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ITERATIVE FUNCTIONAL-DIFFERENTIAL SYSTEM WITH RETARDED ARGUMENT*

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Abstract. Existence, uniqueness and data dependence results of solution to the Cauchy problem for iterative functional-differential system with delays are obtained using weakly Picard operator theory.

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Keywords. Iterative functional-differential equation, weakly Picard operator, delay, data dependence.

1. INTRODUCTION

The aim of this paper is to study the following iterative system with delays

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(x_1(t-\tau_1)), x_2(x_2(t-\tau_2))),\ t \in [t_0, b],\ i = 1, 2, \tag{1.1}$$

with the initial conditions

$$x_i(t) = \varphi_i(t),\ t \in [t_0 - \tau_i, t_0],\ i = 1, 2, \tag{1.2}$$

where

(H$_1$) $t_0 < b,\ \tau_1, \tau_2 > 0,\ \tau_1 < \tau_2$;
(H$_2$) $f_i \in C([t_0, b] \times ([t_0 - \tau_1, b] \times [t_0 - \tau_2, b])^2, \mathbb{R}),\ i = 1, 2$;
(H$_3$) $\varphi_1 \in C([t_0 - \tau_1, t_0], [t_0 - \tau_1, b]),\ \varphi_2 \in C([t_0 - \tau_2, t_0], [t_0 - \tau_2, b])$;
(H$_4$) there exists $L_{f_i} > 0$ such that:

$$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \le L_{f_i}(\sum_{k=1}^{4} |u_k - v_k|),$$

for all $t \in [t_0, b], (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in ([t_0 - \tau_1, b] \times [t_0 - \tau_2, b])^2$, $i = 1, 2$.

By a solution of (1.1)–(1.2) we understand a function (x_1, x_2) with

$$x_1 \in C([t_0 - \tau_1, b], [t_0 - \tau_1, b]) \cap C^1([t_0, b], [t_0 - \tau_1, b])$$
$$x_2 \in C([t_0 - \tau_2, b], [t_0 - \tau_2, b]) \cap C^1([t_0, b], [t_0 - \tau_2, b])$$

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which satisfies (1.1)–(1.2).

The problem (1.1)–(1.2) is equivalent with the following fixed point equations:

(3a)

$$x_1(t)=\begin{cases}\varphi_1(t),\ t\in[t_0-\tau_1,t_0],\\ \varphi_1(t_0)+\int_{t_0}^t f_1(s,x_1(s),x_2(s),x_1(x_1(s-\tau_1)),x_2(x_2(s-\tau_2)))ds, t\in[t_0,b],\end{cases}$$

(3b)

$$x_2(t)=\begin{cases}\varphi_2(t),\ t\in[t_0-\tau_2,t_0],\\ \varphi_2(t_0)+\int_{t_0}^t f_2(s,x_1(s),x_2(s),x_1(x_1(s-\tau_1)),x_2(x_2(s-\tau_2)))ds, t\in[t_0,b],\end{cases}$$

where $x_1\in C([t_0-\tau_1,b],[t_0-\tau_1,b]), x_2\in C([t_0-\tau_2,b],[t_0-\tau_2,b])$.

On the other hand, the system (1.1) is equivalent with

(4a)

$$x_1(t)=\begin{cases}x_1(t),\ t\in[t_0-\tau_1,t_0],\\ x_1(t_0)+\int_{t_0}^t f_1(s,x_1(s),x_2(s),x_1(x_1(s-\tau_1)),x_2(x_2(s-\tau_2)))ds, t\in[t_0,b],\end{cases}$$

(4b)

$$x_2(t)=\begin{cases}x_2(t),\ t\in[t_0-\tau_2,t_0],\\ x_2(t_0)+\int_{t_0}^t f_2(s,x_1(s),x_2(s),x_1(x_1(s-\tau_1)),x_2(x_2(s-\tau_2)))ds, t\in[t_0,b],\end{cases}$$

and $x_1\in C([t_0-\tau_1,b],[t_0-\tau_1,b]),\ x_2\in C([t_0-\tau_2,b],[t_0-\tau_2,b])$.

We shall use the weakly Picard operators technique to study the systems (3a)–(3b) and (4a)–(4b).

The literature in differential equations with modified arguments, especially of retarded type, is now very extensive. We refer the reader to the following monographs: J. Hale [2], Y. Kuang [4], V. Mureşan [3], I. A. Rus [7] and to our papers [5], [6]. The case of iterative system with retarded arguments has been studied by many authors: I. A. Rus and E. Egri [10], J. G. Si, W. R. Li and S. S. Cheng [11], S. Stanek [12]. So our paper complement in this respect the existing literature.

Let us mention that the results from this paper are obtained as a concequence of those from [10] where is considered the case of boundary value problems.

2. WEAKLY PICARD OPERATORS

In this paper we need some notions and results from the weakly Picard operator theory (for more details see I. A. Rus [9], [8], M. Serban [13]).

Let (X,d) be a metric space and $A:X\to X$ an operator. We shall use the following notations:

$F_A:=\{x\in X\mid A(x)=x\}$ - the fixed point set of A;

$I(A):=\{Y\subset X\mid A(Y)\subset Y, Y\neq\emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1}:=A\circ A^n,\ A^0=1_X,\ A^1=A,\ n\in\mathbb{N}$;

$P(X):=\{Y\subset X\mid Y\neq\emptyset\}$ - the set of the parts of X;

$H(Y,Z) := \max\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\}$ -the Pompeiu–Housdorff functional on $P(X)\times P(X)$.

Definition 2.1. *Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$,

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Remark 2.2. Accordingly to the definition, the contraction principle insures that, if $A : X \to X$ is a α -contraction on the complet metric space X, then it is a Picard operator.

