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DATA DEPENDENCE OF THE FIXED POINTS SET FOR A LOTKA-VOLTERRA SYSTEM

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ABSTRACT. In this paper we study data dependence of the fixed point set for a Lotka-Volterra system using the weakly Picard operators technique.

1. INTRODUCTION

The purpose of this paper is to study the following Lotka-Volterra system with delays

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1, 2; \tag{1}$$

$$\begin{cases} x_1(t) = \varphi(t),\ t \in [t_0 - \tau_1, t_0], \\ x_2(t) = \psi(t),\ t \in [t_0 - \tau_2, t_0], \end{cases} \tag{2}$$

where

(H$_1$) $\tau_1 \leq \tau_2,\ t_0 < b$;

(H$_2$) $f_i \in C([t_0, b] \times \mathbb{R}^4, \mathbb{R})$;

(H$_3$) there exists $L_f > 0$ such that:

$$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \leq L_f(\sum_{k=1}^{4} |u_k - v_k|),$$

for all $t \in [t_0, b],\ u_k, v_k \in \mathbb{R},\ k = \overline{1, 4},\ i = 1, 2$;

(H$_4$) $\varphi \in C([t_0 - \tau_1, t_0], \mathbb{R}),\ \psi \in C([t_0 - \tau_2, t_0], \mathbb{R})$.

The problem (1)–(2) with $x_1 \in C[t_0 - \tau_1, b] \cap C^1[t_0, b],\ x_2 \in C[t_0 - \tau_2, b] \cap C^1[t_0, b]$ is equivalent with

$$x_1(t) = \begin{cases} \varphi(t),\ t \in [t_0 - \tau_1, t_0], \\ \varphi(t_0) + \int_{t_0}^{t} f_1(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2))ds,\ t \in [t_0, b], \end{cases}$$

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(3)

$$x_2(t) = \begin{cases} \psi(t),\ t \in [t_0 - \tau_2, t_0], \\ \psi(t_0) + \int_{t_0}^{t} f_2(s, x_1(s), x_2(s), x_1(s-\tau_1), x_2(s-\tau_2))ds,\ t \in [t_0, b], \end{cases}$$

where $x_1 \in C[t_0 - \tau_1, b]$ and $x_2 \in C[t_0 - \tau_2, b]$.

The system (1) is equivalent with

$$x_1(t) = \begin{cases} x_1(t),\ t \in [t_0 - \tau_1, t_0], \\ x_1(t_0) + \int_{t_0}^{t} f_1(s, x_1(x), x_2(s), x_1(s-\tau_1), x_2(s-\tau_2))ds,\ t \in [t_0, b], \end{cases}$$

(4)

$$x_2(t) = \begin{cases} x_2(t),\ t \in [t_0 - \tau_2, t_0], \\ x_2(t_0) + \int_{t_0}^{t} f_2(s, x_1(x), x_2(s), x_1(s-\tau_1), x_2(s-\tau_2))ds,\ t \in [t_0, b]. \end{cases}$$

Consider the following operators

$$A_f, B_f : C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b] \to C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b],$$

where $A_f(x_1, x_2) = (A_{f_1}(x_1, x_2), A_{f_2}(x_1, x_2))$ is defined by the second part of (3) and $B_f(x_1, x_2) = (B_{f_1}(x_1, x_2), B_{f_2}(x_1, x_2))$ is defined by the second part of (4).

Remark 1. ([2]) *Let $\varphi \in C([t_0 - \tau_1, t_0], X)$ and $\psi \in C([t_0 - \tau_2, t_0], X)$. Then we consider $X_\varphi := \{x_1 \in C([t_0 - \tau_1, b], X) \mid x_1 |_{[t_0-\tau_1, t_0]} = \varphi\}$, $X_\psi := \{x_2 \in C([t_0 - \tau_2, b], X) \mid x_2 |_{[t_0-\tau_2, t_0]} = \psi\}$.*
We remark that $X = \bigcup_{\varphi,\psi} X_\varphi \times X_\psi$ is a partition of X and $X_\varphi \times X_\psi$ is an invariant subset of A_f and of B_f for all $\varphi \in C([t_0 - \tau_1, t_0])$ and $\psi \in C([t_0 - \tau_2, t_0])$.

