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NUMERICAL SOLUTIONS OF LOTKA-VOLTERRA SYSTEM WITH DELAY BY SPLINE FUNCTIONS OF EVEN DEGREE

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. This paper presents a numerical method for the approximate solution of a Lotka-Volterra system with delay. This method is essentially based on the natural spline functions of even degree introduced by using the derivative-interpolating conditions on simple knots.

1. Introduction

In recent years many papers were devoted to the problem of approximate integration of system of differential equation by spline functions. The theory of spline functions presents a special interest and advantage in obtaining numerical solutions of differential equations.

The splines functions of even degree are defined in a similar manner with that for odd degree spline functions, but using the derivative-interpolating conditions. These spline functions preserve all the remarkable extremal and convergence properties of the odd degree splines, and are very suitable for the numerical solutions of the differential equation problems, especially for the delay differential equations with initial conditions.

In this paper we consider a spline approximation method for the numerical solution of a Lotka-Volterra system with delay. The purpose of the present study is to extend the results of [1], [2], [3], [5] from the delay differential equations to the delay

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differential system. In the same manner we shall develop some theory and algorithms for the numerical solutions of a class of delay Lotka-Volterra system.

2. Basic definitions and properties of even degree splines

Let Δ_n be the following partition of the real axis

$$\Delta_n : -\infty = t_0 < a = t_1 < ... < t_n = b < t_{n+1} = +\infty$$

and let m, n be two given natural numbers, satisfying the conditions $n \geq 1,\ m \leq n+1$. One denotes by I_k the following subintervals

$$I_k := [t_k, t_{k+1}[,\ k = \overline{1,n},\ I_0 :=]t_0, t_1[.$$

Definition 1. [3] *For the couple (m, Δ_n) a function $s : \mathbb{R} \to \mathbb{R}$ is called a natural spline function of even degree $2m$ if the following conditions are satisfied:*

1^0 $s \in C^{2m-1}(\mathbb{R})$,

2^0 $s\,|_{I_k} \in \mathcal{P}_{2m},\ k = \overline{1,n}$,

3^0 $s\,|_{I_0} \in \mathcal{P}_m,\ s\,|_{I_n} \in \mathcal{P}_m$,

where $\mathcal{P}_k$ represents the set of algebraic polynomials of degree $\leq k$.

We denote by $\mathcal{S}_{2m}(\Delta_n)$ the linear space of natural polynomial splines of even degree $2m$ with the simple knots $t_1, ..., t_n$.

We now show that $\mathcal{S}_{2m}(\Delta_m)$ is a finite dimensional linear space of functions and we give a basis of it.

Theorem 1. [3] *Any element $s \in \mathcal{S}_{2m}(\Delta_n)$ has the following representation*

$$s(t) = \sum_{i=0}^{m} A_i t^i + \sum_{k=1}^{n} a_k (t - t_k)_+^{2m},$$

where the real coefficients $(A_i)_0^m$ are arbitrary, and the coefficients $(a_k)_1^n$ satisfy the conditions

$$\sum_{k=1}^{n} a_k t_k^i = 0,\ i = \overline{0, m-1}.$$

Remark 1. [3] *If $n + 1 = m$, then $a_k = 0$, $k = \overline{1,n}$.*

Theorem 2. [3] *Suppose that $n+1 \geq m$, and let $f : [t_1, t_n] \to \mathbb{R}$ be a given function such that $f'(t_k) = y'_k$, $k = \overline{1,n}$, and $f(t_1) = y_1$, where y'_k, $k = \overline{1,n}$, and y_1 are given real numbers. Then there exists a unique spline function $s_f \in \mathcal{S}_{2m}(\Delta_n)$, such that the following derivative-interpolating conditions*

$$s_f(t_1) = y_1, \tag{2.1}$$

$$s'_f(t_k) = y'_k, \ k = \overline{1,n}, \tag{2.2}$$

hold.