Theorem 2.3. (Data dependence theorem). *Let (X,d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

(i) *the operator A is a α -contraction;*

(ii) $F_B \neq \emptyset$;

(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta,\ \forall x \in X.$$

Then if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \tfrac{\eta}{1-\alpha}.$$

Definition 2.4. *Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Theorem 2.5. *Let (X,d) be a metric space and $A : X \to X$ an operator. The operator A is weakly Picard operator if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

where Λ is the indices set of partition, such that:

(a) $X_\lambda \in I(A),\ \lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all $\lambda \in \Lambda$.*

Definition 2.6. *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

It is clear that $A^\infty(X) = F_A$.

Definition 2.7. *Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \leq c d(x, A(x)),\ \forall x \in X.$$

Example 2.8. Let (X,d) be a complete metric space and $A : X \to X$ a continuous operator. We suppose that there exists $\alpha \in [0,1)$ such that

$$d(A^2(x), A(x)) \leq \alpha(x, A(x)),\ \forall x \in X.$$

Then A is c-weakly Picard operator with $c = \frac{1}{1-\alpha}$.

Theorem 2.9. *Let (X,d) be a metric space and $A_i : X \to X,\ i = 1,2$. Suppose that*

(i) *the operator A_i is c_i-weakly Picard operator, $i = 1,2$;*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \leq \eta,\ \forall x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2).$$

Theorem 2.10. (Fibre contraction principle). *Let (X,d) and (Y,ρ) be two metric spaces and $A : X \times Y \to X \times Y,\ A = (B,C),\ (\ B : X \to X,\ C : X \times Y \to Y\ )$ a triangular operator. We suppose that*

(i) *(Y,ρ) is a complete metric space;*
(ii) *the operator B is Picard operator;*
(iii) *there exists $l \in [0,1)$ such that $C(x,\cdot) : Y \to Y$ is a l-contraction, for all $x \in X$;*
(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is Picard operator.

3. CAUCHY PROBLEM

In what follows we consider the fixed point equations (3a) and (3b).

Let

$$A_f : C([t_0-\tau_1, b],[t_0-\tau_1, b]) \times C([t_0-\tau_2, b],[t_0-\tau_2, b]) \to C([t_0-\tau_1, b],\mathbb{R}) \times C([t_0-\tau_2, b],\mathbb{R}),$$

given by the relation

$$A_f(x_1, x_2) = (A_{f_1}(x_1, x_2), A_{f_2}(x_1, x_2)),$$

where $A_{f_1}(x_1, x_2)(t) :=$ the right hand side of (3a) and $A_{f_2}(x_1, x_2)(t) :=$ the right hand side of (3b).

Let $L_1, L_2 > 0$, $L = \max\{L_1, L_2\}$ and

$$\begin{aligned} &C_L([t_0-\tau_1, b],[t_0-\tau_1, b]) \times C_L([t_0-\tau_2, b],[t_0-\tau_2, b]) := \\ &= \{(x_1, x_2) \in C([t_0-\tau_1, b],[t_0-\tau_1, b]) \times C([t_0-\tau_2, b],[t_0-\tau_2, b]) : \\ &\quad |x_i(t_1) - x_i(t_2)| \leq L_i\, |t_1 - t_2|\,,\ \forall (t_1, t_2) \in [t_0 - \tau_2, b],\ i = 1,2\}. \end{aligned}$$

It is clear that $C_L([t_0-\tau_1, b],[t_0-\tau_1, b]) \times C_L([t_0-\tau_2, b],[t_0-\tau_2, b])$ is a complete metric space with respect to the metric

$$d(x, \overline{x}) := \max_{t_0 \leq t \leq b} |x(t) - \overline{x}(t)|\,.$$

We remark that $C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])$ is a closed subset in $C([t_0-\tau_1,b],[t_0-\tau_1,b])\times C([t_0-\tau_2,b],[t_0-\tau_2,b])$.

We have

THEOREM 3.1. *We suppose that*

(i) *the conditions* (H_1)–(H_4) *are satisfied;*

(ii) $\varphi_1\in C_L([t_0-\tau_1,t_0],[t_0-\tau_1,b])$, $\varphi_2\in C_L([t_0-\tau_2,t_0],[t_0-\tau_2,b])$;

(iii) m_{f_i} *and* $M_{f_i}\in\mathbb{R}$, $i=1,2$ *are such that*

(iiia) $m_{f_i}\le f_i(t,u_1,u_2,u_3,u_4)\le M_{f_i},\forall t\in[t_0,b],(u_1,u_2,u_3,u_4),(v_1,v_2,v_3,v_4)\in([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2$,

(iiib)

$$\begin{array}{ll} t_0-\tau_i\le\varphi_i(t_0)+m_{f_i}(b-t_0) & \text{for } m_{f_i}<0,\\ t_0-\tau_i\le\varphi_i(t_0) & \text{for } m_{f_i}\ge 0,\\ b\ge\varphi_i(t_0) & \text{for } M_{f_i}\le 0,\\ b\ge\varphi_i(t_0)+M_{f_i}(b-t_0) & \text{for } M_{f_i}>0,\end{array}$$

(iiic) $L+M_{f_i}<1$;

(iv) $(b-t_0)(L_{f_1}+L_{f_2})(L+2)<1$.

Then the Cauchy problem (1.1)–(1.2) *has, in* $C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])$ *a unique solution. Moreover the operator*

$$A_f:C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])\to$$
$$C_L([t_0-\tau_1,b],C_L([t_0-\tau_1,b],[t_0-\tau_1,b]))\times C_L([t_0-\tau_2,b],C_L([t_0-\tau_2,b],[t_0-\tau_2,b]))$$

is a c-Picard operator with $c=\frac{1}{(b-t_0)(L_{f_1}+L_{f_2})(L+2)}$.

Proof. **(a)** $C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])$ is an invariant subset for A_f.

Indeed,

$$t_0-\tau_i\le A_{f_i}(x_1,x_2)(t)\le b,$$

$(x_1,x_2)(t)\in[t_0-\tau_1,b]\times[t_0-\tau_2,b]$, $t\in[t_0,b]$, $i=1,2$.

