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FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH MAXIMA OF MIXED TYPE

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Abstract. In this paper we study the following second order functional-differential equations with maxima, of mixed type,

$$-x''(t) = f(t, x(t), \max_{t-h_1 \le \xi \le t} x(\xi), \max_{t \le \xi \le t+h_2} x(\xi)),\ t \in [a, b]$$

with "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a - h_1, a], \\ x(t) = \psi(t),\ t \in [b, b + h_2]. \end{cases}$$

The plan of the paper is the following: 1. Introduction 2. Picard and weakly Picard operator 3. The operator $\max_I$ 4. Existence and uniqueness 5. Inequalities of Čaplygin type 6. Data dependence: monotony 7. Data dependence: continuity 8. Examples.

Key Words and Phrases: Picard operator, weakly Picard operators, equation of mixed type, equations with maxima, fixed points, data dependence.
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1. Introduction

Differential equations with maxima are often met in applications, for instance in the theory of automatic control. The existence and uniqueness of solutions of the equation with maxima is considered in [3], [4], [9]. The asymptotic stability of the solution of this equations and other problems concerning equations with maxima are investigated in [2], [3], [6], [15], [16].

The main goal of the presented paper is to study a second order functional-differential equations with maxima, of mixed type, using the theory of weakly Picard operators ([10]-[14]).

We consider the following functional-differential equation

$$-x''(t) = f(t, x(t), \max_{t-h_1\leq\xi\leq t} x(\xi), \max_{t\leq\xi\leq t+h_2} x(\xi)),\ t \in [a, b] \tag{1.1}$$

with the "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a - h_1, a], \\ x(t) = \psi(t),\ t \in [b, b + h_2]. \end{cases} \tag{1.2}$$

We suppose that:

(C$_1$) $h_1, h_2,\ a$ and $b \in \mathbb{R},\ a < b,\ h_1 > 0,\ h_2 > 0$;
(C$_2$) $f \in C([a, b] \times \mathbb{R}^3)$;
(C$_3$) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \leq L_f \max_{i=1,2,3} |u_i - v_i|,$$

for all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2, 3$;
(C$_4$) $\varphi \in C[a - h_1, a]$ and $\psi \in C[b, b + h_2]$.

Let G be the Green function of the following problem

$$-x'' = \chi,\ x(a) = 0,\ x(b) = 0,\ \chi \in C[a, b].$$

The problem (1.1)-(1.2), $x \in C[a - h_1, b + h_2] \cap C^2[a, b]$ is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t),\ t \in [a - h_1, a], \\ w(\varphi, \psi)(t)+ \\ \quad + \int_a^b G(t, s)f(s, x(s), \max_{s-h_1\leq\xi\leq s} x(\xi), \max_{s\leq\xi\leq s+h_2} x(\xi))ds,\ t \in [a, b], \\ \psi(t),\ t \in [b, b + h_2], \end{cases} \tag{1.3}$$

$x \in C[a - h_1, b + h_2]$, where

$$w(\varphi, \psi)(t) := \frac{t - a}{b - a}\psi(b) + \frac{b - t}{b - a}\varphi(a).$$

The equation (1.1) is equivalent with

$$x(t) = \begin{cases} x(t),\ t \in [a-h_1, a], \\ w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})(t) + \\ \quad + \int_a^b G(t,s) f(s, x(s), \max\limits_{s-h_1 \le \xi \le s} x(\xi), \max\limits_{s \le \xi \le s+h_2} x(\xi)) ds,\ t \in [a,b], \\ x(t),\ t \in [b, b+h_2], \end{cases} \tag{1.4}$$

$x \in C[a-h_1, b+h_2]$.

In what follow we consider the operators:

$$B_f, E_f : C[a-h_1, b+h_2] \to C[a-h_1, b+h_2]$$

defined by

$$B_f(x)(t) := \text{second part of (1.3)}$$

and

$$E_f(x)(t) := \text{second part of (1.4)}.$$

Let $X := C[a-h_1, b+h_2]$ and $X_{\varphi,\psi} := \{x \in X |\ x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}$. It is clear that

$$X = \bigcup_{\substack{\varphi \in C[a-h_1,a] \\ \psi \in C[b,b+h_2]}} X_{\varphi,\psi}$$

is a partition of X.

We have

Lemma 1.1. *We suppose that the conditions* (C_1), (C_2) *and* (C_4) *are satisfied. Then*

(a) $B_f(X) \subset X_{\varphi,\psi}$ *and* $B_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$;
(b) $B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$.