Corollary 1. [3] *There exists a unique set of $n+1$ fundamental natural polynomial spline functions $S_k \in \mathcal{S}_{2m}(\Delta_n)$, $k = \overline{1,n}$, and $s_0 \in \mathcal{S}_{2m}(\Delta_n)$ satisfying the conditions:*

$$\begin{aligned} s_0(t_1) &= 1, \quad s'_0(t_k) = 0, \qquad k = \overline{1,n}, \\ S_k(t_1) &= 0, \quad S'_k(t_i) = \delta_{ik}, \quad i,k = \overline{1,n}. \end{aligned}$$

It is clear that the functions $\{s_0, S_k, \ k = \overline{1,n}\}$, form a basis of the linear space $\mathcal{S}_{2m}(\Delta_n)$, and for s_f we obtain the representation

$$s_f(t) = s_0(t) f(t_1) + \sum_{k=1}^{n} S_k(t) f'(t_k).$$

But because $s_0(t) = 1$, it follows that

$$s_f(t) = f(t_1) + \sum_{k=1}^{n} S_k(t) f'(t_k).$$

Let us introduce the following sets of functions

$$\begin{aligned} &W_2^{m+1}(\Delta_n) := \{g : [a,b] \to \mathbb{R} \mid g^{(m)}\text{abs.cont.on } I_k \text{ and } g^{(m+1)} \in L_2[a,b]\}, \\ &W_2^{m+1}[a,b] := \{g : [a,b] \to \mathbb{R} \mid g^{(m)}\text{abs.cont.on } [a,b] \text{ and } g^{(m+1)} \in L_2[a,b]\}, \\ &\overset{0}{W}{}_{2,f}^{m+1}(\Delta_n) := \{g \in W_2^{m+1}(\Delta_n) \mid g'(t_k) = f'(t_k), \ k = \overline{1,n}\}, \\ &W_{2,f}^{m+1}(\Delta_n) := \{g \in W_2^{m+1}(\Delta_n) \mid g(t_0) = f(t_0)\}. \end{aligned}$$

Theorem 3. [3] *(Minimal norm property). If* $s \in \mathcal{S}_{2m}(\Delta_n) \cap W^{m+1}_{\overset{0}{2,f}}(\Delta_n)$*, then*

$$\left\| s^{(m+1)} \right\|_2 \leq \left\| g^{(m+1)} \right\|_2, \ \forall g \in W^{m+1}_{\overset{0}{2,f}}(\Delta_n),$$

holds, $\|\cdot\|_2$ *being the usual* L_2*-norm.*

For any function $f \in W_2^{m+1}(\Delta_n)$, we have the following corollaries.

Corollary 2. [3] $\left\| f^{(m+1)} \right\|_2^2 = \left\| s_f^{(m+1)} \right\|_2^2 + \left\| f^{(m+1)} - s_f^{(m+1)} \right\|_2^2.$

Corollary 3. [3] $\left\| s_f^{(m+1)} \right\|_2 \leq \left\| f^{(m+1)} \right\|_2.$

Corollary 4. [3] $\left\| f^{(m+1)} - s_f^{(m+1)} \right\|_2 \leq \left\| f^{(m+1)} \right\|_2.$

Remark 2. [3] *If* $\widetilde{s} := s_f + p_m$*, where* $p_m \in \mathcal{P}_m$*, it follows* $\left\| \widetilde{s}^{(m+1)} \right\|_2 \leq \left\| f^{(m+1)} \right\|_2.$

Theorem 4. [3] *(Best approximation property). If* $f \in W_2^{m+1}(\Delta_n)$ *and* $s_f \in \mathcal{S}_{2m}(\Delta_n)$ *is the derivative-interpolating spline function of even degree, then, for any* $s \in \mathcal{S}_{2m}(\Delta_n)$ *the relation*

$$\left\| s_f^{(m+1)} - f^{(m+1)} \right\|_2 \leq \left\| s^{(m+1)} - f^{(m+1)} \right\|_2$$

holds.