From (iiia) we have m_{f_i} and $M_{f_i}\in\mathbb{R}$ such that

$$m_{f_i}\le f_i(t,u_1,u_2,u_3,u_4)\le M_{f_i},$$

$\forall t\in[t_0,b]$, $(u_1,u_2,u_3,u_4),(v_1,v_2,v_3,v_4)\in([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2$, $i=1,2$.

This implies that

$\int_{t_0}^t m_{f_i}\mathrm{d}s\le\int_{t_0}^t f_i(s,x_1(s),x_2(s),x_1(x_1(s-\tau_1)),x_2(x_2(s-\tau_2)))\mathrm{d}s\le\int_{t_0}^t M_{f_i}\mathrm{d}s$, $\forall t\in[t_0,b]$, that is

$$\varphi_i(t_0)+m_{f_i}(b-t_0)\le A_{f_i}(x_1,x_2)(t)\le\varphi_i(t_0)+M_{f_i}(b-t_0),t\in[t_0,b].$$

Therefor if condition (iii) holds, we have satisfied the invariance property for the operator A_f in $C([t_0-\tau_1,b],[t_0-\tau_1,b])\times C([t_0-\tau_2,b],[t_0-\tau_2,b])$.

Now, consider $t_1,t_2\in[t_0-\tau_1,t_0]$:

$$|A_{f_1}(x_1,x_2)(t_1)-A_{f_1}(x_1,x_2)(t_2)|=|\varphi_1(t_1)-\varphi_1(t_2)|\le L_1\,|t_1-t_2|\,,$$

because $\varphi_1\in C_L([t_0-\tau_1,t_0],[t_0-\tau_1,b])$.

Similarly, for $t_1, t_2 \in [t_0 - \tau_2, t_0]$:

$$|A_{f_2}(x_1, x_2)(t_1) - A_{f_2}(x_1, x_2)(t_2)| = |\varphi_2(t_1) - \varphi_2(t_2)| \le L_2 |t_1 - t_2|,$$

that follows from (ii), too.

On the other hand, if $t_1, t_2 \in [t_0, b]$, we have

$$\begin{aligned}
&|A_{f_i}(x_1, x_2)(t_1) - A_{f_i}(x_1, x_2)(t_2)| = \\
&= \left| \varphi_i(t_1) - \varphi_i(t_2) + \int_{t_0}^{t_1} f_i(s, x_1(s), x_2(s), x_1(x_1(s-\tau_1)), x_2(x_2(s-\tau_2)))\mathrm{d}s - \right. \\
&\quad \left. - \int_{t_0}^{t_2} f_i(s, x_1(s), x_2(s), x_1(x_1(s-\tau_1)), x_2(x_2(s-\tau_2)))\mathrm{d}s \right| \le \\
&\le L_i |t_1 - t_2| + M_{f_i} |t_1 - t_2| \le (L + M_{f_i}) |t_1 - t_2|, \ i = 1, 2.
\end{aligned}$$

So we can affirm that $\forall t_1, t_2 \in [t_0, b], \ t_1 \le t_2$, and doe to (iii), A_f is L-Lipshitz.

Thus, according to the above, we have $C_L([t_0 - \tau_1, b], [t_0 - \tau_1, b]) \times C_L([t_0 - \tau_2, b], [t_0 - \tau_2, b]) \in I(A_f)$.

(b) A_f is a L_{A_f} -contraction with $L_{A_f} = (b - t_0)(L_{f_1} + L_{f_2})(L + 2)$.

For $t \in [t_0 - \tau_1, t_0]$, we have $|A_{f_1}(x_1, x_2)(t) - A_{f_1}(\overline{x}_1, \overline{x}_2)(t)| = 0$.

For $t \in [t_0 - \tau_2, t_0]$, we have $|A_{f_2}(x_1, x_2)(t) - A_{f_2}(\overline{x}_1, \overline{x}_2)(t)| = 0$.

For $t \in [t_0, b]$:

$$\begin{aligned}
&|A_{f_1}(x_1, x_2)(t) - A_{f_1}(\overline{x}_1, \overline{x}_2)(t)| = \\
&= \left| \int_{t_0}^{t} [f_1(s, x_1(s), x_2(s), x_1(x_1(s-\tau_1)), x_2(x_2(s-\tau_2))) \right. \\
&\quad \left. - f_1(s, \overline{x}_1(s), \overline{x}_2(s), \overline{x}_1(\overline{x}_1(s-\tau_1)), \overline{x}_2(\overline{x}_2(s-\tau_2)))]\mathrm{d}s \right| \\
&\le L_{f_1}(|x_1(s) - \overline{x}_1(s)| + |x_2(s) - \overline{x}_2(s)| + |x_1(x_1(s-\tau_1)) - \overline{x}_1(\overline{x}_1(s-\tau_1))| \\
&\quad + |x_2(x_2(s-\tau_2)) - \overline{x}_2(\overline{x}_2(s-\tau_2))|)(b - t_0) \\
&\le (b - t_0)L_{f_1}[\|x_1 - \overline{x}_1\|_C + \|x_2 - \overline{x}_2\|_C + |x_1(x_1(s-\tau_1)) - x_1(\overline{x}_1(s-\tau_1))| \\
&\quad + |x_1(\overline{x}_1(s-\tau_1)) - \overline{x}_1(\overline{x}_1(s-\tau_1))| + |x_2(x_2(s-\tau_2)) - x_2(\overline{x}_2(s-\tau_2))| \\
&+ |x_2(\overline{x}_2(s-\tau_2)) - \overline{x}_2(\overline{x}_2(s-\tau_2))|] \le (b - t_0)L_{f_1}[\|x_1 - \overline{x}_1\|_C + \|x_2 - \overline{x}_2\|_C \\
&\quad + L_1 \|x_1 - \overline{x}_1\|_C + \|x_1 - \overline{x}_1\|_C + L_2 \|x_2 - \overline{x}_2\|_C + \|x_2 - \overline{x}_2\|_C] \\
&\le (b - t_0)L_{f_1}(L + 2)(\|x_1 - \overline{x}_1\|_C + \|x_2 - \overline{x}_2\|_C).
\end{aligned}$$

