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# FURTHER RESULTS ON $L^1$-CONVERGENCE OF SOME MODIFIED COMPLEX TRIGONOMETRIC SUMS

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**Abstract.** In this paper we have defined a new class of numerical sequences, which tend to zero, briefly denoted by $\mathbb{K}^2$. Moreover, employing such class of numerical sequences we have studied $L^1$-convergence of some modified complex trigonometric sums introduced previously by others.

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**Keywords.** $L^1$-convergence, null sequence, trigonometric series, modified sums.

## 1. INTRODUCTION

Let $(c_k)$, $k \in \{0, \pm1, \pm2, \dots\}$, be a sequence of complex numbers and let

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

be a formal complex trigonometric series with its partial sums

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \quad n \in \{0, 1, 2, \dots\}. \tag{2}$$

Let

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|dx$$

be the $L^1$-norm of a function $f \in L^1$.

The following interesting statement is a well-known one: If a trigonometric series converges in $L^1$-norm to a function $f \in L^1$, then it is the Fourier series of the function $f$. Riesz (see [5], Vol. II, Chap. VIII, § 22) gave a counter example showing that in the metric $L^1$ we can not expect the converse of above mention statement to be true. This fact motivated the various authors to study the $L^1$-convergence of trigonometric series, introducing the so-called modified cosine and sine sums, since these modified sums approximate their limits better than the classical trigonometric series in the sense that they converge in $L^1$-norm to the sum of the trigonometric series whereas the classical series itself may not.

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C. S. Rees and C. V. Stanojevic [3] for the first time introduced the following type of modified cosine sums

$$f_n(x) = \tfrac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle a_j \cos kx,$$

and obtained a necessary and sufficient condition for the integrability of the limit of these sums, where $\triangle a_j = a_j - a_{j+1}$.

Then several interesting properties (their integrability [2] or $L^1$-convergence [15]) of these sums were investigated imposing several conditions on the coefficients $a_k$ in "old papers" [1], [2], [6] or in some "new papers" [4], [9], [13], [14], [16].

After introducing the sums $f_n(x)$, B. Ram and S. Kumari [8] seems to be motivated to introduce the set of the sums

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k \cos kx$$

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k \sin kx$$

and studied their $L^1$-convergence under condition that the coefficients $a_n$ belong to the class $\mathbb{R}$.

Definition 1 ([12]). *If $a_k \to 0$ as $k \to \infty$ and*

$$\sum_{k=1}^{\infty} k^2 \left|\triangle^2\left(\frac{a_k}{k}\right)\right| < \infty$$

*then it is said that $(a_k)$ belongs to the class $\mathbb{R}$, where $\triangle^2 a_j = a_j - 2a_{j+1} + a_{j+2}$.*

The above definition were introduced by T. Kano who verified a result which we will formulate it as follows.

Theorem 2 ([12]). *If $(a_k) \in \mathbb{R}$ then the series (1) and (2) are Fourier series or equivalently they represent integrable functions.*

Among others, using this result B. Ram and S. Kumari proved the following result.

Theorem 3 ([8]). *Let $(a_k) \in \mathbb{R}$. Then for $x \in (0, \pi]$*

$$\lim_{n\to\infty} t_n(x) = t(x), \quad t \in L(0,\pi], \tag{3}$$

*and*

$$\|t_n - t\|_{L^1} \to 0, \text{ as } n \to \infty, \tag{4}$$

*where $t_n(x)$ represents either $h_n(x)$ or $g_n(x)$.*

The authors of [9] have introduced the following modified complex trigonometric sums

$$g_n^c(x) = S_n(x) + \frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} + c_{n+1}e^{inx} - c_{-(n+1)}e^{-inx} + (c_n - c_{n+2})E_n(x) + (c_{-(n+2)} - c_{-n})E_{-n}(x)],$$

where $E_n(x) = \sum_{j=0}^{n} e^{ijx}$.

Note that the above sums are indeed the complex form of the modified sine and cosine sums introduced in [10] and [11] respectively.