Remark 3. [3] *If* $s_f - s \in \mathcal{P}_m$ *then*

$$\left\| s_f^{(m+1)} - f^{(m+1)} \right\|_2 = \left\| s^{(m+1)} - f^{(m+1)} \right\|_2.$$

3. The numerical solutions of Lotka-Volterra system with delay by spline functions of even degree

Let us consider the following delay differential system with a constant delay $\omega > 0$

$$\frac{dy^u}{dt} = f^u(t, y^1(t), y^2(t), y^1(t-\omega), y^2(t-\omega)), \ a \leq t \leq b, \ u = 1, 2 \tag{3.1}$$

with initial conditions

$$y^u(t) = \varphi^u(t), \ t \in [a-\omega, a], \ u = 1, 2 \tag{3.2}$$

and we suppose that $f^u : D \subset \mathbb{R}^4 \to \mathbb{R}$, satisfies all the conditions assuring the existence and uniqueness of the solutions $y^u : [a, b] \to \mathbb{R}$ of the problem (3.1)+(3.2).

We propose an algorithm to approximate the solutions y^u of the problem (3.1)+(3.2) by spline functions of even degree $s^u \in \mathcal{S}_{2m}(\Delta_n)$, where Δ_n is a partition of $[a,b]$ and m,n are two integers satisfying the conditions $n \geq 1$ and $m \leq n+1$.

For $t \in [a, a+\omega]$, the problem (3.1)+(3.2) reduces to the following usual initial value problems:

$$\begin{cases} \dfrac{dy^u}{dt} = f^u(t, y^1(t), y^2(t), y^1(t-\omega), y^2(t-\omega)),\ a \leq t \leq a+\omega \\ y^u(t) = \varphi^u(a) = y_1^u\ ,\ u = 1,2 \end{cases}$$

Theorem 5. *If y^u are the exact solutions of the problem (3.1)+(3.2), then, there exists some unique spline functions $s_{y^u} \in \mathcal{S}_{2m}(\Delta_n)$ such that:*

$$\begin{aligned} & s_{y^u}(t_1) = y^u(t_1) = \varphi^u(t_1), \\ & \frac{ds_{y^u}}{dt}(t_k) = \frac{dy^u}{dt}(t_k),\ k = \overline{1,n},\ u = 1,2 \end{aligned} \tag{3.3}$$

The assertion of this theorem is a direct consequence of Theorem 2 by substituting t_1 by a and f by y^u.

Denoting $y_k^u := y^u(t_k),\ \overline{y}_k^u := y^u(t_k - \omega),\ k = \overline{1,n},\ u = 1,2$, we have

$$\begin{aligned} & s_{y^u}(t_1) = y_1^u \\ & \frac{ds_{y^u}}{dt}(t_k) = f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2),\ k = \overline{1,n},\ u = 1,2. \end{aligned}$$

Corollary 5. *If the functions $\{s_0, S_k, k = \overline{1,n}\}$ are the fundamental spline functions in $\mathcal{S}_{2m}(\Delta_n)$, then we can write*

$$s_{y^u}(t) = \varphi^u(a) + \sum_{k=1}^{n} S_k(t) f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2),\ u = 1,2, \tag{3.4}$$

where $y_k^1,\ y_k^2,\ k = \overline{2,n}$, are unknown, and

$$\overline{y}_k^u = \begin{cases} \varphi^u(t_k - \omega), & \text{if } t_k \leq a+\omega, \text{ are known,} \\ y^u(t_k - \omega), & \text{if } t_k > a+\omega, \text{ are unknown.} \end{cases}$$

We shall call the function $s_{y^u}(t)$, the approximating solution of the problem (3.1)+(3.2) and it can be written as follows

$$\begin{aligned} s_{y^u}(t) &= \varphi^u(a)+ \\ &+ \sum_{t_k\leq a+\omega} S_k(t) f^u(t_k, y_k^1, y_k^2, \varphi^1(t_k-\omega), \varphi^2(t_k-\omega)) \\ &+ \sum_{t_k > a+\omega} S_k(t) f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2). \end{aligned} \tag{3.5}$$

For simplicity, in writing (3.5), let us use the following index sets:

$J_1 := \{j \in \mathbb{N} \mid t_j > a+\omega, \exists i : t_j - \omega = t_i\} =: \{j_1, j_2, ..., j_q\}$,

$J_0 := \{i \in \mathbb{N} \mid \exists j \in J_1 : t_j - \omega = t_i\} =: \{i_1, i_2, ..., i_q\}$,

$I := \{j \in \mathbb{N} \mid t_j > a+\omega, \nexists i : t_j - \omega = t_i\} =: \{d_1, d_2, ..., d_p\}$.

