STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

A NUMERICAL METHOD FOR APPROXIMATING THE SOLUTION OF A LOTKA-VOLTERRA SYSTEM WITH TWO DELAYS

DIANA OTROCOL

Abstract. In this paper, using the step method, we established the existence and uniqueness of solution for the system (1.2) with initial condition (1.3). The aim of this paper is to present a numerical method for this system.

1. The statement of the problem

Consider the following Lotka-Volterra type delay differential system:

$$\begin{cases} x_i'(t) = x_i(t)r_i(t)\left\{c_i - a_i x_i(t) - \sum\limits_{j=1}^{n}\sum\limits_{k=0}^{m} a_{ij}^k x_j(\tau_{ij}^k(t))\right\},\ t \geq t_0,\ 1 \leq i \leq n \\ x_i(t) = \phi_i(t) \geq 0,\ t \leq t_0 \text{ and } \phi_i(t_0) > 0,\ 1 \leq i \leq n \end{cases} \tag{1}$$

There have been many studies on this subject (see [2], [5], [7]). In particular, for $n = 2$, $r_i(t) \equiv 1$, $a_i = 0$ and $\tau_{ij}^k(t) = t - \tau_{ij}^k$, $1 \leq i,j \leq 2$, $0 \leq k \leq m$, the fact that time delays are harmless for the uniform persistence of solutions, is established by Wang and Ma for a predator-prey system, by Lu and Takeuchi and Takeuchi for competitive systems.

Recently, Saito, Hara and Ma [7] have derived necessary and sufficient conditions for the permanence (uniform persistence) and global stability of a symmetrical Lotka-Volterra-type predator-prey system with $a_i > 0$, $i = 1, 2$ and two delays.

For a nonautonomous competitive Lotka-Volterra system with no delays, recently Ahmad and Lazer have established the average conditions for the persistence,

Received by the editors: 19.07.2004.

2000 *Mathematics Subject Classification.* 34L05, 47H10.

Key words and phrases. Differential equation, delay, the step method.

which are weaker than those of Gopalsamy and Tineo and Alvarez for periodic or almost-periodic cases.

In this paper, using the step method [6], we established the existence and uniqueness of solution for the following system

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x(t-\tau_1), y(t-\tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x(t-\tau_1), y(t-\tau_2)) \end{cases}, \ t \in [t_0, b], \ t_0 < b \qquad (2)$$

with initial condition

$$\begin{cases} x(t) = \varphi(t), \ t \in [t_0 - \tau_1, t_0] \\ y(t) = \psi(t), \ t \in [t_0 - \tau_2, t_0] \end{cases} \qquad (3)$$

Here τ_1 and τ_2 are constants with $\tau_1 \geq 0, \ \tau_2 \geq 0, \ \tau_1 \leq \tau_2$ and φ, ψ are continuous functions.

On the basis of these results, the aim of this paper is to present a numerical method for obtaining the solutions of system (2) with initial condition (3).

2. **The existence and uniqueness of solution**

We consider the system (2) with initial condition (3) and we established the existence and uniqueness of the solution for the problem (2) + (3).

We have

$$\begin{aligned} x &\in C[t_0 - \tau_1, b] \cap C^1[t_0, b] \\ y &\in C[t_0 - \tau_2, b] \cap C^1[t_0, b] \end{aligned}$$

If we suppose that

(i) $f_i \in C([t_0, b] \times \mathbb{R}^4), \ i = 1, 2$

(ii)$|f_i(t, u_1, v_1, u, v) - f_i(t, u_2, v_2, u, v)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$

$$\forall u_1, u_2, v_1, v_2, u, v \in \mathbb{R}, \forall t \in [t_0, b]$$

(ii')$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \leq L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4|)$

$$\forall u_i, v_i \in \mathbb{R}, i = \overline{1, 4}, \forall t \in [t_0, b]$$

then the following result is given.

Theorem 1. *We consider the system (2) with initial condition (3). If the conditions (i) and (ii) are satisfied, then the problem (2)+(3) has a unique solution.*

Proof. We use the step method.

