A DIFFERENTIAL EQUATION WITH DELAY FROM BIOLOGY

DIANA OTROCOL

ABSTRACT. The purpose of this paper is to present a differential equation with delay from biological excitable medium. Existence, uniqueness and data dependence (monotony, continuity, differentiability with respect to parameter) results for the solution of the Cauchy problem of biological excitable medium are obtained using weakly Picard operator theory.

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1. Introduction

In recent years the theory of excitable medium has rapidly developed and its results have been applied in various areas: chemistry, biology, ecology, electric engineering, populations dynamics, cardiology, neurology. At present, different approaches for the mathematical description of biological excitable medium by means of partial-differential equation, functional-differential, functional and discrete equations are applied. The papers [2], [3], [10] has offered the opportunity for understanding the normal regulation of living systems as well as its anomalies.

The activity of the i-th element of the excitable medium can be described by the following equation:

$$x_i'(t) = a_i f_i(x_1(t-h), \ldots, x_m(t-h)) - b_i x_i(t) \tag{1}$$

where $x_i(t)$ is the activity of the i-th element; a_i is the functional parameter of the i-th element; $f_i(\cdot)$ is the feedback function; b_i is the decay constant, $i = \overline{1,m}$.

The aim of this paper is to study the following problem

$$x_i'(t) = a_i f_i(x_1(t-h), \ldots, x_m(t-h)) - b_i x_i(t), \tag{2}$$

$t \in [t_0, b], i = \overline{1,m}$, with initial conditions

$$x_i(t) = \varphi_i(t),\ t \in [t_0 - h, t_0], \tag{3}$$

where

(H_1) $t_0 < b,\ h > 0,\ t_0, b, h \in R$;
(H_2) $f_i \in C(R^m, R),\ i = \overline{1,m}$;
(H_3) $\varphi_i \in C([t_0 - h, t_0], R),\ i = \overline{1,m}$;
(H_4) there exists $L_f > 0$, such as:

$$|f_i(u_1, \ldots, u_m) - f_i(v_1, \ldots, v_m)| \le L_f \sum_{i=1}^{m} |u_i - v_i|,$$

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for all $u_i, v_i \in R, i=\overline{1,m}$.

By a solution of the problem (2)–(3) we understand the function $x=(x_1,\dots,x_m) \in R^m$ with $x_i \in C([t_0-h,b],R)\cap C^1([t_0,b],R),\ i=\overline{1,m}$ which satisfies (2)–(3).

The problem (2)–(3) is equivalent with the following fixed point system:

$$x_i(t)=\begin{cases}\varphi_i(t),\ t\in[t_0-h,t_0],\\ \varphi_i(t_0)e^{-b_i(t-t_0)}+a_i\int_{t_0}^t e^{b_i(s-t)}f_i(x_1(s-h),\dots,\\ \quad x_m(s-h))ds,\ t\in[t_0,b],\end{cases} \tag{4}$$

where $x_i \in C([t_0-h,b],R),\ i=\overline{1,m}$.

On the other hand, the system (2) is equivalent with

$$x_i(t)=\begin{cases}x_i(t),\ t\in[t_0-h,t_0],\\ x_i(t_0)e^{-b_i(t-t_0)}+a_i\int_{t_0}^t e^{b_i(s-t)}f_i(x_1(s-h),\dots,\\ \quad x_m(s-h))ds,\ t\in[t_0,b],\end{cases} \tag{5}$$

where $x_i \in C([t_0-h,b],R),\ i=\overline{1,m}$.

In this paper we apply the weakly Picard operators technique to study the systems (4) and (5).

2. Weakly Picard operators

I.A. Rus introduced the Picard operators class (PO) and the weakly Picard operators class (WPO) for the operators defined on a metric space and he gave basic notations, definitions and results in this field in many papers [7]–[9]. Some problems concerning this techniques were study in [4], [11], [5], [6].

Let (X,d) be a metric space and $A:X\to X$ an operator. We shall use the following notations:

$F_A := \{x\in X \mid A(x)=x\}$ - the fixed point set of A;

$I(A) := \{Y\subset X \mid A(Y)\subset Y, Y\neq\emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A\circ A^n,\ A^0=1_X,\ A^1=A,\ n\in N$;

$P(X) := \{Y\subset X \mid Y\neq\emptyset\}$ - the set of the parts of X;

$H(Y,Z) := \max\{\sup\limits_{y\in Y}\inf\limits_{z\in Z} d(y,z), \sup\limits_{z\in Z}\inf\limits_{y\in Y} d(y,z)\}$ -the Pompeiu–Housdorff functional on $P(X)\times P(X)$.

