LOTKA-VOLTERRA SYSTEM WITH TWO DELAYS VIA WEAKLY PICARD OPERATORS

Diana Otrocol

Faculty of Mathematics and Computer Science
Babeş-Bolyai University
Str. Kogălniceanu, nr. 1, RO-400084 Cluj-Napoca, Romania
E-mail : dotrocol@math.ubbcluj.ro

Abstract. In this paper we apply the weakly Picard operators technique to study a Lotka-Volterra system with two delays.

1. Introduction

The purpose of this paper is to study the following Lotka-Volterra system with delays

\[ x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t - \tau_1), x_2(t - \tau_2)), \quad t \in [t_0, b], \quad i = 1, 2; \]  

\[ \begin{cases} 
  x_1(t) = \varphi(t), & t \in [t_0 - \tau_1, t_0], \\
  x_2(t) = \psi(t), & t \in [t_0 - \tau_2, t_0], 
\end{cases} \]  

where

(H1) \( \tau_1 \leq \tau_2, \; t_0 < b; \)
(H2) \( X \) is an ordered Banach space, \( f_i \in C([t_0, b] \times X \times X \times X, X); \)
(H3) there exists \( L_f > 0 \) such that:

\[ |f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \leq L_f \left( \sum_{k=1}^{4} |u_k - v_k| \right), \]

for all \( t \in [t_0, b], \; u_k, v_k \in \mathbb{R}, \; k = 1, 2, 3, 4, \; i = 1, 2; \)
(H4) \( \varphi \in C([t_0 - \tau_1, t_0]), \; \psi \in C([t_0 - \tau_2, t_0]). \)

Some problems concerning problem (1)+(2) were studied by Y. Muroya [2], Y. Saito, T. Hara and W. Ma [8], D. Otrocol [3, 4].

The problem (1)+(2) with \( x_1 \in C([t_0 - \tau_1, b] \cap C^1[t_0, b], \; x_2 \in C([t_0 - \tau_2, b] \cap C^1[t_0, b] \)

is equivalent with

\[ x_1(t) = \begin{cases} 
  \varphi(t), & t \in [t_0 - \tau_1, t_0], \\
  \varphi(t_0) + \int_{t_0}^{t} f_1(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2))ds, & t \in [t_0, b], 
\end{cases} \]
\( x_2(t) = \begin{cases} 
\psi(t), & t \in [t_0 - \tau_1, t_0], \\
\psi(t_0) + \int_{t_0}^{t} f_2(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) \, ds, & t \in [t_0, b], 
\end{cases} \)

where \( x_1 \in C[t_0 - \tau_1, b] \) and \( x_2 \in C[t_0 - \tau_2, b] \).

The system (1) is equivalent with

\( x_1(t) = \begin{cases} 
x_1(t_0), & t \in [t_0 - \tau_1, t_0], \\
x_1(t_0) + \int_{t_0}^{t} f_1(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) \, ds, & t \in [t_0, b], 
\end{cases} \)

\( x_2(t) = \begin{cases} 
x_2(t_0), & t \in [t_0 - \tau_2, t_0], \\
x_2(t_0) + \int_{t_0}^{t} f_2(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) \, ds, & t \in [t_0, b]. 
\end{cases} \)

Consider the following operators

\( A_f, B_f : C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b] \to C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b], \)

where \( A_f(x_1, x_2) \) is defined by the second part of (3) and \( B_f(x_1, x_2) \) is defined by the second part of (4).

**Remark 1.** Let \( \varphi \in C([t_0 - \tau_1, t_0], X) \) and \( \psi \in C([t_0 - \tau_2, t_0], X) \). Then we consider \( X_\varphi := \{ x_1 \in C([t_0 - \tau_1, b], X) \mid x_1|_{[t_0-\tau_1,t_0]} = \varphi \} \), \( X_\psi := \{ x_2 \in C([t_0 - \tau_2, b], X) \mid x_2|_{[t_0-\tau_2,t_0]} = \psi \} \).

We remark that \( X = \bigcup_{\varphi, \psi} X_\varphi \times X_\psi \) is a partition of \( X \) and \( X_\varphi \times X_\psi \) is an invariant subset of \( A_f \) and of \( B_f \) for all \( \varphi \in C([t_0 - \tau_1, t_0]) \) and \( \psi \in C([t_0 - \tau_2, t_0]) \).