Thus, we can write (3.5) in the form

$$\begin{aligned} s_{y^u}(t) &= \varphi^u(a)+ \\ &+ \sum_{t_k\leq a+\omega} S_k(t) f^u(t_k, y_k^1, y_k^2, \varphi^1(t_k-\omega), \varphi^2(t_k-\omega)) \\ &+ \sum_{k=1}^{q} S_{j_k}(t) f^u(t_{j_k}, y_{j_k}^1, y_{j_k}^2, y_{i_k}^1, y_{i_k}^2) \\ &+ \sum_{k=1}^{p} S_{d_k}(t) f^u(t_{d_k}, y_{d_k}^1, y_{d_k}^2, \overline{y}_{d_k}^1, \overline{y}_{d_k}^2), \end{aligned} \tag{3.6}$$

where the values $y_k^u,\ k=\overline{2,n}$, and $\overline{y}_k^u,\ k=\overline{1,p},\ u=1,2$ are unknown.

Before giving an algorithm to determine these values, we shall give the following estimation error and convergence theorem.

Theorem 6. [3]*If* $y^u \in W_2^{m+1}[a,b],\ u=1,2$ *are the exact solutions of the problem (3.1)+(3.2) and* s_{y^u} *is the spline approximating solution for* y^u, *the following estimations hold:*

$$\left\| y^{u(k)} - s_{y^u}^{(k)} \right\|_\infty \leq \sqrt{m}(m-1)(m-2)...k\Delta_n^{m-k+\frac{1}{n}} \left\| y^{u(m+1)} \right\|_2,$$

for $k=1,2,...,m$, *where* $\|\Delta_n\| := \max\limits_{i=\overline{2,n}}\{t_i - t_{i-1}\},\ u\ =1,2.$

Corollary 6. [3]*If* $y^u \in W_2^{m+1}[a,b]$, *we have*

$$\|y^u - s_{y^u}\|_\infty \leq (b-a)\sqrt{m}(m-1)! \left\| y^{u(m+1)} \right\|_2 \|\Delta_n\|^{m-\frac{1}{2}},\ u=1,2.$$

Corollary 7. [3] $\lim\limits_{\|\Delta_n\|\to 0} \left\| y^{u(k)} - s_{y^u}^{(k)} \right\|_\infty = 0,\ k=\overline{1,m},\ u=1,2.$

4. Effective development of the algorithm

For any $t \in [a,b]$, we suppose that $y^u(t) \approx s_{y^u}(t),\ u = 1,2.$

If we denote, as usual, $e^u(t) := y^u(t) - s_{y^u}(t),\ t \in [a,b],$

we have

$$|e^u(t)| \leq \sqrt{m}(m-1)!\, \|\Delta_n\|^{m-\frac{1}{2}} \left\| y^{u(m+1)} \right\|_2,$$

or

$$|e^u(t)| = O(\|\Delta_n\|^{m-\frac{1}{2}}),\ \forall t \in [a,b].$$

If we denote

$$\begin{aligned} w_i^u &:= s_{y^u}(t_i), \qquad & e_i^u &:= e^u(t_i) = y^u(t_i) - s_{y^u}(t_i), i = \overline{1,n}, \\ \overline{w}_i^u &:= s_{y^u}(t_i - \omega), & \overline{e}_i^u &:= e^u(t_i - \omega) = y^u(t_i - \omega) - s_{y^u}(t_i - \omega), i = \overline{1,n}, \end{aligned}$$

then we have $y_i^u = w_i^u + e_i^u,\ \overline{y}_i^u = \overline{w}_i^u + \overline{e}_i^u$, where

$$\begin{aligned} w_i^u &= y_1^u + \sum_{k=1}^{n} S_k(t_i) f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2),\ i = \overline{1,n},\ u = 1,2, \qquad (4.1) \\ \overline{w}_i^u &= y_1^u + \sum_{k=1}^{n} S_k(t_i - \omega) f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2),\ i = \overline{1,n},\ u = 1,2. \end{aligned}$$