Definition 1. *([7], [9]) Let (X,d) be a metric space. An operator $A:X\to X$ is a Picard operator (PO) if there exists $x^*\in X$ such that:*

(i) $F_A=\{x^*\}$;

(ii) *the sequence $(A^n(x_0))_{n\in N}$ converges to x^* for all $x_0\in X$.*

Remark 1. *([7], [9]) Accordingly to the definition, the contraction principle insures that, if $A:X\to X$ is an α -contraction on the complete metric space X, then it is a Picard operator.*

Theorem 1. *([7], [9])* (Data dependence theorem). *Let (X,d) be a complete metric space and $A,B:X\to X$ two operators. We suppose that*

(i) *the operator A is a α -contraction;*
(ii) $F_B \neq \emptyset$;
(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta, \ \forall x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Definition 2. *([7], [9]) Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n \in N}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Theorem 2. *([7], [9]) Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is weakly Picard operator if and only if there exists a partition of X,*

$$X = \underset{\lambda \in \Lambda}{\cup} X_\lambda$$

where Λ is the indices set of partition, such that:
(a) *$X_\lambda \in I(A), \ \lambda \in \Lambda;$*
(b) *$A|_{X_\lambda} : X_\lambda \to X_\lambda$ is a Picard operator for all $\lambda \in \Lambda$.*

Definition 3. *([7], [9]) If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X, \ A^\infty(x) := \lim_{n \to \infty} A^n(x).$$

It is clear that $A^\infty(X) = F_A$.

Definition 4. *([7], [9]) Let A be a weakly Picard operator and $c > 0$. The operator A is c -weakly Picard operator if*

$$d(x, A^\infty(x)) \leq cd(x, A(x)), \ \forall x \in X.$$

Example 1. *([7], [9]) Let (X, d) be a complete metric space and $A : X \to X$ a continuous operator. We suppose that there exists $\alpha \in [0, 1)$ such that*

$$d(A^2(x), A(x)) \leq \alpha(x, A(x)), \ \forall x \in X.$$

Then A is c -weakly Picard operator with $c = \dfrac{1}{1-\alpha}$.

Theorem 3. *([7], [9]) Let (X, d) be a metric space and $A_i : X \to X, \ i = 1, 2$. Suppose that*

(i) *the operator A_i is c_i-weakly Picard operator, $i = 1, 2$;*

(ii) *there exists* $\eta > 0$ *such that*

$$d(A_1(x), A_2(x)) \leq \eta,\ \forall x \in X.$$

Then $H(F_{A_1}, F_{A_2}) \leq \eta \max(c_1, c_2)$.

Theorem 4. *(*[7], [9]*)* (Fibre contraction principle). *Let* (X, d) *and* (Y, ρ) *be two metric spaces and* $A : X \times Y \to X \times Y,\ A = (B, C),\ (\ B : X \to X,\ C : X \times Y \to Y\ )$ *a triangular operator. We suppose that*

(i) (Y, ρ) *is a complete metric space;*
(ii) *the operator* B *is Picard operator;*
(iii) *there exists* $l \in [0, 1)$ *such that* $C(x, \cdot) : Y \to Y$ *is a* l*-contraction, for all* $x \in X$*;*
(iv) *if* $(x^*, y^*) \in F_A$*, then* $C(\cdot, y^*)$ *is continuous in* x^**.*

Then the operator A *is Picard operator.*

3. Cauchy problem

We consider the fixed point system (4).

Let $A_f : C([t_0 - h, b], R^m) \to C([t_0 - h, b], R^m)$ given by the relation

$$A_f(x) = (A_{f_1}(x_1, \ldots, x_m), \ldots, A_{f_m}(x_1, \ldots, x_m)),$$

where

$$A_{f_i}(x_1, \ldots, x_m)(t) := \begin{cases} \varphi_i(t),\ t \in [t_0 - h, t_0], \\ \varphi_i(t_0)e^{-b_i(t-t_0)} + a_i \int_{t_0}^{t} e^{b_i(s-t)}\cdot \\ \quad \cdot f_i(x_1(s-h), \ldots, x_m(s-h))ds, t \in [t_0, b]. \end{cases}$$

We consider the Banach space $C([t_0 - h, b], R^m)$ with the Chebyshev norm $\|\cdot\|_C$. Let $X = (C([t_0 - h, b], R^m), \|\cdot\|_C)$.