In this paper we shall prove that, if L_f is small enough, then the operator E_f is weakly Picard operator and we shall study the equation (1.1) in the terms of this operator.

2. Picard and Weakly Picard operators

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}$;

Definition 2.1. *(*[10], [13]*) Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *(*[10], [13]*) Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *(*[10], [13]*) If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. *It is clear that $A^\infty(X) = F_A = \{x \in X \mid A(x) = x\}$.*

Definition 2.5. *(*[10], [13]*) Let A be a weakly Picard operator and $c > 0$. The operator A is c-weakly Picard operator if*

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall x \in X.$$

Theorem 2.6. *(*[10], [13]*) Let (X,d) be a metric space and $A : X \to X$ an operator. The operator A is weakly Picard operator if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

where Λ is the indices set of partition, such that:

(a) $X_\lambda \in I(A),\ \lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all $\lambda \in \Lambda$.*

Example 2.7. *(*[10], [13]*) Let (X,d) be a complete metric space and $A : X \to X$ an α-contraction. Then A is $\frac{1}{1-\alpha}$-PO.*

Example 2.8. *Let (X,d) be a complete metric space and $A : X \to X$ continuous and α-graphic contraction. Then A is $\frac{1}{1-\alpha}$-WPO.*

For more details on WPOs theory see [10], [12], [13].

3. The operator $\max\limits_I$

Let $I : \mathbb{R} \to P_{cp,cv}(\mathbb{R}) := \{Y \subset \mathbb{R}|\ Y \text{ compact and convex}\}$ be a multivalued operator. We suppose that $I(t) = [\alpha(t), \beta(t)]$ where $\alpha \le \beta$ and $\alpha, \beta \in C(\mathbb{R})$.

For $x \in C(\mathbb{R})$ we consider the function $\max\limits_I x$ defined by $(\max\limits_I x)(t) := \max\limits_{\xi \in I(t)} x(\xi)$. We remark that $\max\limits_I x \in C(\mathbb{R})$. So, we have the operator

$$\max_I : C(\mathbb{R}) \to C(\mathbb{R}).$$

Some properties of the operator $\max\limits_I$ are given by

Lemma 3.1. *We have*

(i) $x \le y \Rightarrow \max\limits_I x \le \max\limits_I y$, *i.e. the operator* $\max\limits_I$ *is increasing;*

(ii) $\left|\max\limits_{\xi \in I(t)} x(\xi) - \max\limits_{\xi \in I(t)} y(\xi)\right| \le \max\limits_{\xi \in I(t)} |x(\xi) - y(\xi)|$, *for all* $t \in \mathbb{R}$, $x, y \in C(\mathbb{R})$;

(iii) $\max\limits_{t \in K} \left|\max\limits_{\xi \in I(t)} x(\xi) - \max\limits_{\xi \in I(t)} y(\xi)\right| \le \max\limits_{\xi \in \bigcup\limits_{t \in K} I(t)} |x(\xi) - y(\xi)|$, *for all* $t \in \mathbb{R}, x, y \in C(\mathbb{R})$.

Proof.

(ii) Let $\xi \in I(t)$. We have

$$x(\xi) \le x(\xi) - y(\xi) + y(\xi) \le y(\xi) + |x(\xi) - y(\xi)|.$$

Then

$$\max_{\xi \in I(t)} x(\xi) \le \max_{\xi \in I(t)} y(\xi) + \max_{\xi \in I(t)} |x(\xi) - y(\xi)|$$

and

$$\left|\max_{\xi \in I(t)} x(\xi) - \max_{\xi \in I(t)} y(\xi)\right| \le \max_{\xi \in I(t)} |x(\xi) - y(\xi)|, \text{ for all } t \in \mathbb{R}, x, y \in C(\mathbb{R}).$$

(iii) Follows from (ii).

□

4. Existence and uniqueness

Our first result is the following

Theorem 4.1. *We suppose that:*

(a) *the conditions* (C_1)-(C_4) *are satisfied;*

(C$_5$) $\frac{L_f}{8}(b-a)^2 < 1$.

Then the problem (1.1)-(1.2) has a unique solution which is the uniform limit of the successive approximations.

Proof. Consider the Banach space $(C[a-h_1, b+h_2], \|\cdot\|)$ where $\|\cdot\|$ is the Chebyshev norm, $\|\cdot\| := \max_{a-h_1 \leq t \leq b+h_2} |x(t)|$.

