The convergence of Mann iteration with delay

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Abstract

We show the convergence of Mann iteration with delay for various classes of non-Lipschitzian operators.

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1 Introduction

Let $X$ be a real Banach space, $B$ be a nonempty, convex subset of $X$, and $T : B \to B$ be an operator. Let $u_1 \in B$ be given and $s > 0$ a fixed number. We consider the following iteration, to which we further refer as Mann iteration with delay, see [5]:

$$u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n Tu_{n-s},$$

the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$  

We are inspired for such delays from economics and biology problems in which fixed point are required. Usually, $T$ is a contraction. It is wellknown that Mann iteration is desirable when $T$ is not a contraction.

The operator $J : X \to 2^X$ given by $Jx := \{f \in X^* : \langle x, f \rangle = \|x\|, \|f\| = \|x\|\}, \forall x \in X$, is called the normalized duality mapping. The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$. It is easy to see that we have $\langle j(x), y \rangle \leq \|x\| \|y\|, \forall x, y \in X, \forall j(x) \in J(x)$.

Definition 1 Let $X$ be a real Banach space. Let $B$ be a nonempty subset. A map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0, 1)$ and a $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq (1 - k) \|x - y\|^2, \forall x, y \in B.$$  

A map $S : B \to B$ is called strongly accretive if there exists $k \in (0, 1)$ and a $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle \geq k \|x - y\|^2, \forall x, y \in B.$$  

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In (3) when $k = 1$, then $T$ is called pseudocontractive. In (4) when $k = 0$, then $S$ is called accretive. Let us denote by $I$ the identity map.

**Lemma 2** If $X$ is a real Banach space, then the following relation is true

$$
\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \forall x, y \in X, \forall j(x + y) \in J(x + y).
$$

(5)

**Lemma 3** [6] Let $(\rho_n)_n$ be a nonnegative sequence which satisfies the following inequality

$$
\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \varepsilon_n,
$$

(6)

where $\lambda_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\varepsilon_n = o(\lambda_n)$. Then $\lim_{n \to \infty} \rho_n = 0$.

**Lemma 4** Let $s \geq 0$ be a fixed number, $\{a_n\}$ a nonnegative sequence which satisfies the following inequality

$$
a_{n+1} \leq (1 - \alpha_n)a_{n-s} + \sigma_n,
$$

(6)

where $\alpha_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sigma_n = o(\alpha_n)$. Then $\lim_{n \to \infty} a_n = 0$.

**Proof.** Note that sequence $\{a_n\}$ is the reunion of $s + 1$ independent subsequences. If all such subsequences converges to zero then $\{a_n\}$ shall converge. The generic subsequence satisfies (6). Set $\rho_n := a_{n-s}$, $\lambda_n := \alpha_n$, $\varepsilon_n := \sigma_n$, and use Lemma 3 to obtain $\lim_{n \to \infty} a_n = 0$. ■

2 **Main result**

**Theorem 5** Let $s \geq 0$ be a fixed number, $X$ a real Banach space with a uniformly convex dual $X^*$, $B$ a nonempty closed convex bounded subset of $X$, and $T : B \to B$ a continuous strongly pseudocontractive mapping. Then the Mann iteration with delay $\{x_n\}_{n=1}^{\infty}$ defined by (1) converges strongly to the unique fixed point of $T$.

**Proof.** Corollary 1 of [2] assures the existence of a fixed point. The uniqueness of the fixed point comes from (3). Because $X^*$ is uniformly convex the duality map is singled valued (see, e.g., [1]). Let $x^*$ be the fixed point of $T$. Using (1), (3) and Lemma 2 we get

$$
\|u_{n+1} - x^*\|^2 = \|(1 - \alpha_n)(u_{n-s} - x^*) + \alpha_n(Tu_{n-s} - Tx^*)\|^2
\leq (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n(Tu_{n-s} - Tx^*, J(u_{n+1} - x^*))
= (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n(Tu_{n-s} - Tx^*, J(u_{n-s} - x^*)) +
+ 2\alpha_n(Tu_{n-s} - Tx^*, J(u_{n+1} - x^*) - J(u_{n-s} - x^*))
\leq (1 - \alpha_n)^2 \|u_{n-s} - x^*\|^2 + 2\alpha_n(1 - k) \|u_{n-s} - x^*\|^2 + 2\alpha_n\sigma_n
$$

where

$$
\sigma_n = \langle Tu_{n-s} - Tx^*, J(u_{n+1} - x^*) - J(u_{n-s} - x^*) \rangle.
$$
Now we shall show \( \sigma_n \rightarrow 0 \) as \( n \rightarrow \infty \). Observe that \( (\|Tu_{n-s} - Tx^*\|) \) is bounded. We prove now that

\[
J(u_{n+1} - x^*) - J(u_{n-s} - x^*) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\] (7)