We have the following result

Theorem 5. *We suppose that*

(i) *the conditions* (H$_1$)–(H$_4$) *are satisfied;*
(ii) $L_f(b - t_0)\sum\limits_{i=1}^{m} \frac{a_i}{b_i} e^{-b_i t_0} < 1.$

Then the Cauchy problem (2)–(3) *has in* $C([t_0 - h, b], R^m)$ *a unique solution. Moreover, the operator* $A_f : C([t_0 - h, b], R^m) \to C([t_0 - h, b], R^m)$ *is* c*-Picard with*

$$c = \frac{1}{1 - L_f(b - t_0)\sum\limits_{i=1}^{m} \frac{a_i}{b_i} e^{-b_i t_0}}.$$

Proof. For $t \in [t_0 - h, t_0]$, we have
$|A_{f_i}(x_1, \ldots, x_m)(t) - A_{f_i}(\overline{x}_1, \ldots, \overline{x}_m)(t)| = 0, \ i = \overline{1, m}.$
For $t \in [t_0, b]$, we have

$$|A_{f_i}(x_1, \ldots, x_m)(t) - A_{f_i}(\overline{x}_1, \ldots, \overline{x}_m)(t)| =$$
$$= a_i \left| \int_{t_0}^{t} e^{b_i(s-t)} [f_i(x_1(s-h), \ldots, x_m(s-h)) - \right.$$
$$-f_i(\overline{x}_1(s-h), \ldots, \overline{x}_m(s-h))]ds| \leq$$
$$\leq \frac{a_i}{b_i} e^{-b_i t_0}(b - t_0) L_f(\|x_1 - \overline{x}_1\|_C + \ldots + \|x_m - \overline{x}_m\|_C), i = \overline{1, m}.$$

Then

$$\|A_f(x_1, \ldots, x_m) - A_f(\overline{x}_1, \ldots, \overline{x}_m)\|_C \leq$$
$$\leq L_f(b - t_0) \sum_{i=1}^{m} \tfrac{a_i}{b_i} e^{-b_i t_0} \, \|(x_1, \ldots, x_m) - (\overline{x}_1, \ldots, \overline{x}_m)\|_C \, .$$

So A_f is c-Picard operator with $c = \frac{1}{1 - L_{A_f}}$, where $L_{A_f} = L_f(b - t_0) \sum_{i=1}^{m} \frac{a_i}{b_i} e^{-b_i t_0}$. □

In what follows, we consider the following operator

$$B_f : C([t_0 - h, b], R^m) \to C([t_0 - h, b], R^m)$$

given by

$$B_f(x) = (B_{f_1}(x_1, \ldots, x_m), \ldots, B_{f_m}(x_1, \ldots, x_m)),$$

where

$$B_{f_i}(x_1, \ldots, x_m)(t) := \begin{cases} x_i(t), \ t \in [t_0 - h, t_0], \\ x_i(t_0) e^{-b_i(t - t_0)} + a_i \int_{t_0}^{t} e^{b_i(s-t)} \cdot \\ \quad \cdot f_i(x_1(s-h), \ldots, x_m(s-h)) ds, t \in [t_0, b]. \end{cases}$$

Theorem 6. *In the condition of Theorem 5, $B_f : C([t_0 - h, b], R^m) \to C([t_0 - h, b], R^m)$ is WPO.*

Proof. The operator B_f is a continuous operator but it is not a contraction. Let take the following notation:

$$X_{\varphi_i} := \{x_i \in C([t_0 - h, b], R) | \ x_i|_{[t_0 - h, t_0]} = \varphi_i, \ i = \overline{1, m}\}.$$

Then we can write

$$C([t_0 - h, b], R^m) = \bigcup_{\varphi_i \in C([t_0 - h, t_0], R)} X_{\varphi_1} \times \ldots \times X_{\varphi_m}, \ i = \overline{1, m}. \tag{6}$$

We have that $X_{\varphi_1} \times \ldots \times X_{\varphi_m} \in I(B_f)$ and $B_f|_{X_{\varphi_1} \times \ldots \times X_{\varphi_m}}$ is a Picard operator, because it is the operator which appears in the proof of the Theorem 5. By applying the Theorem 2, we obtain that B_f is WPO. □