In what follows, we suppose that in (3.1)+(3.2) the functions

$$\begin{aligned} & f^u : D \subset \mathbb{R}^5 \to \mathbb{R}\ (D \subset [a,b] \times \mathbb{R}^4), \\ & \frac{\partial f^u(t,u_1,u_2,u_3,u_4)}{\partial u_1},\ \frac{\partial f^u(t,u_1,u_2,u_3,u_4)}{\partial u_2}, \\ & \frac{\partial f^u(t,u_1,u_2,u_3,u_4)}{\partial u_3},\ \frac{\partial f^u(t,u_1,u_2,u_3,u_4)}{\partial u_4} \end{aligned}$$

are continuous. Thus,

$$\begin{aligned} f^u(t_k, y_k^1, y_k^2, \overline{y}_k^1, \overline{y}_k^2) &= f^u(t_k, w_k^1 + e_k^1, w_k^2 + e_k^2, \overline{w}_k^1 + \overline{e}_k^1, \overline{w}_k^2 + \overline{e}_k^2) \\ &= f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2) + e_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_1} \\ &+ e_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_2} + \overline{e}_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_3} \\ &+ \overline{e}_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_4} \end{aligned}$$

where

$$\min(w_k^u, w_k^u + e_k^u) < \xi_k^u < \max(w_k^u, w_k^u + e_k^u),$$

$$\min(\overline{w}_k^u, \overline{w}_k^u + \overline{e}_k^u) < \eta_k^u < \max(\overline{w}_k^u, \overline{w}_k^u + \overline{e}_k^u),\ u = 1, 2.$$

We can write the system (4.1) in the form

$$w_i^u = y_1^u + \sum_{k=1}^n S_k(t_i) f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2) + E_i^u,\ i = \overline{1, n},\ u = 1, 2$$

$$\overline{w}_i^u = y_1^u + \sum_{k=1}^n S_k(t_i - \omega) f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2) + \overline{E}_i^u,\ i = \overline{1, n},\ u = 1, 2$$

where

$$\begin{aligned} E_i^u &= \sum_{k=1}^n S_k(t_i) e_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_1} + \sum_{k=1}^n S_k(t_i) \overline{e}_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_2} \\ &+ \sum_{k=1}^n S_k(t_i) e_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_3} + \sum_{k=1}^n S_k(t_i) \overline{e}_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_4} \\ &= O(\|\Delta_n\|^{m-\frac{1}{2}}), \end{aligned}$$

$$\begin{aligned} \overline{E}_i^u &= \sum_{k=1}^n S_k(t_i - \omega) e_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_1} + \\ &+ \sum_{k=1}^n S_k(t_i - \omega) \overline{e}_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_2} + \\ &+ \sum_{k=1}^n S_k(t_i - \omega) e_k^1 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_3} + \\ &+ \sum_{k=1}^n S_k(t_i - \omega) \overline{e}_k^2 \frac{\partial f^u(t_k, \xi_k^1, \xi_k^2, \eta_k^1, \eta_k^2)}{\partial u_4} = \\ &= O(\|\Delta_n\|^{m-\frac{1}{2}}),\ i = \overline{1, n},\ u = 1, 2, \end{aligned}$$

supposing that

$$\begin{aligned} &\left| \frac{\partial f^u(t, u_1, u_2, u_3, u_4)}{\partial u_1} \right| \le M_1, \left| \frac{\partial f^u(t, u_1, u_2, u_3, u_4)}{\partial u_2} \right| \le M_2, \\ &\left| \frac{\partial f^u(t, u_1, u_2, u_3, u_4)}{\partial u_3} \right| \le M_3, \left| \frac{\partial f^u(t, u_1, u_2, u_3, u_4)}{\partial u_4} \right| \le M_4, \end{aligned} \tag{4.2}$$

on D. Obviously, $E_i^u \to 0$ and $\overline{E}_i^u \to 0$ for $\|\Delta_n\| \to 0,\ u = 1, 2.$