$t \in [t_0, t_0 + \tau_1]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), \varphi(t - \tau_1), \psi(t - \tau_2)) \\ y'(t) = f_2(t, x(t), y(t), \varphi(t - \tau_1), \psi(t - \tau_2)) \\ x(t_0) = \varphi(t_0) \\ y(t_0) = \psi(t_0) \end{cases}$$

So we have the Cauchy problem with f_i continuous functions, $i = 1, 2$. But $f_i(t, \cdot, \cdot, u, v) : \mathbb{R} \to \mathbb{R}$ are Lipschitz. Then it results from the Cauchy theorem that:

$$\begin{aligned} \exists!\, x_1 &\in C^1[t_0, t_0 + \tau_1] \\ \exists!\, y_1 &\in C^1[t_0, t_0 + \tau_1] \end{aligned}$$

solution of the problem (2) + (3).

$t \in [t_0 + \tau_1, t_0 + 2\tau_1]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x_1(t - \tau_1), y_1(t - \tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x_1(t - \tau_1), y_1(t - \tau_2)) \\ x(t_0 + \tau_1) = x_1(t_0 + \tau_1) \\ y(t_0 + \tau_1) = y_1(t_0 + \tau_1) \end{cases}$$

$$\begin{aligned} &\Rightarrow \quad \exists! x_2 \in C[t_0 + \tau_1, t_0 + 2\tau_1] \\ &\Rightarrow \quad \exists! y_2 \in C[t_0 + \tau_1, t_0 + 2\tau_1] \end{aligned}$$

solution of the problem (2) + (3).

$t \in [t_0 + n\tau_1, t_0 + \tau_2]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x_n(t - \tau_1), y_n(t - \tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x_n(t - \tau_1), y_n(t - \tau_2)) \\ x(t_0 + n\tau_1) = x_n(t_0 + n\tau_1) \\ y(t_0 + n\tau_1) = y_n(t_0 + n\tau_1) \end{cases}$$

$$\Rightarrow \quad \exists ! x_{n+1} \in C[t_0 + n\tau_1, t_0 + \tau_2]$$

$$\Rightarrow \quad \exists ! y_{n+1} \in C[t_0 + n\tau_1, t_0 + \tau_2]$$

So we obtained:

$$(x(t), y(t)) = \begin{cases} (x_1(t), y_1(t)), & t \in [t_0, t_0 + \tau_1] \\ (x_2(t), y_2(t)), & t \in [t_0 + \tau_1, t_0 + 2\tau_1] \\ \dots \\ (x_{n+1}(t), y_{n+1}(t)), & t \in [t_0 + n\tau_1, t_0 + \tau_2] \end{cases}$$

solution of the problem (2) + (3). □

Remark 1. *We consider the system (2) with initial condition (3). If the conditions (i) si (ii') are satisfied, then the problem (2)+(3) has a unique solution which can be obtained by the method of successive approximations.*

3. **The approximation of the solution**

We consider the system (2) with initial condition (3)

This problem is equivalent with the delayed integral Volterra equations:

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau_1, t_0] \\ \varphi(t_0) + \int_{t_0}^{t} f_1(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

$$y(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau_2, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f_2(s, x(s), y(s), x(s - \tau_1), y(s - \tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

where $f_i \in C([t_0, b] \times \mathbb{R}^4),\ i = 1, 2.$

We suppose that the hypotheses of Remark 1 are satisfied. Then the problem (2) + (3) has a unique solution

$$x \in C[t_0 - \tau_1, t_0] \cap C^1[t_0, b]$$

$$y \in C[t_0 - \tau_2, t_0] \cap C^1[t_0, b].$$

Let (α, β) be the solution, which, by virtue of Remark 1, can be obtained by successive approximation method. So, we have

$$\begin{aligned}\alpha(t) &= \varphi(t),\ t \in [t_0 - \tau_1, t_0] \\ \beta(t) &= \psi(t),\ t \in [t_0 - \tau_2, t_0]\end{aligned}$$