4. Increasing solutions of the system (2)

4.1. Inequalities of Čaplygin type

Theorem 7. *We suppose that*

(a) *the conditions of the Theorem 5 are satisfied;*
(b) $u_i, v_i \in R,\ u_i \le v_i,\ i = \overline{1,m}$ *implies that*

$$f_i(u_1, \ldots, u_m) \le f_i(v_1, \ldots, v_m), i = \overline{1,m}.$$

Let $(x_1, \ldots, x_m)$ *be a solution of the system (2) and* $(y_1, \ldots, y_m)$ *a solution of the inequality system*

$$y_i'(t) \le a_i f_i(y_1(t-h), \ldots, y_m(t-h)) - b_i y_i(t),\ t \in [t_0, b].$$

Then $y_i(t) \le x_i(t),\ t \in [t_0-h, t_0],\ i = \overline{1,m}$ *implies that* $(y_1, \ldots, y_m) \le (x_1, \ldots, x_m)$.

Proof. In the terms of the operator B_f, we have

$$(x_1, \ldots, x_m) = B_f(x_1, \ldots, x_m) \text{ and } (y_1, \ldots, y_m) \le B_f(y_1, \ldots, y_m).$$

However, from the condition (b), we have that B_f^∞ is increasing,

$$(y_1, \ldots, y_m) \le B_f^\infty(y_1, \ldots, y_m) = B_f^\infty(\widetilde{y}_1|_{[t_0-h,t_0]}, \ldots, \widetilde{y}_m|_{[t_0-h,t_0]}) \le$$
$$\le B_f^\infty(\widetilde{x}_1|_{[t_0-h,t_0]}, \ldots, \widetilde{x}_m|_{[t_0-h,t_0]}) = (x_1, \ldots, x_m).$$

Thus $(y_1, \ldots, y_m) \le (x_1, \ldots, x_m)$.

Here, we use the notation $\widetilde{x}_i \in X_{x_i|_{[t_0-h,t_0]}}, i = \overline{1,m}$. □

4.2. Comparison theorem

In what follows we want to study the monotony of the solution of the problem (2)–(3), with respect to φ_i and $f_i,\ i = \overline{1,m}$. We shall use the result below:

Lemma 1. (Abstract comparison Lemma). *Let* $(X, d, \le)$ *be an ordered metric space and* $A, B, C : X \to X$ *be such that:*

(i) $A \le B \le C$;
(ii) *the operators* A, B, C *are WPO;*
(iii) *the operator* B *is increasing*

Then $x \le y \le z \Rightarrow A^\infty(x) \le B^\infty(y) \le C^\infty(z)$.

In this case we can establish the theorem.

Theorem 8. *Let* $f_i^j \in C(R^m, R),\ i = \overline{1,m},\ j = 1, 2, 3.$

We suppose that

(a) $f_i^2(\cdot, \ldots, \cdot) : R^m \to R$ *is increasing,* $i = \overline{1,m}$;
(b) $f_i^1 \le f_i^2 \le f_i^3,\ i = \overline{1,m}.$

Let $x^j = (x_1^j, \dots x_m^j)$ *be a solution of the equation*

$$x_i'(t) = a_i f_i^j(x_1(t-h), \dots, x_m(t-h)) - b_i x_i(t),$$

where $t \in [t_0, b], i = \overline{1,m},\ j = 1,2,3.$

If $x_i^1(t) \le x_i^2(t) \le x_i^3(t),\ t \in [t_0 - h, t_0]$, *then* $x_i^1 \le x_i^2 \le x_i^3,\ i = \overline{1,m}.$

Proof. From Theorem 5, the operators $B_f^j,\ j = 1,2,3$ are weakly Picard operators.

Taking into consideration the condition (a), the operator B_f^2 is increasing.

From (b) we have that $B_f^1 \le B_f^2 \le B_f^3$.