Now, we have to solve the following nonlinear system:

$$\begin{cases} w_i^u = y_1^u + \sum\limits_{k=1}^{n} S_k(t_i) f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2), \ i = \overline{1,n}, \\ \overline{w}_i^u = y_1^u + \sum\limits_{k=1}^{n} S_k(t_i - \omega) f^j(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2), \ i = \overline{1,n}\ . \end{cases} \tag{4.3}$$

Let us denote:

$$w^u := (w_1^u, ..., w_n^u), \ \overline{w}^u := (\overline{w}_1^u, ..., \overline{w}_n^u), \ W^u = (w^u, \overline{w}^u),$$

$$H_i^u(w, \overline{w}) := y_1^u + \sum_{k=1}^{n} S_k(t_i) f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2), \ i = \overline{1,n},$$

$$\overline{H}_i^u(w, \overline{w}) := y_1^u + \sum_{k=1}^{n} S_k(t_i - \omega) f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2), \ i = \overline{1,n},$$

$$\begin{aligned} H^u(W^u) &:= H^u(w^u, \overline{w}^u) \\ &:= (H_1^u(w^u, \overline{w}^u), ..., H_n^u(w^u, \overline{w}^u), \overline{H}_1^u(w^u, \overline{w}^u), ..., \overline{H}_n^u(w^u, \overline{w}^u)) \end{aligned}$$

and

$$A^u = \begin{pmatrix} \frac{\partial H_1^u(w^u,\overline{w}^u)}{\partial w_1^u} & \cdots & \frac{\partial H_1^u(w^u,\overline{w}^u)}{\partial w_n^u} & \frac{\partial H_1^u(w^u,\overline{w}^u)}{\partial \overline{w}_1^u} & \cdots & \frac{\partial H_1^u(w^u,\overline{w}^u)}{\partial \overline{w}_n^u} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial H_n^u(w^u,\overline{w}^u)}{\partial w_1^u} & \cdots & \frac{\partial H_n^{uj}(w^u,\overline{w}^u)}{\partial w_n^u} & \frac{\partial H_n^u(w^u,\overline{w}^u)}{\partial \overline{w}_1^u} & \cdots & \frac{\partial H_n^u(w^u,\overline{w}^u)}{\partial \overline{w}_n^u} \\ \frac{\partial \overline{H}_1^u(w^u,\overline{w}^u)}{\partial w_1^u} & \cdots & \frac{\partial \overline{H}_1^u(w^u,\overline{w}^u)}{\partial w_n^u} & \frac{\partial \overline{H}_1^u(w^u,\overline{w}^u)}{\partial \overline{w}_1^u} & \cdots & \frac{\partial \overline{H}_1^u(w^u,\overline{w}^u)}{\partial \overline{w}_n^u} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \overline{H}_n^u(w^u,\overline{w}^u)}{\partial w_1^u} & \cdots & \frac{\partial \overline{H}_n^u(w^u,\overline{w}^u)}{\partial w_n^u} & \frac{\partial \overline{H}_n^u(w^u,\overline{w}^u)}{\partial \overline{w}_1^u} & \cdots & \frac{\partial \overline{H}_n^u(w^u,\overline{w}^u)}{\partial \overline{w}_n^u} \end{pmatrix}$$

Shortly, we write the system (4.3) by

$$W^u = H^u(W^u) \tag{4.4}$$

In order to investigate the solvability of the nonlinear system (4.4) we shall use a classical theorem.

Theorem 7. [6] *Let* $\Omega \subset \mathbb{R}^{2n+2}$ *be a bounded domain and let* $H^u : \Omega \to \Omega$ *be a vector function defined by*

$$W^u = (w^u, \overline{w}^u) \longmapsto$$
$$(H_1^u(w^u, \overline{w}^u), ..., H_n^u(w^u, \overline{w}^u), \overline{H}_1^u(w^u, \overline{w}^u), ..., \overline{H}_n^u(w^u, \overline{w}^u))$$
$$= H^u(W^u).$$