In the same way

$$|A_{f_2}(x_1, x_2)(t) - A_{f_2}(\overline{x}_1, \overline{x}_2)(t)| \le (b - t_0)L_{f_2}(L + 2)(\|x_1 - \overline{x}_1\| + \|x_2 - \overline{x}_2\|).$$

Then we have the following relation

$$\|A_f(x_1, x_2) - A_f(\overline{x}_1, \overline{x}_2)\|_C \le (b - t_0)(L_{f_1} + L_{f_2})(L + 2) \|(x_1, x_2) - (\overline{x}_1, \overline{x}_2)\|_C$$

So A_f is a c-Picard operator with $c = \frac{1}{1 - L_{A_f}}$. $\square$

In what follows, consider the following operator

$$B_f: \; C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])\to C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]),$$

given by the relation

$$B_f(x_1,x_2) = (B_{f_1}(x_1,x_2), B_{f_2}(x_1,x_2)),$$

where $B_{f_1}(x_1,x_2) :=$ the right hand side of (4a) and $B_{f_2}(x_1,x_2) :=$ the right hand side of (4b).

THEOREM 3.2. *In the conditions of Theorem 3.1, the operator* $B_f : C_L([t_0-\tau_1,b],[t_0-\tau_1,b]) \times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]) \to C_L([t_0-\tau_1,b],[t_0-\tau_1,b]) \times C_L([t_0-\tau_2,b],[t_0-\tau_2,b])$ *is WPO.*

Proof. The operator B_f is a continuous operator but it is not a contraction operator. Let take the following notation:

$$X_{\varphi_1} := \{x_1 \in C([t_0-\tau_1,b],[t_0-\tau_1,b]) |\; x_1|_{[t_0-\tau_1,t_0]} = \varphi_1\},$$
$$X_{\varphi_2} := \{x_2 \in C([t_0-\tau_2,b],[t_0-\tau_2,b]) |\; x_2|_{[t_0-\tau_2,t_0]} = \varphi_2\}.$$

Then we can write

(5)
$$C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]) = \bigcup_{\varphi_i \in C_L([t_0-\tau_i,t_0],[t_0-\tau_i,b])} X_{\varphi_1}\times X_{\varphi_2}.$$

We have that $X_{\varphi_1}\times X_{\varphi_2} \in I(B_f)$ and $B_f|_{X_{\varphi_1}\times X_{\varphi_2}}$ is a Picard operator because is the operator which appears in the proof of Theorem 3.1. By applying Theorem 2.5, we obtain that B_f is WPO. □

4. INCREASING SOLUTION OF (1.1)

4.1. Inequalities of Chapligin type.

THEOREM 4.1. *We suppose that*

(a) *the conditions of the Theorem 3.1 are satisfied;*

(b) $(u_1,u_2,u_3,u_4),(v_1,v_2,v_3,v_4) \in ([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2$, $u_j \le v_j$, $j=\overline{1,4}$, *imply that*

$$f_i(t,u_1,u_2,u_3,u_4) \le f_i(t,v_1,v_2,v_3,v_4),$$

$i=1,2$, *for all* $t\in[t_0,b]$.

Let (x_1,x_2) *be an increasing solution of the system (1.1) and* (y_1,y_2) *an increasing solution for the system of inequalities*

$$y_i'(t) \le f_i(t,y_1(t),y_2(t),y_1(y_1(t-\tau_1)),y_2(y_2(t-\tau_2))),\; t\in[t_0,b],$$

Then

$$y_i(t)\le x_i(t),\; t\in[t_0-\tau_i,t_0],\; i=1,2 \Rightarrow (y_1,y_2)\le(x_1,x_2).$$

Proof. In the terms of the operator B_f, we have

$$(x_1, x_2) = B_f(x_1, x_2) \text{ and } (y_1, y_2) \leq B_f(y_1, y_2).$$

However, from the condition (b), we have that the operator B_f^∞ is increasing,

$$\begin{aligned}(y_1, y_2) &\leq B_f^\infty(y_1, y_2) = B_f^\infty(\widetilde{y}_1|_{[t_0-\tau_1,t_0]}, \widetilde{y}_2|_{[t_0-\tau_2,t_0]}) \\ &\leq B_f^\infty(\widetilde{x}_1|_{[t_0-\tau_1,t_0]}, \widetilde{x}_2|_{[t_0-\tau_2,t_0]}) = (x_1, x_2).\end{aligned}$$

Thus $(y_1, y_2) \leq (x_1, x_2)$.

Here, for $(\widetilde{x}_1, \widetilde{x}_2)$ we used the notation $\widetilde{x}_1 \in X_{x_1|_{[t_0-\tau_1,t_0]}}, \widetilde{x}_2 \in X_{x_1|_{[t_0-\tau_2,t_0]}}$. □

4.2. Comparison theorem. In the next result we want to study the monotony of the solution of the problem (1.1)–(1.2) with respect to φ_i and f_i, $i = 1, 2$. We shall use the result below:

Lemma 4.2. (Abstract comparison lemma). *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ such that:*

(i) $A \leq B \leq C$;
(ii) *the operators A, B, C are WPO;*
(iii) *the operator B is increasing.*

Then

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

In this case we can establish the theorem.

Theorem 4.3. *Let $f_i^j \in C([t_0, b] \times ([t_0-\tau_1, b] \times [t_0-\tau_2, b])^2)$, $i = 1, 2, j = 1, 2, 3$.*

We suppose that

(a) $f_i^2(t, \cdot, \cdot, \cdot, \cdot) : ([t_0-\tau_1, b] \times [t_0-\tau_2, b])^2 \to ([t_0-\tau_1, b] \times [t_0-\tau_2, b])^2$ *are increasing;*
(b) $f_i^1 \leq f_i^2 \leq f_i^3$.