The problem (1.1)-(1.2) is equivalent with the fixed point equation

$$B_f(x) = x, \; x \in C[a-h_1, b+h_2].$$

From the condition (C_3) we have, for $t \in [a,b]$

$$\begin{aligned}
&|B_f(x)(t) - B_f(y)(t)| \leq \\
&\leq L_f \int_a^b G(t,s) \max\left\{ |x(s) - y(s)|, \left| \max_{a-h_1 \leq \xi \leq a} x(\xi) - \max_{a-h_1 \leq \xi \leq a} y(\xi) \right|, \right. \\
&\qquad \left. \left| \max_{b \leq \xi \leq b+h_2} x(\xi) - \max_{b \leq \xi \leq b+h_2} y(\xi) \right| \right\} ds \leq \\
&\leq L_f \int_a^b G(t,s) \max_{a-h_1 \leq \xi \leq b+h_2} |x(s) - y(s)| \, ds \leq \\
&\leq \frac{L_f}{8}(b-a)^2 \|x - y\|.
\end{aligned}$$

This implies that B_f is an α-contraction, with $\alpha = \frac{L_f}{8}(b-a)^2$. The proof follows from the contraction principle. $\square$

Remark 4.2. *From the proof of Theorem 4.1, it follows that the operator B_f is PO. Since*

$$B_f|_{X_{\varphi,\psi}} = E_f|_{X_{\varphi,\psi}}$$

and

$$X := C[a-h_1, b+h_2] = \underset{\varphi,\psi}{\cup} X_{\varphi,\psi}, \; E_f(X_{\varphi,\psi}) \subset X_{\varphi,\psi}$$

hence, the operator E_f is WPO and

$$F_{E_f} \cap X_{\varphi,\psi} = \{x^*_{\varphi,\psi}\}, \forall \varphi \in C[a-h_1, a], \; \forall \psi \in C[b, b+h_2],$$

where $x^*_{\varphi,\psi}$ is the unique solution of the problem (1.1)-(1.2).

Remark 4.3. *E_f is α-graphic contraction, i.e.*

$$\left\|E_f^2(x) - E_f(x)\right\| \leq \alpha \|x - E_f(x)\|, \; \forall x \in C[a-h_1, b+h_2].$$

5. Inequalities of Čaplygin type

In this section we need the following abstract result

Lemma 5.1. *(see [12]) Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) *A is WPO;*

(ii) *A is increasing.*

Then, the operator A^∞ is increasing.

Now we consider the operators E_f and B_f on the ordered Banach space $(C[a-h_1, b+h_2], \|\cdot\|, \leq)$.

We have

Theorem 5.2. *We suppose that:*

(a) *the conditions $(C_1) - (C_4)$ are satisfied;*

(b) *$\frac{L_f}{8}(b-a)^2 < 1$;*

(c) *$f(t, \cdot, \cdot, \cdot) : \mathbb{R}^3 \to \mathbb{R}$ is increasing, $\forall t \in [a, b]$.*

Let x be a solution of equation (1.1) and y a solution of the inequality

$$-y''(t) \leq f(t, y(t), \max_{t-h_1 \leq \xi \leq t} y(\xi), \max_{t \leq \xi \leq t+h_2} y(\xi)), \ t \in [a, b].$$

Then

$$y(t) \leq x(t), \forall t \in [a-h_1, a] \cup [b, b+h_2] \text{ implies that } y \leq x.$$

Proof. Let us consider the operator $\widetilde{w} : C[a-h_1, b+h_2] \to C[a-h_1, b+h_2]$ defined by

$$\widetilde{w}(z)(t) := \begin{cases} z(t), \ t \in [a-h_1, a], \\ w(z|_{[a-h_1,a]}, z|_{[b,b+h_2]})(t), \ t \in [a, b], \\ z(t), \ t \in [b, b+h_2]. \end{cases}$$

First of all we remark that

$$w(y|_{[a-h_1,a]}, y|_{[b,b+h_2]}) \leq w(x|_{[a-h_1,a]}, x|_{[b,b+h_2]})$$

and

$$\widetilde{w}(y) \leq \widetilde{w}(x).$$

In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \leq E_f(y).$$

On the other hand, from the condition (c) and Lemma 5.1, we have that the operator E_f^∞ is increasing. Hence

$$y \leq E_f(y) \leq E_f^2(y) \leq \ldots \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x.$$

So, $y \leq x$. □

6. Data dependence: monotony

In this section we study the monotony of the solution of the problem (1.1)-(1.2) with respect to φ, ψ and f. For this we need the following result from the WPOs theory.