We note that $(x_1^j, \dots, x_m^j) = B_f^{j\infty}(\widetilde{x}_1^j, \dots, \widetilde{x}_m^j),\ j = 1,2,3.$ Now using the abstract comparison lemma, the proof is complete. □

5. Data dependence: continuity

Consider the Cauchy problem (2)–(3) and suppose the conditions of the Theorem 5 are satisfied. Denote by $x(\cdot, \varphi, f) = (x_1(\cdot;\varphi_1, f_1), \dots, x_m(\cdot;\varphi_m, f_m))$, the solution of this problem. We can state the following result:

Theorem 9. *Let* $\varphi_i^j, f_i^j, i = \overline{1,m},\ j = 1,2$ *be as in the Theorem 5. Furthermore, we suppose that there exists* $\eta_i^1, \eta_i^2, i = \overline{1,m}$ *such that*

(i) $\left|\varphi_i^1(t) - \varphi_i^2(t)\right| \le \eta_i^1,\ \forall t \in [t_0 - h, t_0],\ i = \overline{1,m}$;

(ii) $\left|f_i^1(u_1, \dots, u_m) - f_i^2(u_1, \dots, u_m)\right| \le \eta_i^2,\ i = \overline{1,m},\ u_i \in R.$

Then

$$\left|x_i(t;\varphi_i^1, f_i^1) - x_i(t;\varphi_i^2, f_i^2)\right| \le \frac{\sum_{i=1}^m \eta_i^1 e^{-b_i(t-t_0)} + (b-t_0)\sum_{i=1}^m \frac{a_i}{b_i} e^{-b_i t_0}\eta_i^2}{1 - L_f(b-t_0)\sum_{i=1}^m \frac{a_i}{b_i} e^{-b_i t_0}},$$

where $L_f = \max(L_{f^1}, L_{f^2}), i = \overline{1,m}.$

Proof. Consider the operators $A_{\varphi_i^j, f_i^j}, i = \overline{1,m}, j = 1,2$. From Theorem 5 these operators are contractions.

Additionally

$$\left\|A_{\varphi_i^1, f_i^1}(x_1, \dots, x_m) - A_{\varphi_i^2, f_i^2}(x_1, \dots, x_m)\right\|_C \le$$
$$\le \sum_{i=1}^m \eta_i^1 e^{-b_i(t-t_0)} + (b-t_0)\sum_{i=1}^m \frac{a_i}{b_i} e^{-b_i t_0}\eta_i^2,$$

$\forall x = (x_1, \dots, x_m) \in C([t_0 - h, b], R^m)$.

Now the proof follows from the Theorem 1, with $A := A_{\varphi_i^1, f_i^1},\ B = A_{\varphi_i^2, f_i^2},\ \eta = \sum_{i=1}^m \eta_i^1 e^{-b_i(t-t_0)} + (b-t_0)\sum_{i=1}^m \frac{a_i}{b_i} e^{-b_i t_0}\eta_i^2$ and $\alpha := L_{A_f} = L_f(b-t_0)\sum_{i=1}^m \frac{a_i}{b_i} e^{-b_i t_0}$ where $L_f = \max(L_{f^1}, L_{f^2}), i = \overline{1,m}.$ □

From the Theorem above we have:

Theorem 10. *Let* f_i^1 *and* f_i^2 *be as in the Theorem 5,* $i=\overline{1,m}$. *Let* $S_{B_{f_i^1}}, S_{B_{f_i^2}}$ *be the solution set of system (2) corresponding to* f_i^1 *and* $f_i^2, i=\overline{1,m}$. *Suppose that there exists* $\eta_i>0, i=\overline{1,m}$ *such that*

$$\left|f_i^1(u_1,\ldots,u_m)-f_i^2(u_1,\ldots,u_m)\right|\le\eta_i \tag{7}$$

for all $u_i\in R, i=\overline{1,m}$.

Then

$$H_{\|\cdot\|_C}(S_{B_{f_i^1}},S_{B_{f_i^2}})\le\frac{(b-t_0)\sum\limits_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}\eta_i}{1-L_f(b-t_0)\sum\limits_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}},$$

where $L_f=\max(L_{f^1},L_{f^2})$ *and* $H_{\|\cdot\|_C}$ *denotes the Pompeiu-Housdorff functional with respect to* $\|\cdot\|_C$ *on* $C([t_0-h,b],R^m)$.

Proof. In condition of the Theorem 5, the operators $B_{f_i^1}$ and $B_{f_i^2}, i=\overline{1,m}$ are c_1-WPO and c_2-weakly Picard operators.