Let (x_1^j, x_2^j) be an increasing solution of the systems

$$x_i'(t) = f_i^j(t, x_1(t), x_2(t), x_1(x_1(t-\tau_1)), x_2(x_2(t-\tau_2))), t \in [t_0, b], i = 1, 2, j = 1, 2, 3.$$

If $x_i^1(t) \leq x_i^2(t) \leq x_i^3(t)$, $t \in [t_0 - \tau_i, t_0]$ then $x_i^1 \leq x_i^2 \leq x_i^3$, $i = 1, 2$.

Proof. The operators B_f^j, $j = 1, 2, 3$ are WPO. Taking into consideration the condition (a) the operator B_f^2 is increasing. From (b) we have that $B_f^1 \leq B_f^2 \leq B_f^3$. We note that $(x_1^j, x_2^j) = B_f^{j\infty}(\widetilde{x}_1^j, \widetilde{x}_2^j)$, $j = 1, 2, 3$. Now, using the Abstract comparison lemma, the proof is complete. □

5. DATA DEPENDENCE: CONTINUITY

Consider the Cauchy problem (1.1)–(1.2) and suppose the conditions of Theorem 3.1 are satisfied. Denote by $(x_1, x_2)(\cdot; \varphi_1, \varphi_2, f_1, f_2), i = 1, 2$ the solution of this problem. We can state the following result:

THEOREM 5.1. *Let $\varphi_1^j, \varphi_2^j, f_1^j, f_2^j,\ j = 1, 2$ be as in Theorem 3.1. We suppose that there exists $\eta^1, \eta^2, \eta_i^3, i = 1, 2$ such that*

(i) $\left|\varphi_1^1(t) - \varphi_1^2(t)\right| \le \eta^1,\ \forall t \in [t_0 - \tau_1, t_0]$ *and* $\left|\varphi_2^1(t) - \varphi_2^2(t)\right| \le \eta^2,\ \forall t \in [t_0 - \tau_2, t_0]$;

(ii) $\left|f_i^1(t, u_1, u_2, u_3, u_4) - f_i^2(t, v_1, v_2, v_3, v_4)\right| \le \eta_i^3, i = 1, 2, (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in ([t_0 - \tau_1, b] \times [t_0 - \tau_2, b])^2$.

Then

$$\left|(x_1, x_2)(t; \varphi_1^1, \varphi_2^1, f_1^1, f_2^1) - (x_1, x_2)(t; \varphi_1^2, \varphi_2^2, f_1^2, f_2^2)\right| \le \frac{\eta^1 + \eta^2 + (\eta_1^3 + \eta_2^3)(b - t_0)}{(b - t_0)(L_{f_1} + L_{f_2})(L + 2)},$$

where $L_{f_i} = \max(L_{f_i^1}, L_{f_i^2}), i = 1, 2.$

Proof. Consider the operators $A_{\varphi_1^j, \varphi_2^j, f_1^j, f_2^j}, j = 1, 2$. From Theorem 3.1 these operators are contractions.

Then

$$\left\|A_{\varphi_1^1, \varphi_2^1, f_1^1, f_2^1}(x_1, x_2) - A_{\varphi_1^2, \varphi_2^2, f_1^2, f_2^2}(x_1, x_2)\right\|_C \le \eta^1 + \eta^2 + (\eta_1^3 + \eta_2^3)(b - t_0),$$

$\forall (x_1, x_2) \in C_L([t_0 - \tau_1, b], [t_0 - \tau_1, b]) \times C_L([t_0 - \tau_2, b], [t_0 - \tau_2, b])$.

Now the proof follows from Theorem 2.3, with $A := A_{\varphi_1^1, \varphi_2^1, f_1^1, f_2^1}$, $B = A_{\varphi_1^2, \varphi_2^2, f_1^2, f_2^2}$, $\eta = \eta^1 + \eta^2 + (\eta_1^3 + \eta_2^3)(b - t_0)$ and $\alpha := L_{A_f} = (b - t_0)(L_{f_1} + L_{f_2})(L + 2)$ where $L_{f_i} = \max(L_{f_i^1}, L_{f_i^2}), i = 1, 2.$ □

From the Theorem above we have:

THEOREM 5.2. *Let f_i^1 and f_i^2 be as in Theorem 3.1, $i = 1, 2$. Let $S_{B_{f_i^1}}, S_{B_{f_i^2}}$ be the solution set of the system (1.1) corresponding to f_i^1 and $f_i^2, i = 1, 2$. Suppose that there exists $\eta_i > 0, i = 1, 2$ such that*

$$\left|f_i^1(t, u_1, u_2, u_3, u_4) - f_i^2(t, v_1, v_2, v_3, v_4)\right| \le \eta_i \tag{6}$$

for all $t \in [t_0, b], (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in ([t_0 - \tau_1, b] \times [t_0 - \tau_2, b])^2, i = 1, 2.$

Then

$$H_{\|\cdot\|_C}(S_{B_{f_i^1}}, S_{B_{f_i^2}}) \le \frac{(\eta_1 + \eta_2)(b - t_0)}{1 - (L_{f_1} + L_{f_2})(L + 2)(b - t_0)},$$

where $L_{f_i} := \max(L_{f_i^1}, L_{f_i^2})$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C_L([t_0 - \tau_1, b], [t_0 - \tau_1, b]) \times C_L([t_0 - \tau_2, b], [t_0 - \tau_2, b])$.

Proof. We will look for those c_1 and c_2 for which in condition of Theorem 3.1 the operators $B_{f_i^1}$ and $B_{f_i^2}, i = 1, 2$ are c_1-WPO and c_2-WPO.

