The convergence of Mann iteration with delay

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Abstract

We show the convergence of Mann iteration with delay for various classes of non-Lipschitzian operators.

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1 Introduction

Let X be a real Banach space, B be a nonempty, convex subset of X, and $T : B \to B$ be an operator. Let $u_1 \in B$ be given and $s > 0$ a fixed number. We consider the following iteration, to which we further refer as Mann iteration with delay, see [5]:

$$u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n T u_{n-s}, \tag{1}$$

the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty. \tag{2}$$

We are inspired for such delays from economics and biology problems in which fixed point are required. Usually, T is a contraction. It is wellknown that Mann iteration is desirable when T is not a contraction.

The operator $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X$, is called *the normalized duality mapping*. The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$. It is easy to see that we have $\langle j(x), y\rangle \leq \|x\| \|y\|, \forall x, y \in X, \forall j(x) \in J(x)$.

Definition 1 *Let X be a real Banach space. Let B be a nonempty subset. A map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0, 1)$ and a $j(x - y) \in J(x - y)$ such that*

$$\langle Tx - Ty, j(x - y)\rangle \leq (1 - k) \|x - y\|^2, \forall x, y \in B. \tag{3}$$

A map $S : B \to B$ is called strongly accretive if there exists $k \in (0, 1)$ and a $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y)\rangle \geq k \|x - y\|^2, \forall x, y \in B. \tag{4}$$

In (3) when $k = 1$, then T is called *pseudocontractive*. In (4) when $k = 0$, then S is called *accretive*. Let us denote by I the identity map.

Lemma 2 *If X is a real Banach space, then the following relation is true*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \forall x, y \in X, \forall j(x + y) \in J(x + y). \quad (5)$$

Lemma 3 [6] *Let $(\rho_n)_n$ be a nonnegative sequence which satisfies the following inequality*

$$\rho_{n+1} \le (1 - \lambda_n)\rho_n + \varepsilon_n, \quad (6)$$

where $\lambda_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\varepsilon_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \rho_n = 0$.

Lemma 4 *Let $s \ge 0$ be a fixed number, $\{a_n\}$ a nonnegative sequence which satisfies the following inequality*

$$a_{n+1} \le (1 - \alpha_n)a_{n-s} + \sigma_n, \quad (6)$$

where $\alpha_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sigma_n = o(\alpha_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Note that sequence $\{a_n\}$ is the reunion of $s + 1$ independent subsequences. If all such subsequences converges to zero then $\{a_n\}$ shall converge. The generic subsequence satisfies (6). Set $\rho_n := a_{n-s}$, $\lambda_n := \alpha_n$, $\varepsilon_n := \sigma_n$, and use Lemma 3 to obtain $\lim_{n\to\infty} a_n = 0$. ∎

2 Main result

Theorem 5 *Let $s \ge 0$ be a fixed number, X a real Banach space with a uniformly convex dual X^*, B a nonempty closed convex bounded subset of X, and $T : B \to B$ be a continuous strongly pseudocontractive mapping. Then the Mann iteration with delay $\{x_n\}_{n=1}^{\infty}$ defined by (1) converges strongly to the unique fixed point of T.*

Proof. Corollary 1 of [2] assures the existence of a fixed point. The uniqueness of the fixed point comes from (3). Because X^* is uniformly convex the duality map is singled valued (see, e.g., [1]). Let x^* be the fixed point of T. Using (1), (3) and Lemma 2 we get

$$\begin{aligned}
\|u_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)(u_{n-s} - x^*) + \alpha_n(Tu_{n-s} - Tx^*)\|^2 \\
&\le (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n \langle Tu_{n-s} - Tx^*, J(u_{n+1} - x^*)\rangle \\
&= (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n \langle Tu_{n-s} - Tx^*, J(u_{n-s} - x^*)\rangle + \\
&\quad + 2\alpha_n \langle Tu_{n-s} - Tx^*, J(u_{n+1} - x^*) - J(u_{n-s} - x^*)\rangle \\
&\le (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n(1 - k) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n
\end{aligned}$$

where

$$\sigma_n = \langle Tu_{n-s} - Tx^*, J(u_{n+1} - x^*) - J(u_{n-s} - x^*)\rangle.$$