For $t \in [t_0, b]$ we have:

$$\begin{cases} \alpha_0(t) = \varphi(t) \\ \beta_0(t) = \psi(t) \end{cases} \tag{4}$$

$$\begin{cases} \alpha_1(t) = \varphi(t_0) + \int_{t_0}^{t} f_1(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds \\ \beta_1(t) = \psi(t_0) + \int_{t_0}^{t} f_2(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds \end{cases}$$

$$\cdots$$

$$\begin{cases} \alpha_m(t) = \varphi(t_0) + \int_{t_0}^{t} f_1(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s-\tau_1), \beta_{m-1}(s-\tau_2))ds \\ \beta_m(t) = \psi(t_0) + \int_{t_0}^{t} f_2(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s-\tau_1), \beta_{m-1}(s-\tau_2))ds \end{cases}$$

To obtain the sequence of successive approximations (4), it is necessary to calculate the integrals which appear in the right-hand side. In general, this problem is difficult. We shall use the trapezoidal rule.

Let an interval $[a, b] \subseteq \mathbb{R}$ be given, and the function $f \in C^2[a, b]$.

Divide the interval $[a, b]$ by points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

into n equal parts of length $\Delta x = \dfrac{b-a}{n}$.

Then we have the trapezoidal formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + r_n(f) \tag{5}$$

where $r_n(f)$ is the remainder of the formula.

To evaluate the approximation error of the trapezoidal formula there exists the following result.

Theorem 2. *For every function $f \in C^2[a, b]$, the remainder $r_n(f)$ from the trapezoidal formula (5), satisfies the ineguality:*

$$|r_n(f)| \leq \frac{(b-a)^3}{12n^2} = \max_{x \in [a,b]} |f''(x)| \tag{6}$$

3.1. **The calculation of the integrals which appear in the successive approximations methods.** Now we suppose that $f_i \in C([t_0, b] \times \mathbb{R}^4)$, $i = 1, 2$, and in order to calculate the integral α_m and β_m from (4), we apply the formula (5). Then we divide the interval $[t_0, b]$ by the points:

$$0 = t_0 \leq t_1 < \ldots < t_n = b \tag{7}$$

where: $t_i = t_{i-1} + h$, $h = \dfrac{t - t_0}{2^v}$, $v = 0, 1, 2, ...$, $i = \overline{1, n}$, $n = \left[\dfrac{b}{h}\right]$ ($[\cdot]$ is integer part). Thus we have

$$\begin{aligned}\alpha_m(t_k) = \int_{t_0}^{t_k} & f_1(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s - \tau_1), \beta_{m-1}(s - \tau_2))ds = \\ \frac{t - t_0}{2n} & [f_1(t_0, \alpha_{m-1}(t_0), \beta_{m-1}(t_0), \alpha_{m-1}(t_0 - \tau_1), \beta_{m-1}(t_0 - \tau_2)) + \\ & f_1(t_k, \alpha_{m-1}(t_k), \beta_{m-1}(t_k), \alpha_{m-1}(t_k - \tau_1), \beta_{m-1}(t_k - \tau_2)) + \\ & 2\sum_{i=1}^{n-1} f_1(t_i, \alpha_{m-1}(t_i), \beta_{m-1}(t_i), \alpha_{m-1}(t_i - \tau_1), \beta_{m-1}(t_i - \tau_2))] + \\ & \varphi(t_0) + r_{m,k}(f_1)\end{aligned} \tag{8}$$

where, for the remainder $r_{m,k}(f_1)$, we have the estimation:

$$|r_{m,k}(f_1)| \leq \tfrac{(t-t_0)^3}{12n^2} \max_{s \in [t_0, b]} \left| [f_1(, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s - \tau_1), \beta_{m-1}(s - \tau_2))]''_s \right|,$$

$$k = \overline{0, n}, m \in \mathbb{N}$$

We denote by $\alpha_{m-1}(s) = u$, $\beta_{m-1}(s) = v$, $\alpha_{m-1}(s - \tau_1) = w$, $\beta_{m-1}(s - \tau_2) = z$. Taking into account the fact that:

$$\begin{aligned}& [f_1(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s - \tau_1), \beta_{m-1}(s - \tau_2))]''_s = \\ & \frac{\partial^2 f_1}{\partial s^2} + \frac{\partial^2 f_1}{\partial s \partial u} u' + \frac{\partial^2 f_1}{\partial s \partial v} v' + \frac{\partial^2 f_1}{\partial s \partial w} w' + \frac{\partial^2 f_1}{\partial s \partial z} z' + \\ & \frac{\partial^2 f_1}{\partial s \partial u} u' + \frac{\partial^2 f_1}{\partial u^2} (u')^2 + \frac{\partial^2 f_1}{\partial u \partial v} u'v' + \frac{\partial^2 f_1}{\partial u \partial w} u'w' + \frac{\partial^2 f_1}{\partial u \partial z} u'z' + \\ & \frac{\partial^2 f_1}{\partial s \partial v} v' + \frac{\partial^2 f_1}{\partial u \partial v} u'v' + \frac{\partial^2 f_1}{\partial v^2} (v')^2 + \frac{\partial^2 f_1}{\partial v \partial w} v'w' + \frac{\partial^2 f_1}{\partial v \partial z} v'z' + \\ & \frac{\partial^2 f_1}{\partial s \partial w} w' + \frac{\partial^2 f_1}{\partial u \partial w} u'w' + \frac{\partial^2 f_1}{\partial v \partial w} v'w' + \frac{\partial^2 f_1}{\partial w^2} (w')^2 + \frac{\partial^2 f_1}{\partial w \partial z} w'z' + \\ & \frac{\partial^2 f_1}{\partial s \partial z} z' + \frac{\partial^2 f_1}{\partial u \partial z} u'z' + \frac{\partial^2 f_1}{\partial v \partial z} v'z' + \frac{\partial^2 f_1}{\partial w \partial z} w'z' + \frac{\partial^2 f_1}{\partial z^2} (z')^2 +\end{aligned}$$

$$\frac{\partial f_1}{\partial u}u'' + \frac{\partial f_1}{\partial v}v'' + \frac{\partial f_1}{\partial w}w'' + \frac{\partial f_1}{\partial z}z'' =$$
$$\frac{\partial^2 f_1}{\partial s^2} + 2\frac{\partial^2 f_1}{\partial s \partial u}u' + 2\frac{\partial^2 f_1}{\partial s \partial v}v' + 2\frac{\partial^2 f_1}{\partial s \partial w}w' + 2\frac{\partial^2 f_1}{\partial s \partial z}z' +$$
$$\frac{\partial^2 f_1}{\partial u^2}(u')^2 + \frac{\partial^2 f_1}{\partial v^2}(v')^2 + \frac{\partial^2 f_1}{\partial w^2}(w')^2 + \frac{\partial^2 f_1}{\partial z^2}(z')^2 + 2\frac{\partial^2 f_1}{\partial u \partial v}u'v' +$$
$$2\frac{\partial^2 f_1}{\partial u \partial w}u'w' + 2\frac{\partial^2 f_1}{\partial u \partial z}u'z' + 2\frac{\partial^2 f_1}{\partial v \partial w}v'w' + \frac{\partial^2 f_1}{\partial v \partial z}v'z' +$$
$$2\frac{\partial^2 f_1}{\partial w \partial z}w'z' + \frac{\partial f_1}{\partial u}u'' + \frac{\partial f_1}{\partial v}v'' + \frac{\partial f_1}{\partial w}w'' + \frac{\partial f_1}{\partial z}z''$$

and

$$\alpha_{m-1}(t) = \varphi(t_0) + \int_{t_0}^{t} f_1(s, \alpha_{m-2}(s), \beta_{m-2}(s), \alpha_{m-2}(s-\tau_1), \beta_{m-2}(s-\tau_2))ds$$
$$\alpha'_{m-1}(t) = \int_{t_0}^{t} \frac{\partial f_1(s, \alpha_{m-2}(s), \beta_{m-2}(s), \alpha_{m-2}(s-\tau_1), \beta_{m-2}(s-\tau_2))}{\partial s}ds$$
$$\alpha''_{m-1}(t) = \int_{t_0}^{t} \frac{\partial^2 f_1(s, \alpha_{m-2}(s), \beta_{m-2}(s), \alpha_{m-2}(s-\tau_1), \beta_{m-2}(s-\tau_2))}{\partial s^2}ds$$