Also they have introduced the following definition.

Definition 4. *A sequence* $(c_k)$ *of complex numbers belongs to class* $J^*$ *if* $\lim_{k\to\infty} c_k = 0$, *and there exists a sequence* $(A_k)$ *such that*

$$A_k \downarrow 0, \quad as \quad k \to \infty, \tag{5}$$

$$\sum_{k=1}^{\infty} kA_k < \infty, \tag{6}$$

*and*

$$\left|\triangle\left(\frac{c_k - c_{-k}}{k}\right)\right| \le \frac{A_k}{k}, \quad \forall k. \tag{7}$$

Throughout this paper we will denote by $S_n(x)$ the partial sums of the series (1) and $\lim_{n\to\infty} S_n(x) = f(x)$.

Theorem 5 ([8]). *Let* $(c_k)$ *belongs to the class* $J^*$ *then*

$$\lim_{n\to\infty} g_n^c(x) = f(x), \quad exists\ for \quad |x| \in (0, \pi], \tag{8}$$

$$f \in L^1(0, \pi] \quad and \quad \|g_n^c - f\|_{L^1} \to 0,\ as\ n \to \infty, \tag{9}$$

*and*

$$\|S_n - f\|_{L^1} \to 0,\ as\ n \to \infty. \tag{10}$$

Here and in sequel we will use the notation $\triangle^2 b_k = \triangle b_k - \triangle b_{k+1}, k \in \{1, 2, \dots\}$. Now we are going to introduce the following class of sequences of complex numbers.

Definition 6. *A sequence* $(c_k)$ *of complex numbers belongs to class* $\mathbb{K}^2$ *if* $\lim_{k\to\infty} c_k = 0$, *and there exists a sequence* $(A_k)$ *such that*

$$A_k \downarrow 0, \quad as \quad k \to \infty, \tag{11}$$

$$\sum_{k=1}^{\infty} k^2 A_k < \infty, \tag{12}$$

*and*

$$\max\left\{\left|\triangle^2\left(\frac{c_k}{k}\right)\right|,\left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right|\right\}\le\frac{A_k}{k^2},\quad k\in\{1,2,\dots\}. \tag{13}$$

Next example shows that the there exist sequences that belong or not belong to the class $\mathbb{K}^2$.

Let $(c_k)$ be a sequence defined by its general term $c_k := \frac{1}{n^2}$ , $n \in \{1, 2, \dots\}$. Then, $|\triangle^2\left(\frac{c_{\pm k}}{k}\right)| \le \frac{4}{k^3} = \frac{A_k}{k^2}$, $A_k = \frac{4}{k} \downarrow 0$, and $\sum_{k=1}^{\infty} k^2 A_k = +\infty$, which means that $(c_k) \notin \mathbb{K}^2$.

On the other hand, let $(\overline{c}_k)$ be a sequence defined by its general term $\overline{c}_k = \frac{1}{n^5}$ , $n \in \{1, 2, \dots\}$. Then, $|\triangle^2\left(\frac{c_{\pm k}}{k}\right)| \le \frac{4}{k^6} = \frac{A_k}{k^2}$, $A_k = \frac{4}{k^4} \downarrow 0$, and $\sum_{k=1}^{\infty} k^2 A_k < +\infty$, which means that $(\overline{c}_k) \in \mathbb{K}^2$.

The aim of this paper is to study $L^1$-convergence of sums $g_n^c(x)$ under condition that the coefficients $c_k$ belong to the class $\mathbb{K}^2$.

Closing this section, we recall the well-known equality named as Abel's transformation: Let $n$ be a positive integer, and $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be two sequences. If $S_k = a_1 + a_2 + \cdots + a_n$, then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k(b_k - b_{k+1}) + S_n b_n.$$

## 2. HELPFUL LEMMAS

Let

$$\widetilde{D}_n(x) = \sum_{j=1}^{n} \cos(jx) = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

and

$$\widetilde{D}_n(x) = \sum_{j=1}^{n} \sin(jx) = \frac{\cos\frac{x}{2} - \cos\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}$$

be the Dirichlet's and conjugate Dirichlet's kernels respectively.