In this paper we apply the weakly Picard operators technique to study data dependence of the fixed point set for the system (1).

2. Weakly Picard operators

In this paper we need some notions and results from the weakly Picard operator theory.

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \varnothing\}$- the family of the nonempty invariant subsets of A;

$A^{n+1} := A \circ A^n,\ A^1 = A,\ A^0 = 1_X,\ n \in \mathbb{N}$;

$P(X) := \{Y \subset X \mid Y \neq \varnothing\}$;

$H(Y,Z) := \max\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\}$- the Pompeiu-Hausdorff functional on $P(X)\times P(X)$.

Definition 2. ([3], [4]) *The operator A is a Picard operator* (PO) *if there exists $x^*\in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0\in X$.*

Definition 3. ([4]) *The operator A is a weakly Picard operator* (WPO) *if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x\in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 4. ([5]) *If A is* WPO *then we consider the operator A^∞, $A^\infty : X\to X$, defined by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 5. ([5]) $A^\infty(X) = F_A$.

Remark 6. ([5]) *If A is a* WPO *and $F_A = \{x^*\}$ then by definition the operator A is a PO.*

Remark 7. ([5]) *If A is a* PO *then*

$$F_{A^n} = F_A = \{x^*\}, \textit{ for all } n\in\mathbb{N}^*.$$

Remark 8. ([5]) *If A is a* WPO *then*

$$F_{A^n} = F_A \neq \emptyset, \textit{ for all } n\in\mathbb{N}^*.$$

Definition 9. ([6], [7]) *Let A be an WPO and $c>0$. The operator A is c-WPO if*

$$d(x, A^\infty(x)) \le c d(x, A(x)),\ \forall x\in X.$$

Example 10. ([6], [7]) *Let (X,d) be a complete metric space and $A: X\to X$ an operator. We suppose that there exists $L\in[0,1[$ such that*

$$d(A^2(x), A(x)) \le L d(x, A(x)),\ \forall x\in X.$$

Then A is c-WPO with $c=(1-L)^{-1}$.

Theorem 11. ([4]) *Let (X,d) be a metric space and $A : X\to X$ an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that:
(a) $X_\lambda\in I(A)$, $\lambda\in\Lambda$, $I(A)$-the family of nonempty invariant subsets of A;
(b) $A|_{X_\lambda} : X_\lambda\to X_\lambda$ is a Picard (c-Picard) operator for all $\lambda\in\Lambda$.

For the class of c-WPO we have the following result of data dependence.

Theorem 12. ([7]) *Let (X,d) be a metric space and $A_i : X \to X$, $i = 1,2$. We suppose that*

(i) the operator A_i is c_i-WPO, $i = 1,2$;
(ii) there exists $\eta > 0$ such that

$$d(A_1(x), A_2(x)) \leq \eta,\ \forall x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2).$$

3. Main results

Consider the problem (1)–(2). We have

Theorem 13. ([1]) *We suppose that:*
(a) *the conditions (H_1)-(H_4) are satisfied,*
(b) $\dfrac{4L}{\rho} < 1$.

Then the problem (1)–(2) has a unique solution. Moreover, if (x_1^*, x_2^*) the unique solution of (1)–(2), then

$$(x_1^*, x_2^*) = \lim_{n\to\infty} A_f^n(x_1, x_2),\ \textit{for all } x_1 \in C[t_0 - \tau_1, b],\ x_2 \in C[t_0 - \tau_2, b].$$

Remark 14. ([2]) *From Theorem 13 it follows that the operator $A_f \mid_{X_\varphi \times X_\psi} : X_\varphi \times X_\psi \to X_\varphi \times X_\psi$ is* PO. *But*

$$A_f \mid_{X_\varphi \times X_\psi} = B_f \mid_{X_\varphi \times X_\psi},$$

and

$$X := \bigcup_{\varphi,\psi} X_\varphi \times X_\psi\ ,\ X_\varphi \times X_\psi\ \in I(A_f),\ X_\varphi \times X_\psi\ \in I(B_f).$$

So, the operator B_f is WPO.