Let

$$X_{\varphi_1} := \{x_1 \in C([t_0-\tau_1,b],[t_0-\tau_1,b])|\ x_1|_{[t_0-\tau_1,t_0]} = \varphi_1\},$$
$$X_{\varphi_2} := \{x_2 \in C([t_0-\tau_2,b],[t_0-\tau_2,b])|\ x_2|_{[t_0-\tau_2,t_0]} = \varphi_2\}.$$

It is clear that $B_{f_i^1}|_{X\varphi_1\times X_{\varphi_2}} = A_{f_i^1},\ B_{f_i^2}|_{X\varphi_1\times X_{\varphi_2}} = A_{f_i^2}$. So from Theorem 2.5 and Theorem 3.1 we have

$$\begin{aligned}\left\|B^2_{f_i^1}(x_1,x_2)-B_{f_i^1}(x_1,x_2)\right\|_C &\leq (b-t_0)(L_{f_1^1}+L_{f_2^1})(L+2)\left\|B_{f_i^1}(x_1,x_2)-(x_1,x_2)\right\|_C,\\ \left\|B^2_{f_i^2}(x_1,x_2)-B_{f_i^2}(x_1,x_2)\right\|_C &\leq (b-t_0)(L_{f_1^2}+L_{f_2^2})(L+2)\left\|B_{f_i^2}(x_1,x_2)-(x_1,x_2)\right\|_C,\end{aligned}$$

for all $(x_1,x_2) \in C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]), i = 1,2.$

Now choosing

$$\begin{aligned}\alpha_1 &= (b-t_0)(L_{f_1^1}+L_{f_2^1})(L+2),\\ \alpha_2 &= (b-t_0)(L_{f_1^2}+L_{f_2^2})(L+2),\end{aligned}$$

we get that $B_{f_i^1}$ and $B_{f_i^2}$ are c_1-WPO and c_2-WPO with $c_1 = (1-\alpha_1)^{-1},\ c_2 = (1-\alpha_2)^{-1}$. From (6) we obtain that

$$\left\|B_{f_i^1}(x_1,x_2) - B_{f_i^2}(x_1,x_2)\right\|_C \leq (\eta_1+\eta_2)(b-t_0),$$

for all $(x_1,x_2) \in C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]), i = 1,2.$ Applying Theorem 2.9 we have that

$$H_{\|\cdot\|_C}(S_{B_{f_i^1}}, S_{B_{f_i^2}}) \leq \frac{(\eta_1+\eta_2)(b-t_0)}{1-(b-t_0)(L_{f_1}+L_{f_2})(L+2)},$$

where $L_{f_i} := \max(L_{f_i^1}, L_{f_i^2})$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C_L([t_0-\tau_1,b],[t_0-\tau_1,b])\times C_L([t_0-\tau_2,b],[t_0-\tau_2,b]),\ i=1,2.$ □

6. DATA DEPENDENCE: DIFFERENTIABILITY

Consider the following Cauchy problem with parameter

(7) $x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(x_1(t-\tau_1)), x_2(x_2(t-\tau_2)); \lambda),\ t \in [t_0,b], i = 1,2,$

(8) $$x_i(t) = \varphi_i(t),\ t \in [t_0-\tau_i, t_0], i = 1,2.$$

Suppose that we have satisfied the following conditions:

(C$_1$) $t_0 < b, \tau_1, \tau_2 > 0, \tau_1 < \tau_2, J \subset \mathbb{R}$ a compact interval;
(C$_2$) $\varphi_i \in C_L([t_0-\tau_i,t_0],[t_0-\tau_i,b]),\ i=1,2;$
(C$_3$) $f_i \in C^1([t_0,b]\times([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2\times J,\mathbb{R})\ i=1,2;$
(C$_4$) there exists $L_{f_i} > 0$ such that

$$\left|\frac{\partial f_i(t,u_1,u_2,u_3,u_4;\lambda)}{\partial u_i}\right| \leq L_{f_i}$$

for all $t\in[t_0,b], (u_1,u_2,u_3,u_4)\in([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2,\ i=1,2, \lambda\in J;$

(C$_5$) m_{f_i} and $M_{f_i} \in \mathbb{R},\ i=1,2$ are such that

(a) $m_{f_i} \le f_i(t,u_1,u_2,u_3,u_4) \le M_{f_i}, \forall t \in [t_0,b],\ (u_1,u_2,u_3,u_4),$
$(v_1,v_2,v_3,v_4) \in ([t_0-\tau_1,b]\times[t_0-\tau_2,b])^2,$

(b)

$$\begin{array}{ll} t_0-\tau_i \le \varphi_i(t_0)+m_{f_i}(b-t_0) & \text{for } m_{f_i}<0,\\ t_0-\tau_i \le \varphi_i(t_0) & \text{for } m_{f_i}\ge 0,\\ b \ge \varphi_i(t_0) & \text{for } M_{f_i}\le 0,\\ b\ge \varphi_i(t_0)+M_{f_i}(b-t_0) & \text{for } M_{f_i}>0,\end{array}$$

(c) $L+M_{f_i}<1$;

(C_6) $(b-t_0)(L_{f_1}+L_{f_2})(L+2)<1$.

Then, from Theorem 3.1, we have that the problem (1.1)–(1.2) has a unique solution $(x_1^*(\cdot,\lambda), x_2^*(\cdot,\lambda))$.