If the functions H^u*, and* $\dfrac{\partial H^u}{\partial W^u}$*, are continuous in* Ω*, then there exists in* Ω *a fixed point* W^{u*} *of* H^u*, i.e.* $W^{u*} = H^u(W^{u*})$*, which can be found by iterations.* $W^{u*} = \lim\limits_{n\to\infty} W^{u(n)}$*,* $W^{u(k)} := H^u(W^{u(k-1)})$*,* $k = 1, 2, ...,$ $W^{u(0)} \in \Omega$ *(arbitrary). If in addition* $\|A\| \le L < 1$*, for any iteration* $W^{u(k)}$*, the following estimation holds:*

$$\left\| W^u - W^{u(k)} \right\| \le \frac{L^k}{1-L} \left\| W^{u(1)} - W^{u(0)} \right\|.$$

Taking in consideration the expression of H^u, the matrix A^u is $A^u = SF^u$, where

$$S = \begin{pmatrix} S_1(t_1) & \cdots & S_n(t_1) & S_1(t_1) & \cdots & S_n(t_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_1(t_n) & \cdots & S_n(t_n) & S_1(t_n) & \cdots & S_n(t_n) \\ S_1(t_1-\omega) & \cdots & S_n(t_1-\omega) & S_1(t_1-\omega) & \cdots & S_n(t_1-\omega) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_1(t_n-\omega) & \cdots & S_n(t_n-\omega) & S_1(t_n-\omega) & \cdots & S_n(t_n-\omega) \end{pmatrix}$$

and F is the diagonal matrix with the following elements:

$$\frac{\partial f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2)}{\partial w_k^u}, \ \frac{\partial f^u(t_k, w_k^1, w_k^2, \overline{w}_k^1, \overline{w}_k^2)}{\partial \overline{w}_k^u}, k = \overline{1,n}, \ u = 1, 2.$$

Theorem 8. *Suppose that there exists the constants* M*,* N *such that (4.2) holds and*

$$|f^u(t, u_1, u_2, u_3, u_4)| \le N_u, \ \forall (t, u_1, u_2, u_3, u_4) \in D, \ u = 1, 2.$$

If $M_u \le \|S\|^{-1}$*, then the system (4.3) has a solution which can be found by iterations.*

5. Numerical example

Example 1. *Consider the following Lotka-Volterra delay differential system*

$$\begin{cases} \dfrac{dy^1}{dt} = y^1\left[y^1(t-1) + y^2(t-1) + 1 - e^{t-1} - e^{2t-2}\right] \\ \dfrac{dy^2}{dt} = y^2\left[y^2(t-1) + 2 - e^{2t-2}\right] \end{cases}, t \in [0,b],$$

with initial conditions

$$\begin{cases} y^1(t) = \varphi^1(t) = e^t,\ t \in [-1,0] \\ y^2(t) = \varphi^2(t) = e^{2t},\ t \in [-1,0] \end{cases},$$

and the corresponding exact solutions

$$(y^1(t), y^2(t)) = (e^t, e^{2t}).$$

In the below table are given the actual errors for the considered examples. The table list

$$\max\left\{\left|w_i^u - y^u(t_i)\right|,\ i = \overline{1,n};\ \left|\overline{w}_i^u - y^u(t_j - \omega)\right|,\ j \in I;\right.$$
$$\left.\left|s_{y^u}(a + 0.1i) - y(a + 0.1i)\right|,\ i = \overline{1, 10(b-a)}\right\},$$

for $m = 1, 2, 3$ and the interval $[a,b]$ is $[0,2]$.

$[a,b]$	$[0,2]$		
$n \setminus m$	1	2	3
6	65.6521	5.4291	6.4198
9	12.2874	0.75975	0.25095
11	7.0645	0.39634	0.072303

For $a = 0,\ b = 2,\ \omega = 1,\ m = 1,\ n = 6$ we obtain $r = 3$ (the number of the nodes at the left of $a + \omega$), $p = 3,\ q = 0$. The approximating solution $\widetilde{s}^u$ and the exact solution y^u , $u = 1, 2$, in this case,are plotted in FIGURE 1 and FIGURE 2. For $a = 0,\ b = 2,\ \omega = 1,\ m = 2,\ n = 9$ we obtain $r = 5,\ p = 0,\ q = 4$. The approximating solution $\widetilde{s}^u$ and the exact solution y^u , $u = 1, 2$, in this case, are plotted in FIGURE 3 and FIGURE 4.