Let

$$X_{\varphi_i}:=\{x_i\in C([t_0-h,b],R)|\ x_i|_{[t_0-\tau_1,t_0]}=\varphi_i, i=\overline{1,m}\}.$$

It is clear that $B_{f_i^1}|_{X_{\varphi_i}}=A_{f_i^1},\ B_{f_i^2}|_{X_{\varphi_i}}=A_{f_i^2}$. So, from Theorem 2 and Theorem 5 we have

$$\begin{aligned}&\left\|B^2_{f_i^1}(x_1,\ldots,x_m)-B_{f_i^1}(x_1,\ldots,x_m)\right\|_C\le\\&\le L_{f^1}(b-t_0)\sum_{i=1}^{m}\tfrac{a_i}{b_i}e^{-b_it_0}\left\|B_{f_i^1}(x_1,\ldots,x_m)-(x_1,\ldots,x_m)\right\|_C,\end{aligned}$$

$$\begin{aligned}&\left\|B^2_{f_i^2}(x_1,\ldots,x_m)-B_{f_i^2}(x_1,\ldots,x_m)\right\|_C\le\\&\le L_{f^2}(b-t_0)\sum_{i=1}^{m}\tfrac{a_i}{b_i}e^{-b_it_0}\left\|B_{f_i^2}(x_1,\ldots,x_m)-(x_1,\ldots,x_m)\right\|_C,\end{aligned}$$

for all $(x_1,\ldots,x_m)\in C([t_0-h,b],R^m), i=\overline{1,m}$.

Now, choosing

$$\alpha_1=L_{f^1}(b-t_0)\sum_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}\text{ and}$$

$$\alpha_2=L_{f^2}(b-t_0)\sum_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0},$$

we get that $B_{f_i^1}$ and $B_{f_i^2}$ are c_1-weakly Picard operators and c_2-weakly Picard operators with $c_1=(1-\alpha_1)^{-1}$ and $c_2=(1-\alpha_2)^{-1}$. From (7) we obtain that

$$\left\|B_{f_i^1}(x_1,\ldots,x_m)-B_{f_i^2}(x_1,\ldots,x_m)\right\|_C\le(b-t_0)\sum_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}\eta_i,$$

$\forall(x_1,\dots,x_m)\in C([t_0-h,b],R^m), i=\overline{1,m}$. Applying Theorem 3 we have that

$$H_{\|\cdot\|_C}(S_{B_{f_i^1}},S_{B_{f_i^2}})\le \frac{(b-t_0)\sum\limits_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}\eta_i}{1-L_f(b-t_0)\sum\limits_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}},$$

where $L_f=\max(L_{f^1},L_{f^2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|_C$ on $C([t_0-h,b],R^m)$. □

6. Data dependence: differentiability

Consider the following differential system with parameter

$$x_i'(t)=a_if_i(x_1(t-h),\dots,x_m(t-h);\lambda)-b_ix_i(t), t\in[t_0,b], i=\overline{1,m}, \quad (8)$$

$$x_i(t)=\varphi_i(t),\ t\in[t_0-h,t_0], i=\overline{1,m}. \quad (9)$$

Suppose that we have satisfied the following conditions:

(C_1) $t_0<b, h>0,\ J\subset R$ a compact interval;
(C_2) $f_i\in C^1(R^m\times J,R),\ i=\overline{1,m}$;
(C_3) $\varphi_i\in C([t_0-h,t_0],R),\ i=\overline{1,m}$;
(C_4) there exists $L_f>0,\ i=\overline{1,m}$ such that

$$\left|\frac{\partial f_i(u_1,\dots,u_m;\lambda)}{\partial u_i}\right|\le L_f, u_i\in R,\ i=\overline{1,m}, \lambda\in J;$$

(C_5) $L_f(b-t_0)\sum\limits_{i=1}^{m}\frac{a_i}{b_i}e^{-b_it_0}<1$.

Then, from Theorem 5, we have that the problem (2)–(3) has a unique solution, $(x_1^*(\cdot,\lambda),\dots,x_m^*(\cdot,\lambda))$.

We prove that $x_i^*(\cdot,\lambda)\in C^1(J),\ \forall\ t\in[t_0-h,b], i=\overline{1,m}$.

For this we consider the system

$$x_i'(t,\lambda)=a_if_i(x_1(t-h;\lambda),\dots,x_m(t-h;\lambda);\lambda)-b_ix_i(t;\lambda), \quad (10)$$

$t\in[t_0,b],\ \lambda\in J,\ x_i\in C([t_0-h,b]\times J,R)\cap C^1([t_0,b]\times J,R),\ i=\overline{1,m}$.