The following statements are needed for the proof of the main result.

Lemma 7 ([7]). *Let $r$ be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $M_{r\varepsilon} > 0$ such that for all $\varepsilon \le |x| \le \pi$ and all $n \ge 1$,*

(i) $|E_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|}$,

(ii) $|E_{-n}^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|}$,

(iii) $|D_n^{(r)}(x)| \le \frac{2M_{r\varepsilon} n^r}{|x|}$,

(iv) $|\widetilde{D}_n^{(r)}(x)| \le \frac{M_{r\varepsilon} n^r}{|x|}$,

Lemma 8 ([9]). *For $n \ge 1$, we have*

(i) $\left\|\frac{E_n(x)}{2\sin x}\right\|_{L^1} = o(n)$ *as* $n \to \infty$,

(ii) $\left\|\frac{E_{-n}(x)}{2\sin x}\right\|_{L^1} = o(n)$ *as* $n \to \infty$,

(iii) $\left\|\frac{e^{inx}}{2\sin x}\right\|_{L^1} = o(\log n)$ *as* $n \to \infty$.

Lemma 9. *Let $r$ be a non-negative integer and $0 < \varepsilon < \pi$. Then there exists $M_{r\varepsilon} > 0$ such that for all $\varepsilon \le |x| \le \pi$ and all $n \ge 1$,*

(i) $|\overline{E}'_n(x)| \le \frac{M_{r\varepsilon} n^2}{|x|}$,

(ii) $|\overline{E}'_{-n}(x)| \le \frac{M_{r\varepsilon} n^2}{|x|}$,

*where* $\overline{E}_n(x) = \sum_{m=1}^{n} E_m(x)$.

*Proof.* (i) Under conditions of this Lemma and Lemma 7 we have

$$\begin{aligned} |\overline{E}'_n(x)| &\le \sum_{m=1}^{n} |E'_m(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^{n} m \\ &= \frac{M_{r\varepsilon}}{|x|} \cdot \frac{n(n+1)}{2} \le \frac{M_{r\varepsilon} n^2}{|x|}, \end{aligned}$$

for $0 < \varepsilon \le |x| \le \pi$.

(ii) Similarly we have obtained

$$|\overline{E}'_{-n}(x)| \le \sum_{m=1}^{n} |E'_{-m}(x)| \le \frac{M_{r\varepsilon}}{|x|} \sum_{m=1}^{n} m \le \frac{M_{r\varepsilon} n^2}{|x|},$$

for $0 < \varepsilon \le |x| \le \pi$. □

## 3. MAIN RESULTS

The following theorem presents the main result.

Theorem 10. *Let $(c_k)$ belongs to the class $\mathbb{K}^2$. Then*

$$\lim_{n\to\infty} g_n^c(x) = f(x), \quad \textit{exists for} \quad |x| \in (0, \pi], \tag{14}$$

$$f \in L^1(0, \pi] \quad \textit{and} \quad \|g_n^c - f\|_{L^1} \to 0, \ \textit{as } n \to \infty, \tag{15}$$

*and*

$$\|S_n(f) - f\|_{L^1} \to 0, \ \textit{as } n \to \infty. \tag{16}$$

*Proof.* Firstly, we will show that $f(x)$ exists on $(0, \pi]$. Indeed, it is clear that we can write

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = c_0 + \sum_{k=1}^{n} \left( \frac{c_k}{k} k e^{ikx} + \frac{c_{-k}}{k} k e^{-ikx} \right)$$

Applying twice the Abel's transformation we obtain

$$S_n(x) = c_0 - i\left[\sum_{k=1}^{n-2} \triangle^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x) + \triangle\left(\frac{c_{n-1}}{n-1}\right)\overline{E}'_n(x)\right] - i\frac{c_n}{n}E'_n(x)$$
$$+i\left[\sum_{k=1}^{n-2} \triangle^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x) + \triangle\left(\frac{c_{-(n-1)}}{n-1}\right)\overline{E}'_{-n}(x)\right] + i\frac{c_{-n}}{n}E'_{-n}(x).$$