We will prove that

$$x_i^*(\cdot,\lambda)\in C^1(J),\ \text{for all } t\in[t_0-\tau_i,t_0],\ i=1,2.$$

For this we consider the system

(9) $x_i'(t,\lambda) = f_i(t,x_1(t;\lambda),x_2(t;\lambda),x_1(x_1(t-\tau_1;\lambda);\lambda),x_2(x_2(t-\tau_2;\lambda);\lambda);\lambda),$

$t\in[t_0,b],\ \lambda\in J,\ x_i\in C([t_0-\tau_i,b]\times J,[t_0-\tau_i,b]\times J)\cap C^1([t_0,b]\times J,[t_0-\tau_i,b]\times J),\ i=1,2.$

THEOREM 6.1. *Consider the problem (9)–(8), and suppose the conditions* (C_1)–(C_6) *holds. Then,*

(i) *(9)–(8) has a unique solution* (x_1^*,x_2^*), *in* $C([t_0-\tau_1,b]\times J,[t_0-\tau_1,b])\times C([t_0-\tau_2,b]\times J,[t_0-\tau_2,b])$;

(ii) $x_i^*(\cdot,\lambda)\in C^1(J)$, *for all* $t\in[t_0-\tau_i,t_0],\ i=1,2.$

Proof. The problem (9)–(8) is equivalent with the following functional integral equations

(10a)

$$x_1(t;\lambda)=\begin{cases}\varphi_1(t),\ t\in[t_0-\tau_1,t_0]\\ \varphi_1(t)+\int_{t_0}^t f_1(s,x_1(s;\lambda),x_2(s;\lambda),x_1(x_1(s-\tau_1;\lambda);\lambda),x_2(x_2(s-\tau_2;\lambda);\lambda);\lambda)ds,\ t\in[t_0,b]\end{cases}$$

(10b)

$$x_2(t;\lambda)=\begin{cases}\varphi_2(t),\ t\in[t_0-\tau_2,t_0]\\ \varphi_2(t)+\int_{t_0}^t f_2(s,x_1(s;\lambda),x_2(s;\lambda),x_1(x_1(s-\tau_1;\lambda);\lambda),x_2(x_2(s-\tau_2;\lambda);\lambda);\lambda)ds,\ t\in[t_0,b]\end{cases}$$

Now, let take the operator

$$\begin{aligned}A\ :\ & C_L([t_0-\tau_1,b]\times J,[t_0-\tau_1,b]\times J)\times C_L([t_0-\tau_2,b]\times J,[t_0-\tau_2,b]\times J)\to\\ & C_L([t_0-\tau_1,b]\times J,[t_0-\tau_1,b]\times J)\times C_L([t_0-\tau_2,b]\times J,[t_0-\tau_2,b]\times J),\end{aligned}$$

given by the relation

$$A(x_1,x_2)=(A_1(x_1,x_2),A_2(x_1,x_2)),$$

where $A_1(x_1,x_2)(t;\lambda) :=$ the right hand side of (10a) and $A_2(x_1,x_2)(t;\lambda):=$ the right hand side of (10b).

Let $X = C_L([t_0 - \tau_1, b] \times J, [t_0 - \tau_1, b]) \times C_L([t_0 - \tau_2, b] \times J, [t_0 - \tau_2, b])$.

It is clear from the proof of Theorem 3.1 that in the conditions (C_1)–(C_6) the operator

$$A : (X, \|\cdot\|_C) \to (X, \|\cdot\|_C)$$

is a PO.

Let (x_1^*, x_2^*) be the unique fixed point of A.

We consider the subset $X_1 \subset X$,

$$X_1 := \{(x_1, x_2) \in X | \ \tfrac{\partial x_1}{\partial t} \in [t_0 - \tau_1, t_0], \ \tfrac{\partial x_2}{\partial t} \in [t_0 - \tau_2, t_0]\}.$$

We remark that $(x_1^*, x_2^*) \in X_1, A(X_1) \subset X_1$ and $A : (X_1, \|\cdot\|_C) \to (X_1, \|\cdot\|_C)$ is PO.

Let $Y := C([t_0 - \tau_1, b] \times J) \times C([t_0 - \tau_2, b] \times J)$.

Supposing that there exists $\dfrac{\partial x_1^*}{\partial \lambda}$ and $\dfrac{\partial x_2^*}{\partial \lambda}$, from (10a)–(10b) we have that

$$\begin{aligned}
\frac{\partial x_i^*}{\partial \lambda} \quad &= \int_{t_0}^{t} \frac{\partial f_i(s, x_1^*(s;\lambda), x_2^*(s;\lambda), x_1^*(x_1^*(s-\tau_1;\lambda);\lambda), x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x_1^*(s,\lambda)}{\partial \lambda} \mathrm{d}s \\
&+ \int_{t_0}^{t} \frac{\partial f_i(s, x_1^*(s;\lambda), x_2^*(s;\lambda), x_1^*(x_1^*(s-\tau_1;\lambda);\lambda), x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x_2^*(s,\lambda)}{\partial \lambda} \mathrm{d}s \\
&+ \int_{t_0}^{t} \frac{\partial f_i(s, x_1^*(s;\lambda), x_2^*(s;\lambda), x_1^*(x_1^*(s-\tau_1;\lambda);\lambda), x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda)}{\partial u_3} \\
&\cdot \left[\frac{\partial x_1^*(x_1^*(s-\tau_1;\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x_1^*(s-\tau_1;\lambda)}{\partial \lambda} + \frac{\partial x_1^*(x_1^*(s-\tau_1;\lambda);\lambda)}{\partial \lambda} \right] \mathrm{d}s \\
&+ \int_{t_0}^{t} \frac{\partial f_i(s, x_1^*(s;\lambda), x_2^*(s;\lambda), x_1^*(x_1^*(s-\tau_1;\lambda);\lambda), x_2^*(x_2^*(s-\tau_2;\lambda);\lambda);\lambda)}{\partial u_4} \cdot \\
&\cdot \left[\frac{\partial x_2^*(x_2^*(s-\tau_1;\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x_2^*(s-\tau_2;\lambda)}{\partial \lambda} + \frac{\partial x_2^*(x_2^*(s-\tau_2;\lambda);\lambda)}{\partial \lambda} \right] \mathrm{d}s \\
&+ \int_{t_0}^{t} \frac{\partial f_i(s, x_1^*(s;\lambda), x_2^*(s;\lambda), x_1^*(x_1^*(s-\tau_1;\lambda);\lambda), x_2^*(x_2^*(s-\tau_2;\lambda);\lambda);\lambda)}{\partial \lambda} \mathrm{d}s,
\end{aligned}$$

$t \in [t_0, b], \lambda \in J, i = 1, 2$.