Theorem 11. *Consider the problem (10)–(9), and suppose the conditions* (C_1)–(C_5) *hold. Then,*

(i) *(10)–(9) has a unique solution* $(x_1^*(\cdot,\lambda),\dots,x_m^*(\cdot,\lambda))$*, in* $C([t_0-h,b]\times J,R^m)$;
(ii) $(x_1^*(\cdot,\lambda),\dots,x_m^*(\cdot,\lambda))\in C^1(J), \forall t\in[t_0-h,b], i=\overline{1,m}$.

Proof. The problem (10)–(9) is equivalent with the following functional-integral equation

$$x_i(t;\lambda) = \begin{cases} \varphi_i(t), \ t \in [t_0 - h, t_0] \\ \varphi_i(t)e^{-b_i(t-t_0)} + a_i \int_{t_0}^t e^{b_i(s-t)} f_i(x_1(s-h;\lambda), \dots, \\ \quad x_m(s-h;\lambda);\lambda), t \in [t_0, b]. \end{cases} \tag{11}$$

Now let take the operator

$$A : C([t_0 - h, b] \times J, R^m) \to C([t_0 - h, b] \times J, R^m),$$

given by

$$A(x_1, \dots, x_m) = (A_1(x_1, \dots, x_m), \dots, A_m(x_1, \dots, x_m)),$$

where

$$A_i(x_1, \dots, x_m)(t;\lambda) := \begin{cases} \varphi_i(t), \ t \in [t_0 - h, t_0] \\ \varphi_i(t)e^{-b_i(t-t_0)} + a_i \int_{t_0}^t e^{b_i(s-t)} \cdot \\ \quad \cdot f_i(x_1(s-h;\lambda), \dots, x_m(s-h;\lambda);\lambda), t \in [t_0, b] \end{cases}$$

Let $X = C([t_0 - \tau_1, b] \times J, R^m)$.

It is clear, from the proof of the Theorem 5, that in the condition (C_1)–(C_5), the operator$A : (X, \|\cdot\|_C) \to (X, \|\cdot\|_C)$ is Picard operator.

Let $(x_1^*, \dots, x_m^*)$ be the unique fixed point of A.

Supposing that there exists $\dfrac{\partial x_i^*}{\partial \lambda}$, $i = \overline{1, m}$, from (11), we have that

$$\begin{aligned} \frac{\partial x_i^*}{\partial \lambda} = a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x_1^*(s-h,\lambda)}{\partial \lambda} ds + \dots \\ + a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial u_m} \cdot \frac{\partial x_m^*(s-h,\lambda)}{\partial \lambda} ds + \\ + a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial \lambda} ds, \end{aligned}$$

for all $t \in [t_0, b], \lambda \in J, i = \overline{1, m}$.

This relation suggest us to consider the following operator

$$C : X \times X \to X,$$
$$(x_1, \dots, x_m, u_1, \dots, u_m) \to C(x_1, \dots, x_m, u_1, \dots, u_m),$$

where $C_i(x_1, \dots, x_m, u_1, \dots, u_m)(t;\lambda) = 0$ for $t \in [t_0 - h, t_0], \lambda \in J, i = \overline{1, m}$ and

$$\begin{aligned} & C_i(x_1, \dots, x_m, u_1, \dots, u_m)(t;\lambda) := \\ & = a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial u_1} \cdot u_1(s-h;\lambda) ds + \dots \\ & \quad + a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial u_m} \cdot u_m(s-h;\lambda) ds + \\ & \quad + a_i \int_{t_0}^t e^{b_i(s-t)} \frac{\partial f_i(x_1^*(s-h;\lambda), \dots, x_m^*(s-h;\lambda);\lambda)}{\partial \lambda} ds, \end{aligned}$$

for $t \in [t_0, b], \lambda \in J, i = \overline{1, m}$.

In this way we have the triangular operator

$$D : X \times X \to X \times X,$$
$$(x_1, \ldots, x_m, u_1, \ldots, u_m) \to (A(x_1, \ldots, x_m), C(x_1, \ldots, x_m, u_1, \ldots, u_m)),$$

where A is Picard operator and $C(x_1, \ldots, x_m, \cdot, \ldots, \cdot) : Y \to Y$ is L_C -contraction with $L_C = L_f(b - t_0)\sum_{i=1}^{m} \frac{a_i}{b_i} e^{-b_i t_0}$.