Based on Lemmas 7 and 9 we clearly have

$$\begin{aligned}
&|S_n(x)| \le \\
&\le |c_0| + \sum_{k=1}^{n-2}\left[\left|\triangle^2\left(\frac{c_k}{k}\right)\right||\overline{E}'_k(x)| + \left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right||\overline{E}'_{-k}(x)|\right] \\
&\quad + \left[\left|\triangle\left(\frac{c_{n-1}}{n-1}\right)\right||\overline{E}'_n(x)| + \left|\triangle\left(\frac{c_{-(n-1)}}{n-1}\right)\right||\overline{E}'_{-n}(x)|\right] \\
&\quad + \frac{|c_n|}{n}|E'_n(x)| + \frac{|c_{-n}|}{n}|E'_{-n}(x)| \\
&\le |c_0| + \frac{M_{r\varepsilon}}{|x|}\left\{\sum_{k=1}^{n-2} k^2\left[\left|\triangle^2\left(\frac{c_k}{k}\right)\right| + \left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right|\right]\right. \\
&\quad \left. + n^2\left[\left|\triangle\left(\frac{c_{n-1}}{n-1}\right)\right| + \left|\triangle\left(\frac{c_{-(n-1)}}{n-1}\right)\right|\right] + |c_n| + |c_{-n}|\right\} \\
&\le |c_0| + \frac{2M_{r\varepsilon}}{|x|}\left\{\sum_{k=1}^{n-2} A_k + 2\sum_{k=n-1}^{\infty} k^2\left[\left|\triangle^2\left(\frac{c_k}{k}\right)\right| + \left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right|\right] + 2\overline{M}\right\} \\
&\le |c_0| + \frac{2M_{r\varepsilon}}{|x|}\left\{5\sum_{k=1}^{\infty} k^2 A_k + 2\overline{M}\right\} < +\infty,
\end{aligned}$$

since $(c_k) \in \mathbb{K}^2$, where $\overline{M}$ is a positive constant.

Subsequently, $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} g_n^c(x) = f(x)$ exists, because of the boundedness of the functions $\frac{e^{inx}}{\sin x}$, $\frac{E_n(x)}{\sin x}$, $\frac{E_{-n}(x)}{\sin x}$ on $(0,\pi]$, and thus (14) holds true.

Now we are going to prove (15). Indeed, for $x \neq 0$ we have

$$\begin{aligned}
f(x) - g_n^c(x) = &\sum_{k=n+1}^{\infty}\left(\frac{c_k}{k}ke^{ikx} + \frac{c_{-k}}{k}ke^{-ikx}\right) \\
&-\frac{i}{2\sin x}[c_n e^{i(n+1)x} - c_{-n}e^{-i(n+1)x} + c_{n+1}e^{inx} \\
&-c_{-(n+1)}e^{-inx} + (c_n - c_{n+2})E_n(x) + (c_{-(n+2)} - c_{-n})E_{-n}(x)].
\end{aligned}$$

