Nonlinear Analysis Forum **10**(2), pp. 1–0, 2005

LOTKA-VOLTERRA SYSTEM WITH TWO DELAYS VIA WEAKLY PICARD OPERATORS

Diana Otrocol

Faculty of Mathematics and Computer Science
Babeş-Bolyai University
Str. Kogălniceanu, nr. 1, RO-400084 Cluj-Napoca, Romania
E-mail : dotrocol@math.ubbcluj.ro

Abstract. In this paper we apply the weakly Picard operators technique to study a Lotka-Volterra system with two delays.

1. Introduction

The purpose of this paper is to study the following Lotka-Volterra system with delays

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1, 2; \tag{1}$$

$$\begin{cases} x_1(t) = \varphi(t),\ t \in [t_0 - \tau_1, t_0], \\ x_2(t) = \psi(t),\ t \in [t_0 - \tau_2, t_0], \end{cases} \tag{2}$$

where

(H$_1$) $\tau_1 \le \tau_2,\ t_0 < b$;
(H$_2$) X is an ordered Banach space, $f_i \in C([t_0, b] \times X \times X \times X \times X, X)$;
(H$_3$) there exists $L_f > 0$ such that:

$$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \le L_f \left(\sum_{k=1}^{4} |u_k - v_k| \right),$$

for all $t \in [t_0, b],\ u_k, v_k \in \mathbb{R},\ k = 1, 2, 3, 4,\ i = 1, 2$;
(H$_4$) $\varphi \in C[t_0 - \tau_1, t_0],\ \psi \in C[t_0 - \tau_2, t_0]$.

Some problems concerning problem (1)+(2) were studied by Y. Muroya [2], Y. Saito, T. Hara and W. Ma [8], D. Otrocol [3, 4].

The problem (1)+(2) with $x_1 \in C[t_0 - \tau_1, b] \cap C^1[t_0, b],\ x_2 \in C[t_0 - \tau_2, b] \cap C^1[t_0, b]$ is equivalent with

$$x_1(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau_1, t_0], \\ \varphi(t_0) + \displaystyle\int_{t_0}^{t} f_1(s, x_1(s), x_2(s), x_1(s-\tau_1), x_2(s-\tau_2))\mathrm{d}s, & t \in [t_0, b], \end{cases}$$

Received and Accepted Apr. 2005.
2000 Mathematics Subject Classification: Primary 34L05; Secondary 47H10.
Key words and phrases: Picard operators, weakly Picard operators, Lotka-Volterra system, fixed points, delay.

$$x_2(t)=\begin{cases}\psi(t), & t\in[t_0-\tau_2,t_0],\\ \psi(t_0)+\displaystyle\int_{t_0}^{t} f_2(s,x_1(s),x_2(s),x_1(s-\tau_1),x_2(s-\tau_2))\mathrm{ds}, & t\in[t_0,b],\end{cases} \tag{3}$$

where $x_1\in C[t_0-\tau_1,b]$ and $x_2\in C[t_0-\tau_2,b]$.

The system (1) is equivalent with

$$x_1(t)=\begin{cases}x_1(t), & t\in[t_0-\tau_1,t_0],\\ x_1(t_0)+\displaystyle\int_{t_0}^{t} f_1(s,x_1(x),x_2(s),x_1(s-\tau_1),x_2(s-\tau_2))\mathrm{ds}, & t\in[t_0,b],\end{cases}$$

$$x_2(t)=\begin{cases}x_2(t), & t\in[t_0-\tau_2,t_0],\\ x_2(t_0)+\displaystyle\int_{t_0}^{t} f_2(s,x_1(x),x_2(s),x_1(s-\tau_1),x_2(s-\tau_2))\mathrm{ds}, & t\in[t_0,b].\end{cases} \tag{4}$$

Consider the following operators

$$A_f,B_f: C[t_0-\tau_1,b]\times C[t_0-\tau_2,b]\to C[t_0-\tau_1,b]\times C[t_0-\tau_2,b],$$

where $A_f(x_1,x_2)$ is defined by the second part of (3) and $B_f(x_1,x_2)$ is defined by the second part of (4).

Remark 1. Let $\varphi\in C([t_0-\tau_1,t_0],X)$ and $\psi\in C([t_0-\tau_2,t_0],X)$. Then we consider $X_\varphi:=\{x_1\in C([t_0-\tau_1,b],X)\mid x_1\mid_{[t_0-\tau_1,t_0]}=\varphi\}$, $X_\psi:=\{x_2\in C([t_0-\tau_2,b],X)\mid x_2\mid_{[t_0-\tau_2,t_0]}=\psi\}$.
We remark that $X=\bigcup\limits_{\varphi,\psi} X_\varphi\times X_\psi$ is a partition of X and $X_\varphi\times X_\psi$ is an invariant subset of A_f and of B_f for all $\varphi\in C([t_0-\tau_1,t_0])$ and $\psi\in C([t_0-\tau_2,t_0])$.