In what follow we shall use the c-WPOs technique to give some data dependence results, using Theorem 12.

We consider the following systems

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1,2, \tag{5}$$

$$x_i'(t) = g_i(t, x_1(t), x_2(t), x_1(t-\tau_1), x_2(t-\tau_2)),\ t \in [t_0, b],\ i = 1,2. \tag{6}$$

with the initial conditions (2).

From Remark 14 and Theorem 12, we have

Theorem 15. *Let f_i and g_i be as in Theorem 13. Let $S_1, S_2 \subset C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b] \to C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b]$ be the solution sets for the systems (5) and (6) coresponding to f_i and g_i, $i = 1, 2$. If $\eta_i > 0$, $i = 1, 2$ is such that*

$$|f_i(t, u_1, u_2, u_3, u_4) - g_i(t, u_1, u_2, u_3, u_4)| \leq \eta_i,$$

for all $t \in [t_0, b]$, $i = 1, 2$, $u_j \in \mathbb{R}$, $j = \overline{1, 4}$ then

$$H(S_1, S_2) \leq \frac{\eta_i (b - t_0) \rho}{\rho - 4L},$$

where $L = \max(L_{f_i}, L_{g_i})$.

Proof. We consider the operators

$$B_f, B_g : C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b] \to C[t_0 - \tau_1, b] \times C[t_0 - \tau_2, b],$$

where

$$\begin{aligned} B_f(x_1, x_2) &= (B_f^1(x_1, x_2), B_f^2(x_1, x_2)), \\ B_g(x_1, x_2) &= (B_g^1(x_1, x_2), B_g^2(x_1, x_2)), \end{aligned} \tag{7}$$

are defined by

$$B_f^1(x_1, x_2)(t) = \begin{cases} x_1(t), \ t \in [t_0 - \tau_1, t_0], \\ x_1(t_0) + \int_{t_0}^t f_1(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) ds, t \in [t_0, b], \end{cases}$$

$$B_f^2(x_1, x_2)(t) = \begin{cases} x_2(t), \ t \in [t_0 - \tau_2, t_0], \\ x_2(t_0) + \int_{t_0}^t f_2(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) ds, t \in [t_0, b], \end{cases}$$

and

$$B_g^1(x_1, x_2)(t) = \begin{cases} x_1(t), \ t \in [t_0 - \tau_1, t_0], \\ x_1(t_0) + \int_{t_0}^t g_1(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) ds, \ t \in [t_0, b], \end{cases}$$

$$B_g^2(x_1, x_2)(t) = \begin{cases} x_2(t), \ t \in [t_0 - \tau_2, t_0], \\ x_2(t_0) + \int_{t_0}^t g_2(s, x_1(s), x_2(s), x_1(s - \tau_1), x_2(s - \tau_2)) ds, \ t \in [t_0, b]. \end{cases}$$

Then, from (7) we obtain

$$\begin{aligned} (x_1, x_2) &= B_f(x_1, x_2), \\ (x_1, x_2) &= B_g(x_1, x_2). \end{aligned}$$

In the conditions of Theorem 12, B_f, B_g are c_i-weakly Picard operators with

$$c_i = (1 - \alpha_i)^{-1},$$

where $\alpha_i = \frac{4L}{\rho}$.

Then

$$H(S_1, S_2) \leq \frac{\eta_i(b - t_0)\rho}{\rho - 4L},$$

where $L = \max(L_{f_i}, L_{g_i})$. □

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