Again, applying twice the Abel's transformation to the above equality we obtain

$$\begin{aligned}
&f(x)-g_n^c(x)=\\
&=-i\lim_{p\to\infty}\left\{\sum_{k=n+1}^{p-2}\left[\triangle^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x)-\triangle^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x)\right]\right.\\
&\quad+\triangle\left(\frac{c_{p-1}}{p-1}\right)\overline{E}'_{p-1}(x)-\triangle\left(\frac{c_{-(p-1)}}{p-1}\right)\overline{E}'_{-(p-1)}(x)+\frac{c_p}{p}E'_p(x)\\
&\quad-\frac{c_{-p}}{p}E'_{-p}(x)-\triangle\left(\frac{c_n}{n}\right)\overline{E}'_n(x)+\triangle\left(\frac{c_{-n}}{n}\right)\overline{E}'_{-n}(x)-\frac{c_{n+1}}{n+1}E'_n(x)\\
&\quad\left.+\frac{c_{-(n+1)}}{n+1}E'_{-n}(x)\right\}-\frac{i}{2\sin x}[c_ne^{i(n+1)x}-c_{-n}e^{-i(n+1)x}+c_{n+1}e^{inx}\\
&\quad-c_{-(n+1)}e^{-inx}+(c_n-c_{n+2})E_n(x)+(c_{-(n+2)}-c_{-n})E_{-n}(x)]\\
&=-i\left\{\sum_{k=n+1}^{\infty}\left[\triangle^2\left(\frac{c_k}{k}\right)\overline{E}'_k(x)-\triangle^2\left(\frac{c_{-k}}{k}\right)\overline{E}'_{-k}(x)\right]-\triangle\left(\frac{c_n}{n}\right)\overline{E}'_n(x)\right.\\
&\quad\left.+\triangle\left(\frac{c_{-n}}{n}\right)\overline{E}'_{-n}(x)-\frac{c_{n+1}}{n+1}E'_n(x)+\frac{c_{-(n+1)}}{n+1}E'_{-n}(x)\right\}\\
&\quad-\frac{i}{2\sin x}[c_ne^{i(n+1)x}-c_{-n}e^{-i(n+1)x}+c_{n+1}e^{inx}-c_{-(n+1)}e^{-inx}\\
&\quad+(c_n-c_{n+2})E_n(x)+(c_{-(n+2)}-c_{-n})E_{-n}(x)].
\end{aligned}$$

Hence, using Lemmas 7 and 9 we get

$$\begin{aligned}
&|f(x)-g_n^c(x)|\le\\
&\le\sum_{k=n+1}^{\infty}\left[\left|\triangle^2\left(\frac{c_k}{k}\right)\right||\overline{E}'_k(x)|+\left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right||\overline{E}'_{-k}(x)|\right]\\
&\quad+\left|\triangle\left(\frac{c_n}{n}\right)\right||\overline{E}'_n(x)|+\left|\triangle\left(\frac{c_{-n}}{n}\right)\right||\overline{E}'_{-n}(x)|+\left|\frac{c_{n+1}}{n+1}\right||E'_n(x)|\\
&\quad\left.+\left|\frac{c_{-(n+1)}}{n+1}\right||E'_{-n}(x)|\right\}+\frac{1}{2|\sin x|}[|c_n|+|c_{-n}|+|c_{n+1}|\\
&\quad+|c_{-(n+1)}|+(|c_n|+|c_{n+2}|)|E_n(x)|+(|c_{-(n+2)}|+|c_{-n}|)|E_{-n}(x)|]\\
&\le\frac{M_{r\varepsilon}}{|x|}\left\{\sum_{k=n+1}^{\infty}k^2\left[\left|\triangle^2\left(\frac{c_k}{k}\right)\right|+\left|\triangle^2\left(\frac{c_{-k}}{k}\right)\right|\right]\right.\\
&\quad\left.+n^2\left[\left|\triangle\left(\frac{c_n}{n}\right)\right|+\left|\triangle\left(\frac{c_{-n}}{n}\right)\right|\right]+(n+1)\left(|c_{n+1}|+\left|c_{-(n+1)}\right|\right)\right\}\\
&\quad+\frac{|c_n|+|c_{-n}|+|c_{n+1}|+|c_{-(n+1)}|}{2|\sin x|}
\end{aligned}$$

$$+(|c_n|+|c_{n+2}|)\left|\frac{E_n(x)}{2\sin x}\right|+(|c_{-(n+2)}|+|c_{-n}|)\left|\frac{E_{-n}(x)}{2\sin x}\right|$$

$$\leq \frac{M_{r\varepsilon}}{|x|}\left[2\sum_{k=n}^{\infty}A_k+(n+1)\left(|c_{n+1}|+\left|c_{-(n+1)}\right|\right)\right]$$

$$+\frac{|c_n|+|c_{-n}|+|c_{n+1}|+|c_{-(n+1)}|}{2|\sin x|}$$

$$+(|c_n|+|c_{n+2}|)\left|\frac{E_n(x)}{2\sin x}\right|+(|c_{-(n+2)}|+|c_{-n}|)\left|\frac{E_{-n}(x)}{2\sin x}\right|.$$