This remark suggests us to consider the theory of weakly Picard operators.

### 2. Weakly Picard operators

Ioan A. Rus introduced the Picard operators class (PO) and the weakly Picard operators class (WPO) for the operators defined on a metric space and he gave basic notations, definitions and many results in this field in many papers [5-7].

In what follows we shall consider some of these results that are useful in our paper.

Let \( (X, d) \) be a metric space and \( A : X \to X \) be an operator. We denote

\[
\begin{align*}
P(X) & := \{ Y \subset X \mid Y \neq \emptyset \}; \\
F_A & := \{ x \in X \mid A(x) = x \} - \text{the fixed point set of } A; \\
I(A) & := \{ Y \in P(X) \mid A(Y) \subset Y \}; \\
A^{n+1} & := A \circ A^n, \quad A^0 = 1_X, \quad A^1 = A, \quad n \in \mathbb{N}.
\end{align*}
\]
Definition 1. The operator $A$ is a Picard operator (PO) if there exists $x^* \in X$ such that:
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges to $x^*$ for all $x_0 \in X$.

Definition 2. The operator $A$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on $x$) is a fixed point of $A$.

Definition 3. If $A$ is WPO, then we consider the operator $A^\infty : X \to X$, defined by
$$A^\infty(x) := \lim_{n \to \infty} A^n(x).$$

Remark 2. $A^\infty(X) = F_A$.

Remark 3. If $A$ is a WPO and $F_A = \{x^*\}$, then by definition the operator $A$ is a PO.

Remark 4. If $A$ is a PO, then
$$F_{A^n} = F_A = \{x^*\}, \text{ for all } n \in \mathbb{N}.$$

Remark 5. If $A$ is a WPO, then
$$F_{A^n} = F_A \neq \emptyset, \text{ for all } n \in \mathbb{N}.$$

Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We have

Lemma 1. Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator such that:
(i) $A$ is monotone increasing;
(ii) $A$ is WPO.
Then the operator $A^\infty$ is monotone increasing.

Lemma 2. (Abstract comparison lemma) Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ such that:
(i) $A \leq B \leq C$;
(ii) the operators $A, B, C$ are WPOs;
(iii) the operator $B$ is monotone increasing.
Then
$$x \leq y \leq z \implies A^\infty(x) \leq A^\infty(y) \leq A^\infty(z).$$

Remark 6. Let $A, B, C$ as in the Lemma 2. Moreover, we suppose that $F_B = \{x_B^*\}$ i.e., $B$ is a Picard operator. Then we have
$$A^\infty(x) \leq x_B^* \leq C^\infty(x), \forall x \in X.$$ But $A^\infty(X) = F_A$, $C^\infty(X) = F_C$. Thus we have
$$F_A \leq x_B^* \leq F_C.$$
Lemma 3. (Abstract Gronwall lemma) Let \((X,d,\leq)\) be an ordered metric space and \(A : X \to X\) an operator. We suppose that:

(i) \(A\) is a Picard operator;
(ii) \(A\) is monotone increasing.

If we denote by \(x^*_A\), the unique fixed point of \(A\), then

(a) \(x \leq A(x) \implies x \leq x^*_A\);
(b) \(x \geq A(x) \implies x \geq x^*_A\).

3. Main results

Consider the problem (1)+(2). We have

Theorem 4. [4, Theorem 1] We suppose that:

(a) the conditions \((H_1)-(H_4)\) are satisfied,
(b) \(4L + 1 < 1\).

Then the problem (1)+(2) has a unique solution. Moreover, if \((x_1^*, x_2^*)\) is the unique solution of (1)+(2), then

\[
(x_1^*, x_2^*) = \lim_{n \to \infty} A^n_f(x_1, x_2), \text{ for all } x_1 \in C[t_0 - \tau_1, b], x_2 \in C[t_0 - \tau_2, b].
\]

Remark 7. From Theorem 4 it follows that the operator \(A_f |_{X_\varphi \times X_\psi} : X_\varphi \times X_\psi \to X_\varphi \times X_\psi\) is PO. But

\[
A_f |_{X_\varphi \times X_\psi} = B_f |_{X_\varphi \times X_\psi},
\]

and

\[
X := \bigcup_{\varphi, \psi} X_\varphi \times X_\psi, \quad X_\varphi \times X_\psi \in I(A_f), \quad X_\varphi \times X_\psi \in I(B_f).
\]

So, the operator \(B_f\) is WPO.