This remark suggests us to consider the theory of weakly Picard operators.

2. Weakly Picard operators

Ioan A. Rus introduced the Picard operators class (PO) and the weakly Picard operators class (WPO) for the operators defined on a metric space and he gave basic notations, definitions and many results in this field in many papers [5-7].

In what follows we shall consider some of these results that are useful in our paper.

Let (X,d) be a metric space and $A:X\to X$ be an operator. We denote
$P(X):=\{Y\subset X\mid Y\neq\emptyset\}$;
$F_A:=\{x\in X\mid A(x)=x\}$ - the fixed point set of A;
$I(A):=\{Y\in P(X)\mid A(Y)\subset Y\}$;
$A^{n+1}:=A\circ A^n,\ A^0=1_X,\ A^1=A,\ n\in\mathbb{N}$.

Definition 1. The operator A is a Picard operator (PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;

(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2. The operator A is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 3. If A is WPO, then we consider the operator A^∞, $A^\infty : X \to X$, defined by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2. $A^\infty(X) = F_A$.

Remark 3. If A is a WPO and $F_A = \{x^*\}$, then by definition the operator A is a PO.

Remark 4. If A is a PO, then

$$F_{A^n} = F_A = \{x^*\}, \text{ for all } n \in \mathbb{N}^*.$$

Remark 5. If A is a WPO, then

$$F_{A^n} = F_A \neq \emptyset, \text{ for all } n \in \mathbb{N}^*.$$

Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator.

We have

Lemma 1. *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator such that:*

(i) *A is monotone increasing;*

(ii) *A is WPO.*

Then the operator A^∞ is monotone increasing.

Lemma 2. (Abstract comparison lemma) *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C, : X \to X$ such that:*

(i) *$A \leq B \leq C$;*

(ii) *the operators A, B, C are WPOs;*

(iii) *the operator B is monotone increasing.*

Then

$$x \leq y \leq z \Longrightarrow A^\infty(x) \leq A^\infty(y) \leq A^\infty(z).$$

Remark 6. Let A, B, C as in the Lemma 2. Moreover, we suppose that $F_B = \{x_B^*\}$,i.e., B is a Picard operator. Then we have

$$A^\infty(x) \leq x_B^* \leq C^\infty(x), \ \forall x \in X.$$

But $A^\infty(X) = F_A$, $C^\infty(X) = F_C$. Thus we have

$$F_A \leq x_B^* \leq F_C.$$

Lemma 3. (Abstract Gronwall lemma) *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) *A is a Picard operator;*

(ii) *A is monotone increasing.*

If we denote by x_A^*, the unique fixed point of A, then

(a) $x \leq A(x) \Longrightarrow x \leq x_A^*$;

(b) $x \geq A(x) \Longrightarrow x \geq x_A^*$.

3. Main results

Consider the problem (1)+(2). We have

Theorem 4. [4, Theorem 1] *We suppose that:*

(a) *the conditions (H_1)-(H_4) are satisfied,*

(b) $\dfrac{4L}{4L+1} < 1$.

Then the problem (1)+(2) has a unique solution. Moreover, if (x_1^*, x_2^*) is the unique solution of (1)+(2), then

$$(x_1^*, x_2^*) = \lim_{n\to\infty} A_f^n(x_1, x_2), \text{ for all } x_1 \in C[t_0 - \tau_1, b],\, x_2 \in C[t_0 - \tau_2, b].$$

Remark 7. From Theorem 4 it follows that the operator $A_f \mid_{X_\varphi \times X_\psi} : X_\varphi \times X_\psi \to X_\varphi \times X_\psi$ is PO. But

$$A_f \mid_{X_\varphi \times X_\psi} = B_f \mid_{X_\varphi \times X_\psi},$$

and

$$X := \bigcup_{\varphi,\psi} X_\varphi \times X_\psi,\ X_\varphi \times X_\psi \in I(A_f),\ X_\varphi \times X_\psi \in I(B_f).$$

So, the operator B_f is WPO.

Theorem 5. *We suppose that*

(a) *the conditions (H_1)-(H_4) are satisfied,*

(b) *$f_i(t, \cdot, \cdot, \cdot, \cdot)$ is monotone increasing for all $t \in [t_0, b]$, $i = 1, 2$.*

Let $(x_1^1, x_2^1), (x_1^2, x_2^2)$ be two solutions of the equation (1). If $x_1^1(t) \leq x_1^2(t)$ with $t \in [t_0 - \tau_1, b]$ and $x_2^1(t) \leq x_2^2(t)$ with $t \in [t_0 - \tau_2, b]$, then $x_1^1 \leq x_1^2$ and $x_2^1 \leq x_2^2$.