and denoting by

$$M_0 = \max_{\substack{|\alpha| \le 2 \\ s \in [t_0, b] \\ |u|, |v|, |w|, |z| \le R}} \left| \frac{\partial^{|\alpha|} f_1(s, u, v, w, z)}{\partial s^{\alpha_1} \partial u^{\alpha_2} \partial v^{\alpha_3} \partial w^{\alpha_4} \partial z^{\alpha_5}} \right|,$$

we obtain

$$|\alpha_{m-1}(t)| \le (t-t_0)M_0; \ \left|\alpha'_{m-1}(t)\right| \le (t-t_0)M_0; \ \left|\alpha''_{m-1}(t)\right| \le (t-t_0)M_0.$$

Again from here we have:

$$[f_1(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s-\tau_1), \beta_{m-1}(s-\tau_2))]''_s \le M_1$$

where $M_1 = M_0 + 12(t-t_0)M_0^2 + 16(t-t_0)M_0^3$ and M_1does not depend on m and k.

For the remainder $r_{m,k}(f_1)$, from the formula (8) we have:

$$|r_{m,k}(f_1)| \le \frac{(t-t_0)^3}{12n^2}M_1, \ m = 0, 1, 2, ..., \ k = \overline{0, n}. \tag{9}$$

In this way we have obtained a formula for the approximative calculation of the integrals from (4).

3.2. **The approximate calculation of the terms of the successive approximations sequence.** Using the approximation (4) and the formula (8) with the remainder estimation (9), we shall present further down an algorithm for the approximate solution of system (2) with initial condition (3).

So, we have:

$$\alpha_1(t_k) = \int_{t_0}^{t_k} f_1(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \alpha_0(t_0), \beta_0(t_0), \alpha_0(t_0-\tau_1), \beta_0(t_0-\tau_2))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \alpha_0(t_i), \beta_0(t_i), \alpha_0(t_i-\tau_1), \beta_0(t_i-\tau_2))+$$

$$f_1(t_k, \alpha_0(t_k), \beta_0(t_k), \alpha_0(t_k-\tau_1), \beta_0(t_k-\tau_2))]+$$

$$\varphi(t_0) + r_{1,k}(f_1) =$$

$$\widetilde{\alpha}_1(t_k) + r_{1,k}(f_1), \ k = \overline{0, n}$$

$$\alpha_2(t_k) = \int_{t_0}^{t_k} f_1(s, \alpha_1(s), \beta_1(s), \alpha_1(s-\tau_1), \beta_1(s-\tau_2))ds =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \widetilde{\alpha}_1(t_0) + r_{1,0}(f_1), \widetilde{\beta}_1(t_0) + r_{1,0}(f_1),$$

$$\widetilde{\alpha}_1(t_0-\tau_1) + r_{1,0}(f_1), \widetilde{\beta}_1(t_0-\tau_2) + r_{1,0}(f_1))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_1(t_i) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i) + r_{1,i}(f_1),$$

$$\widetilde{\alpha}_1(t_i-\tau_1) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i-\tau_2) + r_{1,i}(f_1))+$$

$$f_1(t_k, \widetilde{\alpha}_1(t_k) + r_{1,n}(f_1), \widetilde{\beta}_1(t_k) + r_{1,n}(f_1),$$

$$\widetilde{\alpha}_1(t_k-\tau_1) + r_{1,n}(f_1), \widetilde{\beta}_1(t_k-\tau_2) + r_{1,n}(f_1))]+$$

$$\varphi(t_0) + r_{2,k}(f_1) =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \widetilde{\alpha}_1(t_0), \widetilde{\beta}_1(t_0), \widetilde{\alpha}_1(t_0-\tau_1), \widetilde{\beta}_1(t_0-\tau_2))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_1(t_i), \widetilde{\beta}_1(t_i), \widetilde{\alpha}_1(t_i-\tau_1), \widetilde{\beta}_1(t_i-\tau_2))+$$