Therefore, using Lemma 8 we obtain

$$\|f-g_n^c\|_{L^1}\leq M_{r\varepsilon}\left[2\sum_{k=n}^{\infty}A_k\int_0^{\pi}\frac{dx}{|x|}+(n+1)\left(|c_{n+1}|+\left|c_{-(n+1)}\right|\right)\int_0^{\pi}\frac{dx}{|x|}\right]$$

$$+\frac{|c_n|+|c_{-n}|+|c_{n+1}|+|c_{-(n+1)}|}{2}\int_0^{\pi}\frac{dx}{|\sin x|}$$

$$+(|c_n|+|c_{n+2}|)\int_0^{\pi}\left|\frac{E_n(x)}{2\sin x}\right|dx$$

$$+(|c_{-(n+2)}|+|c_{-n}|)\int_0^{\pi}\left|\frac{E_{-n}(x)}{2\sin x}\right|dx$$

$$\leq M_{r\varepsilon}\left[2\sum_{k=n}^{\infty}A_k\log k+(n+1)\left(|c_{n+1}|+\left|c_{-(n+1)}\right|\right)o(\log n)\right]$$

$$+\frac{|c_n|+|c_{-n}|+|c_{n+1}|+|c_{-(n+1)}|}{2}o(\log n)$$

$$+(|c_n|+|c_{n+2}|)o(n)+(|c_{-(n+2)}|+|c_{-n}|)o(n).$$

Now we note that

$$\sum_{k=n}^{\infty}A_k\log k\leq\sum_{k=n}^{\infty}k^2A_k=o(1),$$

and

$$(n+1)\left|c_{\pm(n+1)}\right|\log n\leq(n+1)^3\left|\frac{c_{\pm(n+1)}}{n+1}\right|$$

$$\leq(n+1)^3\sum_{k=n+1}^{\infty}\left|\triangle\left(\frac{c_{\pm k}}{k}\right)\right|$$

$$\leq(n+1)^3\sum_{k=n+1}^{\infty}\sum_{j=k}^{\infty}\left|\triangle^2\left(\frac{c_{\pm j}}{j}\right)\right|$$

$$=(n+1)^3\sum_{j=n+1}^{\infty}(j-n)\left|\triangle^2\left(\frac{c_{\pm j}}{j}\right)\right|$$

$$\leq \sum_{j=n+1}^{\infty} j^4 \left|\triangle^2\left(\frac{c_{\pm j}}{j}\right)\right|$$
$$\leq \sum_{j=n+1}^{\infty} j^4 \frac{A_j}{j^2} = \sum_{j=n+1}^{\infty} j^2 A_j = o(1)$$

as $n \to \infty$.

Subsequently, we get

$$\|f - g_n^c\|_{L^1} = o(1) \quad \text{as} \quad n \to \infty.$$

Using the latest equality and the fact that $g_n^c(x)$ is a polynomial it follows that $f \in L^1(0, \pi]$.