Theorem 5. We suppose that

(a) the conditions \((H_1)-(H_4)\) are satisfied,
(b) \(f_i(t, \cdot, \cdot, \cdot)\) is monotone increasing for all \(t \in [t_0, b]\), \(i = 1, 2\).

Let \((x_1^1, x_2^1), (x_1^2, x_2^2)\) be two solutions of the equation (1). If \(x_1^1(t) \leq x_1^2(t)\) with \(t \in [t_0 - \tau_1, b]\) and \(x_2^1(t) \leq x_2^2(t)\) with \(t \in [t_0 - \tau_2, b]\), then \(x_1^1 \leq x_1^2\) and \(x_2^1 \leq x_2^2\).

Proof. From Remark 7 we have that \(B_f\) is WPO.

Let \((x_1^1, x_2^1), (x_1^2, x_2^2)\) be two solutions of the equations (1), i.e., two fixed points of \(B_f\).

We suppose that

\[
x_1^1 \big|_{[t_0 - \tau_1, t_0]} \leq x_1^2 \big|_{[t_0 - \tau_1, t_0]},
\]
\[
x_2^1 \big|_{[t_0 - \tau_2, t_0]} \leq x_2^2 \big|_{[t_0 - \tau_2, t_0]}.
\]

Then there exist \(\tilde{x}_1^1 \in X_{x_1^1|_{[t_0 - \tau_1, t_0]}}\) and \(\tilde{x}_2^1 \in X_{x_2^1|_{[t_0 - \tau_2, t_0]}}\) such as \((\tilde{x}_1^1, \tilde{x}_2^1) \leq (\tilde{x}_1^2, \tilde{x}_2^2)\).

It is clear that \((x_1^1, x_2^1) = B_f^\varphi(x_1^1, x_2^1)\).
From the condition (b), the operator $B_f$ is monotone increasing. By Lemma 1 we have that

$$(x_1^1, x_1^2) \leq (x_2^1, x_2^2) \implies B_f^\infty(x_1^1, x_1^2) \leq B_f^\infty(x_2^1, x_2^2).$$

So, $(x_1^1, x_1^2) \leq (x_2^1, x_2^2)$.

**Theorem 6.** Consider the following differential equations

$$x'_i(t) = f_i^j(t, x_i(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)), \; t \in [t_0, b], \; i = 1, 2, j = 1, 2, 3. \; (5)$$

We suppose that

(a) $f_i^j$ satisfies the conditions $(H_1)$-$H_3), \; i = 1, 2, j = 1, 2, 3$ and $f_i^1 \leq f_i^2 \leq f_i^3$,

(b) $f_i^j(t, \cdot, \cdot, \cdot, \cdot, \cdot)$ is monotone increasing.

Let $(x_1^1, x_1^2)$ be a solution of the system $(5)$, $j = 1, 2, 3.$ If $x_1^1(t) \leq x_1^2(t) \leq x_1^3(t)$ with $t \in [t_0-\tau_1, b]$ and $x_2^1(t) \leq x_2^2(t) \leq x_2^3(t)$ with $t \in [t_0-\tau_2, b]$, then $x_1^1 \leq x_1^2 \leq x_1^3$ and $x_2^1 \leq x_2^2 \leq x_2^3$.

**Proof.** We consider the operators $B_f^j$, $j = 1, 2, 3.$ These operators are WPOs. Condition (b) implies that the operator $B_f^j$ is monotone increasing.

We suppose that

$$x_1^1 |_{[t_0-\tau_1, t_0]} \leq x_1^2 |_{[t_0-\tau_1, t_0]} \leq x_1^3 |_{[t_0-\tau_1, t_0]};$$

$$x_2^1 |_{[t_0-\tau_2, t_0]} \leq x_2^2 |_{[t_0-\tau_2, t_0]} \leq x_2^3 |_{[t_0-\tau_2, t_0]}.$$ 

Then there exist $\tilde{x}_1^1 \in X_{x_1^1 |_{[t_0-\tau_1, t_0]}}$ and $\tilde{x}_2^1 \in X_{x_2^1 |_{[t_0-\tau_2, t_0]}}$ such as $(\tilde{x}_1^1, \tilde{x}_1^2) \leq (\tilde{x}_2^1, \tilde{x}_2^2) \leq (x_1^1, x_1^2).$ It is clear that $(\tilde{x}_1^1, \tilde{x}_1^2) = B_f^\infty(\tilde{x}_2^1, \tilde{x}_2^2), \; j = 1, 2, 3.$

It follows from Lemma 2 that $B_f^1(x_1^1, x_1^2) \leq B_f^2(x_1^1, x_1^2) \leq B_f^3(x_1^1, x_1^2)$.