Proposition 12.3 on page 115, of [2] assures that, when \( X^* \) is uniformly convex, then \( J \) is uniformly continuous on every bounded set of \( X \). To prove (7) it is sufficient to see that

\[
\|u_{n+1} - u_{n-s}\| = \alpha_n \|u_{n-s} - Tu_{n-s}\| \leq \alpha_n(\|u_{n-s}\| + \|Tu_{n-s}\|) \leq 2\alpha_n M \rightarrow 0
\]

where \( M = \sup\{\|u_{n-s}\|, \|Tu_{n-s}\|\} \). The sequences \((u_{n-s})_n, (Tu_{n-s})_n\) are bounded being in the bounded set \( B \). Hence (7) holds. Then

\[
\|u_{n+1} - x^*\|^2 \leq ((1 - \alpha_n)^2 + 2\alpha_n(1 - k)) \|u_{n-s} - x^*\|^2 + 2\alpha_n \sigma_n \quad \text{(8)}
\]

\[
= (1 - 2\alpha_n + \alpha_n^2 + 2\alpha_n - 2\alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n \sigma_n
\]

\[
= (1 + \alpha_n^2 - 2\alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n \sigma_n.
\]

The condition \( \lim_{n \rightarrow \infty} \alpha_n = 0 \) implies the existence of an \( n_0 \) such that for all \( n \geq n_0 \) we have

\[
\alpha_n < k
\] (9)

Substituting (9) into (8) we get \( 1 + \alpha_n^2 - 2\alpha_n k < 1 - \alpha_n k \). Finally, the above inequality yields

\[
\|u_{n+1} - x^*\|^2 \leq (1 - \alpha_n k) \|u_{n-s} - x^*\|^2 + 2\alpha_n \sigma_n.
\]

Setting \( a_n = \|u_{n-s} - x^*\|^2 \), \( \lambda_n = \alpha_n k \in (0, 1) \), and using Lemma 4, we obtain \( \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \|u_{n-s} - x^*\|^2 = 0 \) i.e.

\[
\lim_{n \rightarrow \infty} \|u_{n-s} - x^*\| = 0.
\]

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3 The accretive and strongly accretive cases

Let \( I \) denote the identity map.

**Remark 6** The operator \( T \) is a (strongly) pseudocontractive map if and only if \((I - T)\) is a (strongly) accretive map.

**Remark 7**

1. Let \( T, S : X \rightarrow X \), and \( f \in X \) be given. A fixed point for the map \( Tx = f + (I - S)x, \forall x \in X \) is a solution for \( Sx = f \).

2. Let \( f \in X \) be a given point. If \( S \) is an accretive map then \( T = f - S \) is a strongly pseudocontractive map.
Consider Mann iteration with delay and set $Tx = f + (I - S)x$ to obtain
\[ u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n (f + (I - S)u_{n-s}) . \]  
(10)

Remarks 6 and 7 and Theorem 5 lead to the following results.

**Corollary 8** Let $X$ be a real Banach space with a uniformly convex dual $X^*$, and $S : X \to X$ a continuous and strongly accretive map with $(I - S)(X)$ bounded, $\{\alpha_n\}$ satisfies (2), and $u_0 = x_0 \in X$, then, the Mann iteration with delay (10) converges to the solution of $Sx = f$.

Let $S$ be an accretive operator. The operator $Tx = f - Sx$ is strongly pseudocontractive, for a given $f \in X$. A solution for $Tx = x$ becomes a solution for $x + Sx = f$. Consider Mann iteration with delay, set $Tx := f - Sx$ such that
\[ u_{n+1} = (1 - \alpha_n)u_{n-s} + \alpha_n (f - S)u_{n-s}) . \]  
(11)

Again, using the Remarks 6 and 7 and Theorem 5 we obtain the following result.

**Corollary 9** Let $X$ be a real Banach space with a uniformly convex dual $X^*$, and $B$ a nonempty, convex, closed subset of $X$. Let $S : B \to B$ be a continuous and accretive operator with $(I - S)(X)$ bounded, $\{\alpha_n\}$ satisfies (2). Then, the Mann iteration with delay (11) converges to the solution of $x + Sx = f$.

**References**