From Theorem 4 we have that the operator D is Picard operator, i.e. the sequences

$$(x_{1,n+1}, \ldots, x_{m,n+1}) := A(x_{1,n}, \ldots, x_{m,n}),$$
$$(u_{1,n+1}, \ldots, u_{m,n+1}) := C(x_{1,n}, \ldots, x_{m,n}, u_{1,n}, \ldots, u_{m,n}),$$

$n \in N$, converges uniformly, with respect to $t \in X$, $\lambda \in J$, to $(x_1^*, \ldots, x_m^*, u_1^*, \ldots, u_m^*) \in F_D$, for all $(x_{1,0}, \ldots, x_{m,0}) \in X$, $(u_{1,0}, \ldots, u_{m,0}) \in X$.

If we take

$$x_{1,0} = 0, \ldots, \ x_{m,0} = 0,$$
$$u_{1,0} = \frac{\partial x_{1,0}}{\partial \lambda} = 0, \ldots, u_{m,0} = \frac{\partial x_{m,0}}{\partial \lambda} = 0,$$

then

$$u_{1,1} = \frac{\partial x_{1,1}}{\partial \lambda}, \ldots, u_{m,1} = \frac{\partial x_{m,1}}{\partial \lambda}.$$

By induction we prove that

$$u_{1,n} = \frac{\partial x_{1,n}}{\partial \lambda}, \cdots, u_{m,n} = \frac{\partial x_{m,n}}{\partial \lambda}, \ \forall n \in N.$$

So

$$x_{1,n} \overset{unif}{\to} x_1^*, \cdots, x_{m,n} \overset{unif}{\to} x_m^*, \text{ as } n \to \infty,$$
$$\frac{\partial x_{1,n}}{\partial \lambda} \overset{unif}{\to} u_1^*, \cdots, \frac{\partial x_{m,n}}{\partial \lambda} \overset{unif}{\to} u_m^*, \text{ as } n \to \infty.$$

From a Weierstrass argument we have that there exists $\frac{\partial x_i^*}{\partial \lambda}$, $i = \overline{1, m}$ and

$$\frac{\partial x_1^*}{\partial \lambda} = u_1^*, \ldots, \frac{\partial x_m^*}{\partial \lambda} = u_m^*.$$

□

References

1. J. K. Hale, *Introduction to functional differential equations*, Springer-Verlag, 1993.
2. B. N. Hidirov, *Regulation mechanism of living system*, J. Scientiae Mathematicae Japonicae **64**(2006), No.2, 497-504.
3. M. B. Hidirova, *Dynamics of biological excitable medium*, J. Scientiae Mathematicae Japonicae **64**(2006), No.2, 755-763.
4. V. Mureşan, *Functional-Integral Equations*, Mediamira, Cluj-Napoca, 2003.
5. D. Otrocol, *Data dependence for the solution of a Lotka-Volterra system with two delays*, Mathematica **48**(2006), No.71, 61-68.
6. D. Otrocol, *Lotka-Volterra system with two delays via weakly Picard operators*, Nonlinear Analysis Forum **10**(2005), No.2, 193-199.

7. I. A. Rus, *Picard operators and applications*, J. Scientiae Mathematicae Japonicae **58**(2003), No.1, 191-219.
8. I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001.
9. I. A. Rus, *Weakly Picard operators and applications*, Seminar of Fixed Point Theory, Cluj-Napoca, **2**(2001), 41-58.
10. M. Saidalieva, *Modelling of regulation mechanism of cellular communities*, J. Scientiae Mathematicae Japonicae **8**(2003), 463-469.
11. M. A. Şerban, *Fiber φ-contractions*, Studia Univ. "Babeş-Bolyai", Mathematica **44**(1999), No.3 , 99-108.

Diana Otrocol received her Ph.D at Babeş-Bolyai University under the direction of Petru Blaga. Now she is a research assistant at the Tiberiu Popoviciu Institute of Numerical Analysis. Her research interests focus on the fixed point theory and numerical analysis.

Tiberiu Popoviciu Institute of Numerical Analysis, Cluj-Napoca, Romania
e-mail: dotrocol@ictp.acad.ro