Proof. From Remark 7 we have that B_f is WPO.

Let $(x_1^1, x_2^1), (x_1^2, x_2^2)$ be two solutions of the equations (1), i.e., two fixed points of B_f.

We suppose that

$$x_1^1 \mid_{[t_0-\tau_1, t_0]} \leq x_1^2 \mid_{[t_0-\tau_1, t_0]}$$
$$x_2^1 \mid_{[t_0-\tau_2, t_0]} \leq x_2^2 \mid_{[t_0-\tau_2, t_0]}.$$

Then there exist $\widetilde{x_1^i} \in X_{x_1^i|_{[t_0-\tau_1,t_0]}}$ and $\widetilde{x_2^i} \in X_{x_2^i|_{[t_0-\tau_2,t_0]}}$ such as $(\widetilde{x_1^1}, \widetilde{x_2^1}) \leq (\widetilde{x_1^2}, \widetilde{x_2^2})$.

It is clear that $(x_1^i, x_2^i) = B_f^\infty(\widetilde{x_1^i}, \widetilde{x_2^i})$.

From the condition (b), the operator B_f is monotone increasing. By Lemma 1 we have that

$$(\widetilde{x_1^1},\widetilde{x_2^1}) \le (\widetilde{x_1^2},\widetilde{x_2^2}) \Longrightarrow B_f^{\infty}(\widetilde{x_1^1},\widetilde{x_2^1}) \le B_f^{\infty}(\widetilde{x_1^2},\widetilde{x_2^2}).$$

So, $(x_1^1, x_2^1) \le (x_1^2, x_2^2)$.

Theorem 6. *Consider the following differential equations*

$$x_i'(t) = f_i^j(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1, 2, j = 1, 2, 3. \quad (5)$$

We suppose that

(a) f_i^j *satisfies the conditions* (H_1)*-*(H_3)*,* $i = 1, 2,\ j = 1, 2, 3$ *and* $f_1^1 \le f_1^2 \le f_1^3$*,* $f_2^1 \le f_2^2 \le f_2^3$*,*

(b) $f_i^2(t, \cdot, \cdot, \cdot, \cdot)$ *is monotone increasing.*

Let (x_1^j, x_2^j) *be a solution of the system (5),* $j = 1, 2, 3$*. If* $x_1^1(t) \le x_1^2(t) \le x_1^3(t)$ *with* $t \in [t_0 - \tau_1, b]$ *and* $x_2^1(t) \le x_2^2(t) \le x_2^3(t)$ *with* $t \in [t_0 - \tau_2, b]$*, then* $x_1^1 \le x_1^2 \le x_1^3$ *and* $x_2^1 \le x_2^2 \le x_2^3$*.*

Proof. We consider the operators $B_f^j,\ j = 1, 2, 3$. These operators are WPOs. Condition (b) implies that the operator B_f^2 is monotone increasing.

We suppose that

$$x_1^1 |_{[t_0-\tau_1,t_0]} \le x_1^2 |_{[t_0-\tau_1,t_0]} \le x_1^3 |_{[t_0-\tau_1,t_0]},$$
$$x_2^1 |_{[t_0-\tau_2,t_0]} \le x_2^2 |_{[t_0-\tau_2,t_0]} \le x_2^3 |_{[t_0-\tau_1,t_0]}\ .$$

Then there exist $\widetilde{x_1^j} \in X_{x_1^j|_{[t_0-\tau_1,t_0]}}$ and $\widetilde{x_2^j} \in X_{x_2^j|_{[t_0-\tau_2,t_0]}}$ such as $(\widetilde{x_1^1},\widetilde{x_2^1}) \le (\widetilde{x_1^2},\widetilde{x_2^2}) \le (\widetilde{x_1^3},\widetilde{x_2^3})$. It is clear that $(x_1^j, x_2^j) = B_f^{j\infty}(\widetilde{x_1^j},\widetilde{x_2^j}),\ j = 1, 2, 3$.

It follows from Lemma 2 that $B_f^{1\infty}(\widetilde{x_1^1},\widetilde{x_2^1}) \le B_f^{2\infty}(\widetilde{x_1^2},\widetilde{x_2^2}) \le B_f^{3\infty}(\widetilde{x_1^3},\widetilde{x_2^3})$. So, $(x_1^1, x_2^1) \le (x_1^2, x_2^2) \le (x_1^3, x_2^3)$.