Lemma 6.1. *(Abstract comparison lemma,* [13]*) Let $(X, d, \leq)$ an ordered metric space and $A, B, C : X \to X$ be such that:*

(i) *the operator A, B, C, are WPOs;*
(ii) *$A \leq B \leq C$;*
(iii) *the operator B is increasing.*

Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

From this abstract result we have

Theorem 6.2. *Let $f_i \in C([a, b] \times \mathbb{R}^3), i = 1, 2, 3,$ be as in Theorem 4.1. We suppose that:*

(i) *$f_1 \leq f_2 \leq f_3$;*
(ii) *$f_2(t, \cdot, \cdot, \cdot) : \mathbb{R}^3 \to \mathbb{R}$ is monotone increasing;*

Let x_i be a solution of the equation

$$-x_i''(t) = f_i(t, x(t), \max_{t-h_1 \leq \xi \leq t} x(\xi), \max_{t \leq \xi \leq t+h_2} x(\xi)),\ t \in [a, b] \text{ and } i = 1, 2, 3.$$

Then, $x_1(t) \leq x_2(t) \leq x_3(t),\ \forall t \in [a - h_1, a] \cup [b, b + h_2]$, implies that $x_1 \leq x_2 \leq x_3$, i.e. the unique solution of the problem (1.1)-(1.2) is increasing with respect to f, φ and ψ.

Proof. From Theorem 4.1, the operators $E_{f_i}, i = 1, 2, 3,$ are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \leq E_{f_2} \leq E_{f_3}.$$

On the other hand we remark that

$$\widetilde{w}(x_1) \leq \widetilde{w}(x_2) \leq \widetilde{w}(x_3)$$

and

$$x_i = E^{\infty}_{f_i}(\widetilde{w}(x_i)),\ i = 1, 2, 3.$$

So, the proof follows from Lemma 6.1. □

7. Data dependence: continuity

Consider the boundary value problem (1.1)-(1.2) and suppose the conditions of the Theorem 4.1 are satisfied. Denote by $x^*(\cdot; \varphi, \psi, f)$, the solution of this problem.

We need the following well known result (see [12]).

Theorem 7.1. *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

(i) *the operator A is a α-contraction;*
(ii) $F_B \neq \emptyset$;
(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta,\ \forall x \in X.$$

Then, if $F_A = \{x^*_A\}$ and $x^*_B \in F_B$, we have

$$d(x^*_A, x^*_B) \leq \frac{\eta}{1 - \alpha}.$$

We state the following result:

Theorem 7.2. *Let $\varphi_i, \psi_i, f_i, i = 1, 2$ be as in the Theorem 4.1. Furthermore, we suppose that there exists $\eta_i > 0, i = 1, 2$ such that*

(i) $|\varphi_1(t) - \varphi_2(t)| \leq \eta_1,\ \forall t \in [a - h_1, a]$ *and* $|\psi_1(t) - \psi_2(t)| \leq \eta_1,\ \forall t \in [b, b + h_2]$;
(ii) $|f_1(t, u_1, u_2, u_3) - f_2(t, u_1, u_2, u_3)| \leq \eta_2, \forall t \in C[a, b], u_i \in \mathbb{R}, i = 1, 2, 3.$

Then

$$\|x^*_1(t; \varphi_1, \psi_1, f_1) - x^*_2(t; \varphi_2, \psi_2, f_2)\| \leq \frac{8\eta_1 + (b - a)^2\eta_2}{8 - L_f(b - a)^2},$$

where $x^*_i(t; \varphi_i, \psi_i, f_i)$ are the solution of the problem (1.1)-(1.2) with respect to φ_i, ψ_i, f_i, $i = 1, 2$, and $L_f = \max(L_{f_1}, L_{f_2})$.

Proof. Consider the operators $B_{\varphi_i,\psi_i,f_i}, i = 1,2$. From Theorem 4.1 these operators are contractions.

Additionally

$$\|B_{\varphi_1,\psi_1,f_1}(x) - B_{\varphi_2,\psi_2,f_2}(x)\| \le \eta_1 + \eta_2\frac{(b-a)^2}{8},$$

$\forall x \in C[a-h_1, b+h_2]$.

