, Acta Sci. Math. (Szeged) **49** , nos. 1-4, pp. 257–269, 1985.
[33] R. Păltănea, *Sur un opérateur polynomial defini sur l'ensemble des fonctions integrables*, "Babes-Bolyai" Univ., Fac. Math., Res. Semin. **2**, pp. 101–106, 1983.
[34] R. S. Phillips, *An inversion formula for Laplace transforms and semi-groups of linear operators*, Ann. of Math. (2) **59**, pp. 325–356, 1954, DOI: 10.2307/1969697.
[35] A. Sahai and G. Prasad, *On simultaneous approximation by modified Lupas operators*, J. Approx. Theory **45** , no. 2, pp. 122–128, 1985.
[36] M. Wagner, *Quasi-Interpolanten zu genuinen Baskakov-Durrmeyer-Typ Operatoren*, Disssertation Bergische Universität Wuppertal, 2013.

Received by the editors: January 20, 2016.