From the abstract Gronwall lemma we have

**Theorem 7.** We suppose that

(a) the conditions $(H_1)$-$H_4)$ are satisfied,

(b) $f_i(t, \cdot, \cdot, \cdot, \cdot, \cdot)$ is monotone increasing for all $t \in [t_0, b], \; i = 1, 2$.

Let $(x_1, x_2)$ be a solution of the system $(1)$ and $(y_1, y_2)$ a solution of the inequality

$$y'_i(t) \leq f_i(t, y_1(t), y_2(t), y_1(t-\tau_1), y_2(t-\tau_2)), \; t \in [t_0, b], \; i = 1, 2.$$ 

Then

$$y_1 |_{[t_0-\tau_1, t_0]} \leq x_1 |_{[t_0-\tau_1, t_0]} \implies y_1 \leq x_1,$$

$$y_2 |_{[t_0-\tau_2, t_0]} \leq x_2 |_{[t_0-\tau_2, t_0]} \implies y_2 \leq x_2.$$
**Proof.** In the terms of the operator $B_f$, we have

$$(x_1, x_2) = B_f(x_1, x_2) \text{ and } (y_1, y_2) \leq B_f(y_1, y_2)$$

On the other hand, from the condition (b), the operator $B_f^\infty$ is monotone increasing and we have

$$(y_1, y_2) \leq B_f^\infty(y_1, y_2) = B_f^\infty\left(\tilde{y}_1 \mid_{[t_0-\tau_1, t_0]}, \tilde{y}_2 \mid_{[t_0-\tau_2, t_0]}\right)$$

$$\leq B_f^\infty\left(\tilde{x}_1 \mid_{[t_0-\tau_1, t_0]}, \tilde{x}_2 \mid_{[t_0-\tau_2, t_0]}\right) = (x_1, x_2).$$

So, $(y_1, y_2) \leq (x_1, x_2)$.

**Example 1.** (D. Otrocol, [3]) Consider the following system with two delays:

$$\begin{align*}
    x_1'(t) & = x_1(t-2) + x_2(t-5) \\
    x_2'(t) & = x_1(t-2) - x_2(t-5), \quad t \in [0, 5]
\end{align*}$$

with initial condition

$$\begin{align*}
    x_1(t) & = 1, \quad t \in [-2, 0] \\
    x_2(t) & = 0, \quad t \in [-5, 0].
\end{align*}$$

The exact solution for this problem is

$$(x_1(t), x_2(t)) = \begin{cases} 
    (t + 1, t), & \text{if } t \in [0, 2] \\
    \left(\frac{t^2}{2} - t + 3, \frac{t^2}{2} - t + 2\right), & \text{if } t \in [2, 4] \\
    \left(\frac{(t-2)^3}{6} - \frac{(t-2)^2}{2} + 3t - 26, \frac{(t-2)^3}{6} - \frac{(t-2)^2}{2} + 3t - \frac{16}{3}\right), & \text{if } t \in [4, 5].
\end{cases}$$

Let $(y_1, y_2)$ be a solution of the inequalities

$$\begin{align*}
    y_1'(t) & \leq y_1(t-2) + y_2(t-5) \\
    y_2'(t) & \leq y_1(t-2) - y_2(t-5), \quad t \in [0, 5].
\end{align*}$$

Then, it follows from Theorem 7 that

$$\begin{align*}
    y_1 \mid_{[-2,0]} & \leq x_1 \mid_{[-2,0]} \Rightarrow y_1 \leq x_1, \\
    y_2 \mid_{[-5,0]} & \leq x_2 \mid_{[-5,0]} \Rightarrow y_2 \leq x_2.
\end{align*}$$
References