Now the proof follows from the Theorem 7.1, with $A := B_{\varphi_1,\psi_1,f_1}, B = B_{\varphi_2,\psi_2,f_2}, \eta = \eta_1 + \eta_2\frac{(b-a)^2}{8}$ and $\alpha := \frac{L_f}{8}(b-a)^2$ where $L_f = \max(L_{f_1}, L_{f_2})$. □

We have

Theorem 7.3. *([13]) Let (X,d) be a metric space and $A_i : X \to X,\ i = 1,2$. Suppose that*

(i) *the operator A_i is c_i-weakly Picard operator, $i=1,2$;*
(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \le \eta,\ \forall x \in X.$$

Then $H(F_{A_1}, F_{A_2}) \le \eta \max(c_1, c_2)$.

In what follow we shall use the c-WPOs techniques to give some data dependence results using Theorem 7.3.

Theorem 7.4. *Let f_1 and f_2 be as in the Theorem 4.1. Let $S_{E_{f_1}}, S_{E_{f_2}}$ be the solution sets of system (1.1) corresponding to f_1 and f_2. Suppose that there exists $\eta > 0$, such that*

$$|f_1(t,u_1,u_2,u_3) - f_2(t,u_1,u_2,u_3)| \le \eta \tag{7.1}$$

for all $t \in [a,b], u_i \in \mathbb{R}, i = 1,2,3$.

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \le \frac{(b-a)^2\eta}{8 - L_f(b-a)^2},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ denotes the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C[a,b]$.

Proof. In the condition of Theorem 4.1, the operators E_{f_1} and E_{f_2} are c_1-WPO and c_2-weakly Picard operators.

Let

$$X_{\varphi,\psi} := \{x \in X|\ x|_{[a-h_1,a]} = \varphi,\ x|_{[b,b+h_2]} = \psi\}.$$

It is clear that $E_{f_1}|_{X_{\varphi,\psi}} = B_{f_1},\ E_{f_2}|_{X_{\varphi,\psi}} = B_{f_2}$. Therefore,

$$\left|E_{f_1}^2(x) - E_{f_1}(x)\right| \leq \frac{1}{8} L_{f_1}(b-a)^2 \left|E_{f_1}(x) - x\right|,$$

$$\left|E_{f_2}^2(x) - E_{f_2}(x)\right| \leq \frac{1}{8} L_{f_2}(b-a)^2 \left|E_{f_2}(x) - x\right|,$$

for all $x \in C[a-h_1, b+h_2]$.

Now, choosing

$$\alpha_i = \frac{1}{8} L_{f_i}(b-a)^2, i = 1, 2,$$

we get that E_{f_1} and E_{f_2} are c_1-weakly Picard operators and c_2-weakly Picard operators with $c_1 = (1-\alpha_1)^{-1}$ and $c_2 = (1-\alpha_2)^{-1}$. From (7.1) we obtain that

$$\|E_{f_1}(x) - E_{f_2}(x)\|_C \leq (b-a)^2 \eta,$$

$\forall x \in C[a-h_1, b+h_2]$. Applying Theorem 7.3 we have that

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{(b-a)^2 \eta}{8 - L_f (b-a)^2},$$

where $L_f = \max(L_{f_1}, L_{f_2})$ and $H_{\|\cdot\|_C}$ is the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C[a-h_1, b+h_2]$. □

8. Examples

Let $p, q, r, g \in C[a, b]$. We consider the following boundary value problem

$$-x''(t) = p(t)x(t) + q(t) \max_{t-h_1 \leq \xi \leq t} x(\xi) + r(t) \max_{t \leq \xi \leq t+h_2} x(\xi) + g(t),\ t \in [a, b], \quad (8.1)$$

with the "boundary" conditions

$$\begin{cases} x(t) = \varphi(t),\ t \in [a-h_1, a], \\ x(t) = \psi(t),\ t \in [b, b+h_2]. \end{cases} \quad (8.2)$$

In this case $f(t, u_1, u_2, u_3) = p(t)u_1 + q(t)u_2 + r(t)u_3 + g(t),\ t \in [a, b], u_i \in \mathbb{R}, i = 1, 2, 3$, and $L_f = \max_{t \in [a,b]} (|p(t)| + |q(t)| + |r(t)|)$.

We suppose that:

(C_1') $h_1, h_2,\ a$ and $b \in \mathbb{R},\ a < b,\ h_1 > 0,\ h_2 > 0$;

(C_2') $p, q, r, g \in C[a, b]$;

(C_3') $\varphi \in C[a-h_1, a]$ and $\psi \in C[b, b+h_2]$.

From this conditions and the above results we have

Theorem 8.1. *We suppose that:*

(a) *the conditions* $(C_1') - (C_3')$ *are satisfied;*

(C_4') $\frac{L_f}{8}(b-a)^2 < 1$.