The relation suggest us to consider the following operator

$$C : X_1 \times Y \to Y, \ (x_1, x_2, u, v) \to C(x_1, x_2, u, v),$$

where

$$C(x_1, x_2, u, v)(t; \lambda) = 0 \text{ for } t \in [t_0 - \tau_i, t_0], \lambda \in J, i = 1, 2$$

and

$$\begin{aligned}
&C(x_1,x_2,u,v)(t;\lambda) := \\
&= \int_{t_0}^{t} \tfrac{\partial f_i(s,x_1^*(s;\lambda),x_2^*(s;\lambda),x_1^*(x_1^*(s-\tau_1;\lambda);\lambda),x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda))}{\partial u_1} u(s;\lambda)\mathrm{d}s \\
&\quad + \int_{t_0}^{t} \tfrac{\partial f_i(s,x_1^*(s;\lambda),x_2^*(s;\lambda),x_1^*(x_1^*(s-\tau_1;\lambda);\lambda),x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda)}{\partial u_2} v(s;\lambda)\mathrm{d}s \\
&\quad + \int_{t_0}^{t} \tfrac{\partial f_i(s,x_1^*(s;\lambda),x_2^*(s;\lambda),x_1^*(x_1^*(s-\tau_1;\lambda);\lambda),x_2^*(x_2^*(s-\tau_2\lambda);\lambda);\lambda)}{\partial u_3} \\
&\quad \cdot \left[\tfrac{\partial x_1(x_1(s-\tau_1;\lambda);\lambda)}{\partial u_1} \cdot u(s-\tau_1;\lambda) + \tfrac{\partial x_1(x_1(s-\tau_1;\lambda);\lambda)}{\partial \lambda}\right] \mathrm{d}s \\
&\quad + \int_{t_0}^{t} \tfrac{\partial f_i(s,x_1(s;\lambda),x_2(s;\lambda),x_1(x_1(s-\tau_1;\lambda);\lambda),x_2(x_2(s\tau_2;\lambda);\lambda);\lambda)}{\partial u_4} \\
&\quad \cdot \left[\tfrac{\partial x_2(x_2(s-\tau_2;\lambda);\lambda)}{\partial u_2} \cdot v(s-\tau_2;\lambda) + \tfrac{\partial x_2(x_2(s-\tau_2;\lambda);\lambda)}{\partial \lambda}\right] \mathrm{d}s \\
&\quad + \int_{t_0}^{t} \tfrac{\partial f_i(s,x_1(s;\lambda),x_2(s;\lambda),x_1(x_1(s-\tau_1;\lambda);\lambda),x_2(x_2(s-\tau_2;\lambda);\lambda);\lambda)}{\partial \lambda}
\end{aligned}$$

for $t \in [t_0,b],\ \lambda \in J,\ i=1,2.$

In this way we have the triangular operator

$$\begin{aligned}
D \quad &: \quad X_1 \times Y \to X_1 \times Y, \\
(x_1,x_2,u,v) \quad &\to \quad (A(x_1,x_2), C(x_1,x_2,u,v)),
\end{aligned}$$

where A is PO and $C(x_1,x_2,\cdot,\cdot) : Y \to Y$ is an L_C-contraction with $L_C = (b-t_0)(\widetilde{L}_{f_1}+\widetilde{L}_{f_2})(L+2)$, where $\widetilde{L}_{f_i} = \max\{L_{f_i}, L \cdot L_{f_i}\},\ i=1,2.$

From the fibre contraction Theorem we have that the operator D is PO, i.e. the sequences

$$\begin{aligned}
(x_{1,n+1},x_{2,n+1}) &:= A(x_{1,n},x_{2,n}),\ n \in \mathbb{N}, \\
(u_{n+1},v_{n+1}) &:= C(x_{1,n},x_{2,n},u_n,v_n),\ n \in \mathbb{N},
\end{aligned}$$

converges uniformly, with respect to $t \in X,\ \lambda \in J$, to $(x_1^*,x_2^*,u^*,v^*) \in F_D$, for all $(x_{1,0},x_{2,0}) \in X_1,\ (u_0,v_0) \in Y$.

If we take

$$x_{1,0}=0,\ x_{2,0}=0,\ u_0 = \frac{\partial x_{1,0}}{\partial \lambda} = 0,\ v_0 = \frac{\partial x_{2,0}}{\partial \lambda} = 0,$$

then

$$u_1 = \tfrac{\partial x_{1,1}}{\partial \lambda},\ v_1 = \tfrac{\partial x_{2,1}}{\partial \lambda}.$$

By induction we prove that

$$\begin{aligned}
u_n &= \tfrac{\partial x_{1,n}}{\partial \lambda},\ \forall n \in \mathbb{N}, \\
v_n &= \tfrac{\partial x_{2,n}}{\partial \lambda},\ \forall n \in \mathbb{N}.
\end{aligned}$$

So

$$x_{1,n} \stackrel{unif}{\to} x_1^* \text{ as } n \to \infty,$$
$$x_{2,n} \stackrel{unif}{\to} x_2^* \text{ as } n \to \infty,$$
$$\frac{\partial x_{1,n}}{\partial \lambda} \stackrel{unif}{\to} u^* \text{ as } n \to \infty,$$
$$\frac{\partial x_{2,n}}{\partial \lambda} \stackrel{unif}{\to} v^* \text{ as } n \to \infty.$$

From a Weierstrass argument we have that there exists $\dfrac{\partial x_i^*}{\partial \lambda}$, $i = 1, 2$ and

$$\frac{\partial x_1^*}{\partial \lambda} = u^*, \ \frac{\partial x_2^*}{\partial \lambda} = v^*.$$

□

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