$$f_1(t_k, \widetilde{\alpha}_1(t_k), \widetilde{\beta}_1(t_k), \widetilde{\alpha}_1(t_k-\tau_1), \widetilde{\beta}_1(t_k-\tau_2))]+$$

$$\varphi(t_0) + \widetilde{r}_{2,k}(f_1) =$$

$$\widetilde{\alpha}_2(t_k) + \widetilde{r}_{2,k}(f_1)$$

Observe that

$$r_{1,k}(f_1) = \alpha_1(t_k) - \widetilde{\alpha}_1(t_k) \text{ and } \widetilde{r}_{2,k}(f_1) = \alpha_2(t_k) - \widetilde{\alpha}_2(t_k).$$

and we pass from

$$f_1(t_i, \widetilde{\alpha}_1(t_i) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i) + r_{1,i}(f_1), \widetilde{\alpha}_1(t_i - \tau_1) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i - \tau_2) + r_{1,i}(f_1))$$

to

$$f_1(t_i, \widetilde{\alpha}_1(t_i), \widetilde{\beta}_1(t_i), \widetilde{\alpha}_1(t_i - \tau_1), \widetilde{\beta}_1(t_i - \tau_2)) \ + \text{ same remainder}$$

so that the remainders cumulated after i it gives us $\widetilde{r}_{2,k}(f_1)$.

We use the Taylor formula with respect to the last four variables from f_1 around $\widetilde{\alpha}_1(t_i)$.

$$|\widetilde{r}_{2,k}(f_1)| \leq \frac{t-t_0}{2n} L \left[|r_{1,0}(f_1)| + \sum_{i=1}^{n-1} |r_{1,i}(f_1)| + |r_{1,n}(f_1)| \right] + |r_{2,k}(f_1)| \leq$$

$$\frac{t-t_0}{2n} L \left(\frac{(t-t_0)^3}{12n^2} M_1 + (n-1)\frac{(t-t_0)^3}{12n^2} M_1 + \frac{(t-t_0)^3}{12n^2} M_1 \right) +$$

$$\frac{(t-t_0)^3}{12n^2} M_1 \leq \frac{t-t_0}{2n} L \frac{(t-t_0)^3}{12n^2} M_1 \, (1 + n - 1 + 1) + \frac{(t-t_0)^3}{12n^2} M_1 \leq$$

$$\frac{(t-t_0)^3}{12n^2} M_1 \left[\frac{(n+1)(t-t_0)}{2n} L + 1 \right] \leq \frac{(t-t_0)^3}{12n^2} M_1 \left[(t-t_0)L + 1 \right]$$

We continue in this manner, for $m = 3, 4, ...$, by induction, and obtain:

$$\alpha_m(t_k) = \frac{t-t_0}{2n} [f_1(t_0, \widetilde{\alpha}_{m-1}(t_0) + \widetilde{r}_{m-1,0}(f_1), \widetilde{\beta}_{m-1}(t_0) + \widetilde{r}_{m-1,0}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_0 - \tau_1) + \widetilde{r}_{m-1,0}(f_1), \widetilde{\beta}_{m-1}(t_0 - \tau_2) + \widetilde{r}_{m-1,0}(f_1)) +$$

$$2 \sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_{m-1}(t_i) + \widetilde{r}_{m-1,i}(f_1), \widetilde{\beta}_{m-1}(t_i) + \widetilde{r}_{m-1,i}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_i - \tau_1) + \widetilde{r}_{m-1,i}(f_1), \widetilde{\beta}_{m-1}(t_i - \tau_2) + \widetilde{r}_{m-1,i}(f_1)) +$$

$$f_1(t_k, \widetilde{\alpha}_{m-1}(t_k) + \widetilde{r}_{m-1,n}(f_1), \widetilde{\beta}_{m-1}(t_k) + \widetilde{r}_{m-1,n}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_k - \tau_1) + \widetilde{r}_{m-1,n}(f_1), \widetilde{\beta}_{m-1}(t_k - \tau_2) + \widetilde{r}_{m-1,n}(f_1))] +$$