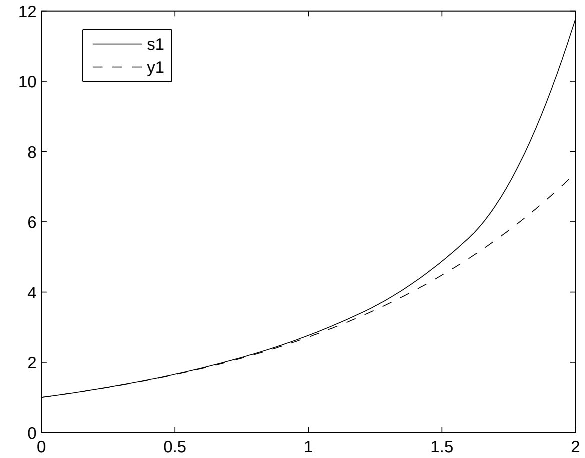

FIGURE 1. Comparison between the approximation solution $\widetilde{s}^1$ and the exact solution y^1 in the first case.

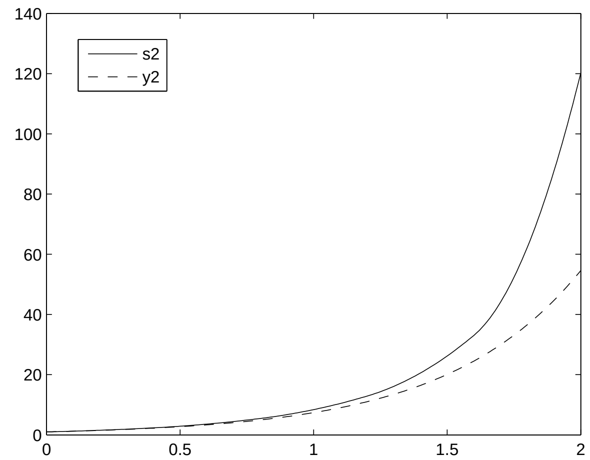

FIGURE 2. Comparison between the approximation solution $\widetilde{s}^2$ and the exact solution y^2 in the first case.

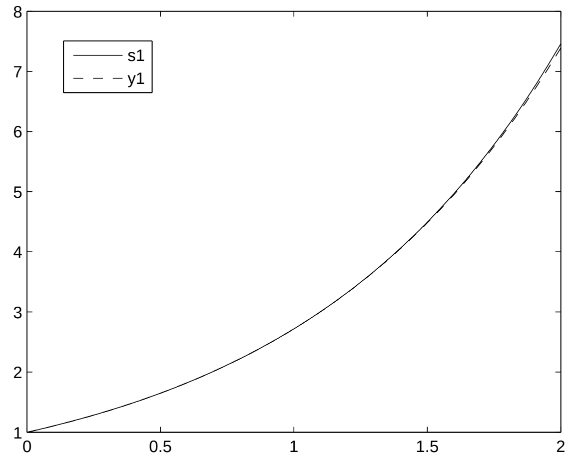

FIGURE 3. Comparison between the approximation solution $\widetilde{s}^1$ and the exact solution y^1 in the second case.

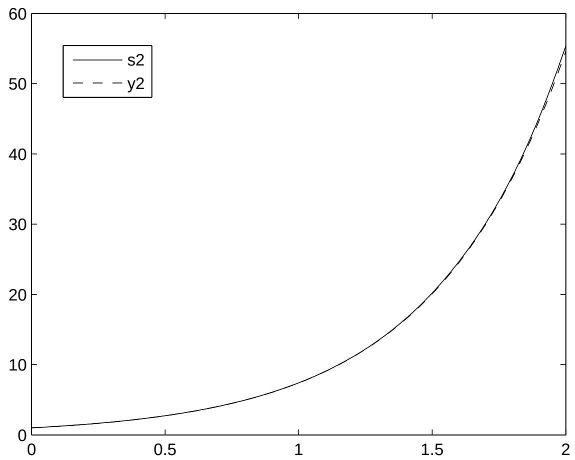

FIGURE 4. Comparison between the approximation solution $\widetilde{s}^2$ and the exact solution y^2 in the second case.

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