Then the problem (8.1)-(8.2) has a unique solution which is the uniform limit of the successive approximations.

Theorem 8.2. *We suppose that:*

(a) *the conditions* $(C_1') - (C_3')$ *are satisfied;*

(b) $\frac{L_f}{8}(b-a)^2 < 1$*;*

(c) $p \geq 0,\ q \geq 0,\ r \geq 0$.

Let x *be a solution of equation (8.1) and* y *a solution of the inequality*

$$-y''(t) \leq p(t)y(t) + q(t) \max_{t-h_1 \leq \xi \leq t} y(\xi) + r(t) \max_{t \leq \xi \leq t+h_2} y(\xi) + g(t),\ t \in [a,b].$$

Then

$$y(t) \leq x(t), \forall t \in [a-h_1, a] \cup [b, b+h_2] \text{ implies that } y \leq x.$$

Theorem 8.3. *Let* $p_i, q_i, r_i, g_i \in C[a,b], i = 1,2,3,$ *be as in Theorem 8.1. We suppose that:*

(i) $p_1 \leq p_2 \leq p_3,\ q_1 \leq q_2 \leq q_3,\ r_1 \leq r_2 \leq r_3,\ g_1 \leq g_2 \leq g_3;$

(ii) $p \geq 0,\ q \geq 0,\ r \geq 0.$

Let $x_i,\ i = 1,2,3,$ *be a solution of the equation*

$$-x_i''(t) = p_i(t)x(t) + q_i(t) \max_{t-h_1 \leq \xi \leq t} x(\xi) + r_i(t) \max_{t \leq \xi \leq t+h_2} x(\xi) + g_i(t),\ t \in [a,b].$$

Then, $x_1(t) \leq x_2(t) \leq x_3(t),\ \forall t \in [a-h_1, a] \cup [b, b+h_2]$*, implies that* $x_1 \leq x_2 \leq x_3$*, i.e. the unique solution of the problem (8.1)-(8.2) is increasing with respect to* $p,\ q,\ r,\ \varphi$ *and* ψ*.*

Theorem 8.4. *Let* $\varphi_i, \psi_i, p_i, q_i, r_i, g_i, i = 1,2$ *be as in the Theorem 8.1. Furthermore, we suppose that there exists* $\eta_i > 0, i = 1,2,$ *such that*

(i) $|\varphi_1(t) - \varphi_2(t)| \leq \eta_1,\ \forall t \in [a-h_1, a]$ *and* $|\psi_1(t) - \psi_2(t)| \leq \eta_1,\ \forall t \in [b, b+h_2]$*;*

(ii) $|f_1(t, u_1, u_2, u_3) - f_2(t, u_1, u_2, u_3)| \leq \eta_2, \forall t \in C[a,b], u_i \in \mathbb{R}, i = 1,2,3.$

Then

$$\|x_1^*(t; \varphi_1, \psi_1, f_1) - x_2^*(t; \varphi_2, \psi_2, f_2)\| \leq \frac{8\eta_1 + (b-a)^2\eta_2}{8 - L_f(b-a)^2},$$

where $x_i^*(t;\varphi_i,\psi_i,f_i)$ *are the solution of the problem (8.1)-(8.2) with respect to* φ_i,ψ_i,f_i, $i=1,2$, *and* $p(t)=\max p_i(t)$, $q(t)=\max q_i(t)$, $r(t)=\max r_i(t)$, $g(t)=\max g_i(t)$, $i=1,2$.

Theorem 8.5. *Let* f_1 *and* f_2 *be as in the Theorem 8.1. Let* $S_{E_{f_1}}, S_{E_{f_2}}$ *be the solution sets of system (8.1) corresponding to* f_1 *and* f_2. *Suppose that there exists* $\eta>0$, *such that*

$$|f_1(t,u_1,u_2,u_3)-f_2(t,u_1,u_2,u_3)|\le\eta \tag{8.3}$$

for all $t\in[a,b], u_i\in\mathbb{R}, i=1,2,3.$

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}},S_{E_{f_2}})\le\frac{(b-a)^2\eta}{8-L_f(b-a)^2},$$

where $p(t)=\max p_i(t)$, $q(t)=\max q_i(t)$, $r(t)=\max r_i(t)$, $g(t)=\max g_i(t)$, $i=1,2$ *and* $H_{\|\cdot\|_C}$ *denotes the Pompeiu-Hausdorff functional with respect to* $\|\cdot\|_C$ *on* $C[a,b]$.

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