$$\varphi(t_0) + r_{m,k}(f_1) =$$

$$\frac{t-t_0}{2n} [f_1(t_0, \widetilde{\alpha}_{m-1}(t_0), \widetilde{\beta}_{m-1}(t_0), \widetilde{\alpha}_{m-1}(t_0 - \tau_1), \widetilde{\beta}_{m-1}(t_0 - \tau_2)) +$$

$$2\sum_{i=1}^{n-1} f_1(t_i,\widetilde{\alpha}_{m-1}(t_i),\widetilde{\beta}_{m-1}(t_i),\widetilde{\alpha}_{m-1}(t_i-\tau_1),\widetilde{\beta}_{m-1}(t_i-\tau_2))+$$

$$f_1(t_k,\widetilde{\alpha}_{m-1}(t_k),\widetilde{\beta}_{m-1}(t_k),\widetilde{\alpha}_{m-1}(t_k-\tau_1),\widetilde{\beta}_{m-1}(t_k-\tau_2))]+$$

$$\varphi(t_0)+\widetilde{r}_{m,k}(f_1)=\widetilde{\alpha}_m(t_k)+r_{m,k}(f_1),\ k=\overline{0,n}.$$

where

$$|\widetilde{r}_{m,k}(f_1)|=|\alpha_m(t_k)-\widetilde{\alpha}_m(t_k)|\le\frac{(t-t_0)^3}{12n^2}M_1\left[(t-t_0)^{m-1}L^{m-1}+...+1\right],\ k=\overline{0,n}.$$

or

$$|\widetilde{r}_{m,k}(f_1)|\le\frac{(t-t_0)^3}{12n^2}M_1\frac{1-(t-t_0)^mL^m}{1-(t-t_0)L}\le\frac{(t-t_0)^3M_1}{12n^2[1-(t-t_0)L]},k=\overline{0,n},\ m\in\mathbb{N}^*.$$

In this way we got the sequence

$$(\widetilde{\alpha}_m(t_k))_{m\in\mathbb{N}},\ k=\overline{0,n}$$

which approximates the sequence of successive approximation (4) on the knots (7), with the error

$$|\alpha_m(t_k)-\widetilde{\alpha}_m(t_k)|\le\frac{(t-t_0)^3M_1}{12n^2[1-(t-t_0)L]} \tag{10}$$

By Picard's theorem [1], we have the following estimation

$$|\alpha(t_k)-\alpha_m(t_k)|\le\frac{(t-t_0)^mL^m}{1-(t-t_0)L}\left\|\alpha_0-\alpha_1\right\|_{C[t_0,b]}. \tag{11}$$

Analogously we calculate for β_m.

In this way there was obtained the main result of our paper:

Theorem 3. *Consider the system (2) with initial condition (3) under the conditions of Remark 1. If the exact solution* (α,β) *is approximated by the sequence* $\left((\widetilde{\alpha}_m(t_k)),(\widetilde{\beta}_m(t_k))\right)_{m\in\mathbb{N}}$, $k=\overline{0,n},\ m<n$ *on the knots (7), by the successive approximations method (4) combined with the trapezoidal rule (5), then the following error estimation holds:*

$$|\alpha(t_k)-\widetilde{\alpha}_m(t_k)|\le\frac{(t-t_0)^3}{1-(t-t_0)L}\left[(t-t_0)^{m-3}L^m\left\|\alpha_0-\alpha_1\right\|_{C[t_0,b]}+\frac{M_1}{12n^2}\right],\quad m=1,2,...,\ k=\overline{0,n} \tag{12}$$

$$\left|\beta(t_k) - \widetilde{\beta}_m(t_k)\right| \leq \frac{(t-t_0)^3}{1-(t-t_0)L}\left[(t-t_0)^{m-3}L^m \left\|\alpha_0 - \alpha_1\right\|_{C[t_0,b]} + \frac{M_2}{12n^2}\right], \qquad (13)$$
$$m = 1, 2, ..., \; k = \overline{0, n}$$