From the abstract Gronwall lemma we have

Theorem 7. *We suppose that*

(a) *the conditions* (H_1)*-*(H_4) *are satisfied,*

(b) $f_i(t, \cdot, \cdot, \cdot, \cdot)$ *is monotone increasing for all* $t \in [t_0, b],\ i = 1, 2$*.*

Let (x_1, x_2) *be a solution of the system (1) and* (y_1, y_2) *a solution of the inequality*

$$y_i'(t) \le f_i(t, y_1(t), y_2(t), y_1(t-\tau_1), y_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1, 2.$$

Then

$$y_1 |_{[t_0-\tau_1,t_0]} \le x_1 |_{[t_0-\tau_1,t_0]} \Longrightarrow y_1 \le x_1,$$
$$y_2 |_{[t_0-\tau_2,t_0]} \le x_2 |_{[t_0-\tau_2,t_0]} \Longrightarrow y_2 \le x_2.$$

Proof. In the terms of the operator B_f, we have

$$(x_1, x_2) = B_f(x_1, x_2) \text{ and } (y_1, y_2) \leq B_f(y_1, y_2)$$

On the other hand, from the condition (b), the operator B_f^∞ is monotone increasing and we have

$$\begin{aligned}(y_1, y_2) &\leq B_f^\infty(y_1, y_2) = B_f^\infty(\widetilde{y_1} \mid_{[t_0-\tau_1, t_0]}, \widetilde{y_2} \mid_{[t_0-\tau_2, t_0]}) \\ &\leq B_f^\infty(\widetilde{x_1} \mid_{[t_0-\tau_1, t_0]}, \widetilde{x_2} \mid_{[t_0-\tau_2, t_0]}) = (x_1, x_2).\end{aligned}$$

So, $(y_1, y_2) \leq (x_1, x_2)$.

Example 1. (D. Otrocol, [3]) Consider the following system with two delays:

$$\begin{cases} x_1'(t) = x_1(t-2) + x_2(t-5) \\ x_2'(t) = x_1(t-2) - x_2(t-5) \end{cases}, \; t \in [0, 5]$$

with initial condition

$$\begin{cases} x_1(t) = 1, & t \in [-2, 0] \\ x_2(t) = 0, & t \in [-5, 0]. \end{cases}$$

The exact solution for this problem is

$$(x_1(t), x_2(t)) = \begin{cases} (t+1, t), & t \in [0, 2] \\ \left(\dfrac{t^2}{2} - t + 3, \dfrac{t^2}{2} - t + 2\right), & t \in [2, 4] \\ \left(\dfrac{(t-2)^3}{6} - \dfrac{(t-2)^2}{2} + 3t - 26, \dfrac{(t-2)^3}{6} - \dfrac{(t-2)^2}{2} + 3t - \dfrac{16}{3}\right), & t \in [4, 5]. \end{cases}$$

Let (y_1, y_2) be a solution of the inegalities

$$\begin{cases} y_1'(t) \leq y_1(t-2) + y_2(t-5) \\ y_2'(t) \leq y_1(t-2) - y_2(t-5) \end{cases}, \; t \in [0, 5].$$

Then, it follows from Theorem 7 that

$$\begin{aligned} y_1 \mid_{[-2,0]} \leq x_1 \mid_{[-2,0]} &\Rightarrow y_1 \leq x_1, \\ y_2 \mid_{[-5,0]} \leq x_2 \mid_{[-5,0]} &\Rightarrow y_2 \leq x_2. \end{aligned}$$

References

[1] V. Muresan, *Ecuatii Diferentiale cu Modificarea Afina a Argumentului*, Transilvania Press, Cluj Napoca, 1997.

[2] Y. Muroya, *Uniform persistence for Lotka-Volterra-type delay differential systems*, Nonlinear Analysis **4** (2003), 689–710.

[3] D. Otrocol, *A numerical method for approximating the solution of a Lotka-Volterra system with two delays* (to appear).

[4] ———, *Data dependence for the solution of a Lotka-Volterra system with two delays* (to appear).

[5] I. A. Rus, *Weakly Picard operators and applications*, Seminar of Fixed Point Theory Cluj Napoca **2** (2001), 41–58.

[6] ———, *Generalized Contractions and Applications*, Cluj University Press, 2001.

[7] ———, *Functional-differential equations of mixed type, via weakly Picard operators*, Seminar of Fixed Point Theory Cluj Napoca **3** (2002), 335–346.

[8] Y. Saito, T. Hara and W. Ma, *Necessary and sufficient conditions for permanence and global stability of a Lotka-Volterra system with two delays*, J. Math. Anal. Appl. **236** (1999), 534–556.