Now we shall show $\sigma_n \to 0$ as $n \to \infty$. Observe that $(\|Tu_{n-s} - Tx^*\|)_n$ is bounded. We prove now that

$$J(u_{n+1} - x^*) - J(u_{n-s} - x^*) \to 0 \text{ as } n \to \infty. \tag{7}$$

Proposition 12.3 on page 115, of [2] assures that, when X^* is uniformly convex, then J is uniformly continuous on every bounded set of X. To prove (7) it is sufficient to see that

$$\|u_{n+1} - u_{n-s}\| = \alpha_n \|u_{n-s} - Tu_{n-s}\| \leq \alpha_n(\|u_{n-s}\| + \|Tu_{n-s}\|) \leq 2\alpha_n M \to 0$$

where $M = \sup\{\|u_{n-s}\|, \|Tu_{n-s}\|\}$. The sequences $(u_{n-s})_n$, $(Tu_{n-s})_n$ are bounded being in the bounded set B. Hence (7) holds. Then

$$\begin{aligned} \|u_{n+1} - x^*\|^2 &\leq ((1-\alpha_n)^2 + 2\alpha_n(1-k)) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n \qquad (8) \\ &= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n - 2\alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n \\ &= (1 + \alpha_n^2 - 2\alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n. \end{aligned}$$

The condition $\lim_{n\to\infty} \alpha_n = 0$ implies the existence of an n_0 such that for all $n \geq n_0$ we have

$$\alpha_n < k \tag{9}$$

Substituting (9) into (8) we get $1 + \alpha_n^2 - 2\alpha_n k < 1 - \alpha_n k$. Finally, the above inequality yields

$$\|u_{n+1} - x^*\|^2 \leq (1 - \alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n.$$

Setting $a_n = \|u_{n-s} - x^*\|^2$, $\lambda_n = \alpha_n k \in (0,1)$, and using Lemma 4, we obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|u_{n-s} - x^*\|^2 = 0$ i.e.

$$\lim_{n\to\infty} \|u_{n-s} - x^*\| = 0.$$

■

3 The accretive and strongly accretive cases

Let I denote the identity map.

Remark 6 *The operator T is a (strongly) pseudocontractive map if and only if $(I - T)$ is a (strongly) accretive map.*

Remark 7

1. *Let $T, S : X \to X$, and $f \in X$ be given. A fixed point for the map $Tx = f + (I - S)x, \forall x \in X$ is a solution for $Sx = f$.*
2. *Let $f \in X$ be a given point. If S is an accretive map then $T = f - S$ is a strongly pseudocontractive map.*

Consider Mann iteration with delay and set $Tx = f + (I - S)x$ to obtain

$$u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n \left(f + (I - S)u_{n-s}\right). \tag{10}$$

Remarks 6 and 7 and Theorem 5 lead to the following results.

Corollary 8 *Let X be a real Banach space with a uniformly convex dual X^*, and $S : X \to X$ a continuous and strongly accretive map with $(I - S)(X)$ bounded, $\{\alpha_n\}$ satisfies (2), and $u_0 = x_0 \in X$, then, the Mann iteration with delay (10) converges to the solution of $Sx = f$.*

Let S be an accretive operator. The operator $Tx = f - Sx$ is strongly pseudocontractive, for a given $f \in X$. A solution for $Tx = x$ becomes a solution for $x + Sx = f$. Consider Mann iteration with delay, set $Tx := f - Sx$ such that

$$u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n \left(f - Su_{n-s}\right). \tag{11}$$

Again, using the Remarks 6 and 7 and Theorem 5 we obtain the following result.

Corollary 9 *Let X be a real Banach space with a uniformly convex dual X^*, and B a nonempty, convex, closed subset of X. Let $S : B \to B$ be a continuous and accretive operator with $(I - S)(X)$ bounded, $\{\alpha_n\}$ satisfies (2). Then, the Mann iteration with delay (11) converges to the solution of $x + Sx = f$.*

References

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