Proof. We have

$$\begin{aligned} |\alpha(t_k) - \widetilde{\alpha}_m(t_k)| &= |\alpha(t_k) - \alpha_m(t_k) + \alpha_m(t_k) - \widetilde{\alpha}_m(t_k)| \leq \\ &\leq |\alpha(t_k) - \alpha_m(t_k)| + |\alpha_m(t_k) - \widetilde{\alpha}_m(t_k)| \end{aligned}$$

which, by virtue of formula (10) and (11), can also be written

$$|\alpha(t_k) - \widetilde{\alpha}_m(t_k)| \leq \frac{(t-t_0)^3 M_1}{12n^2\left[1-(t-t_0)L\right]} + \frac{(t-t_0)^m L^m}{1-(t-t_0)L}\left\|\alpha_0 - \alpha_1\right\|_{C[t_0,b]}$$

and, from here, it results immediately (12) and analogue (13). The theorem is proved. □

Remark 2. *For* $L < \dfrac{1}{t-t_0}$ *the errors from Theorem 3 converges.*

4. **Example**

Consider the following Lotka-Volterra-type predator-prey system with two delays τ_1 and τ_2:

$$\begin{cases} x'(t) = x(t) + x(t-2) + y(t-5) \\ y'(t) = y(t) + x(t-2) - y(t-5) \end{cases}, \; t \geq 0$$

with initial condition

$$\begin{cases} x(t) = 1, \; t \in [-2, 0] \\ y(t) = 0, \; t \in [-5, 0] \end{cases}$$

We apply the step method for this system:

$t \in [0, 2]$

$$\begin{cases} x'(t) = x(t) + 1 \\ x(0) = 1 \end{cases} \Rightarrow x_1(t)$$

$$\begin{cases} y'(t) = y(t) + 1 \\ y(0) = 0 \end{cases} \Rightarrow y_1(t)$$

$t \in [2, 4]$

$$\begin{cases} x'(t) = x(t) + x_1(t-2) \\ x(2) = x_1(2) \end{cases} \Rightarrow x_2(t)$$

$$\begin{cases} y'(t) = y(t) + x_1(t-2) \\ y(2) = y_1(2) \end{cases} \Rightarrow y_2(t)$$

$t \in [4,5]$

$$\begin{cases} x'(t) = x(t) + x_2(t-2) \\ x(4) = x_2(4) \end{cases} \Rightarrow x_3(t)$$

$$\begin{cases} y'(t) = y(t) + x_2(t-2) \\ y(4) = y_2(4) \end{cases} \Rightarrow y_3(t)$$

References

[1] Coman, Gh., Pavel, G., Rus, I., Rus, I. A., *Introducere in teoria ecuatiilor operatoriale*, Editura Dacia, Cluj Napoca, 1976.

[2] Freedman, H. I., Ruan, S., *Uniform persistence in functional differential equations*, J. Differential Equations, 115, 1995.

[3] Iancu, C., *A numerical method for a approximating the solution of an integral equation from biomathematics*, Studia Univ. "Babes-Bolyai", Mathematica, Vol XLIII, Nr. 4, 1998.

[4] Muresan, V., *Ecuatii diferentiale cu modificarea afina a argumentului*, Transilvania Press, Cluj Napoca, 1997.

[5] Muroya, Y., *Uniform persistence for Lotka-Volterra-type delay differential systems*, Nonlinear Analysis, 4, 2003.

[6] Rus, I. A., *Principii si aplicatii ale teoriei punctului fix,* Editura Dacia, Cluj Napoca, 1979.

[7] Saito, Y., Hara, T., Ma, W., *Necessary and sufficient conditions for permanence and global stability of a Lotka-Volterra system with two delays*, J. Math. Anal. Appl., 236, 1999.

"Babeş-Bolyai" University, Department of Applied Mathematics,
Str. M. Kogălniceanu 1, RO-400084 Cluj-Napoca, Romania
E-mail address: dotrocol@math.ubbcluj.ro