Finally, we will prove (16). Namely, using some facts used above we have

$$\int_0^\pi |f(x) - S_n(x)| dx \leq$$
$$\leq \int_0^\pi |f(x) - g_n^c(x)| dx + \int_0^\pi |g_n^c(x) - S_n(x)| dx$$
$$\leq \int_0^\pi |f(x) - g_n^c(x)| dx + \int_0^\pi \Bigg| \frac{i}{2\sin x} [c_n e^{i(n+1)x} - c_{-n} e^{-i(n+1)x} + c_{n+1} e^{inx}$$
$$- c_{-(n+1)} e^{-inx} + (c_n - c_{n+2}) E_n(x) + (c_{-(n+2)} - c_{-n}) E_{-n}(x)] \Bigg| dx$$
$$\leq \int_0^\pi |f(x) - g_n^c(x)| dx + [|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|] \int_0^\pi \frac{dx}{2|\sin x|}$$
$$+ (|c_n| + |c_{n+2}|) \int_0^\pi \left| \frac{E_n(x)}{2\sin x} \right| dx + (|c_{-(n+2)}| + |c_{-n}|) \int_0^\pi \left| \frac{E_{-n}(x)}{2\sin x} \right| dx$$
$$\leq \int_0^\pi |f(x) - g_n^c(x)| dx + [|c_n| + |c_{-n}| + |c_{n+1}| + |c_{-(n+1)}|] o(\log n)$$
$$+ [|c_n| + |c_{n+2}| + |c_{-(n+2)}| + |c_{-n}|] o(n)$$
$$= o(1) + o(1) + o(1) = o(1), \ n \to \infty.$$

The proof of theorem is completed.

□

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## REFERENCES

[1] B. Ram, *Convergence of certain cosine sums in the metric space L*, Proc. Amer. Math. Soc., **66** (1977), pp. 258–260.

[2] B. Ram, *Integrability of Ress-Stanojević sums*, Acta Sci. Math., **42** (1980), pp. 153–155.

[3] C. S. Rees and Č. V. Stanojević, *Necessary and sufficient condition for intergrability of certain cosine sums*, J. Math. Anal. Appl., **43** (1973), pp. 579–586.

[4] L. Leindler, *Additions to the Telyakovskiĭ class $S$*, J. Ineq. Pure and Appl. Math., **4** (2003) Art. 35.

[5] N. K. Bary, *A treatise on trigonometric series* (London: Pergamon Press) Vol.II, 1964.

[6] S. A. Telyakovskiĭ , *On a sufficient condition of Sidon for integrability of trigonometric series*, Math. Zametki, (in Russian) **14** (1973), pp. 317–328.

[7] S. S. Bhatia and B. Ram, *The extensions of the F. Móricz's theorems*, Proc. Amer. Math. Soc. **124** (1996), no. 6, pp. 1821–904.

[8] S. Kumari and B. Ram, *$L^1$-convergence of certain trigonometric sums*, Indian J. Pure Appl. Math. **20** (9), 908–914, 1989.

[9] J. Kaur and S. S. Bhatia, *Convergence of modified complex trigonometric sums in the metric space $L$*, Lobachevskii J. Math., **31** (2010), No. 3, pp. 290–294.

[10] J. Kaur and S. S. Bhatia, *A class of $L^1$ convergence of new modified cosine sum*, Southeast Asian Bull. Math., **36** (6) (2012), pp. 831–836.

[11] K. Kaur, S. S. Bhatia and B. Ram, *Integrability and $L^1$ convergence of modified sine sums*, Georgian. Math. J. **11** (1) (2004), pp. 99–104.

[12] T. Kano, *Coefficients of some trigonometric series*, J. Fac. Sci. Shinshu Univ., **3** (1968), pp. 153–162.

[13] Xh. Z. Krasniqi, *$L^1$-convergence of a set of modified sums*, An. Univ. Oradea Fasc. Mat., Tomus **XXII** (2015), No. 1, pp. 41–46.

[14] Xh. Z. Krasniqi, *Some new modified cosine sums and $L^1$-convergence of cosine trigonometric series*, Arch. Math. (Brno), Tomus **49** (2013), pp. 43–50.

[15] Xh. Z. Krasniqi, *On $L^1$-convergence of Rees-Stanojevic's sums with coefficients from the class $K$*, Matematiche (Catania), Vol. **LXV**–Fasc. II (2010), pp. 25–32.

[16] Ž. Tomovski, *On a theorem of S. S. Bhatia and B. Ram*, Math. Commun, **8**, 187–193, 2003.


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