# ULAM STABILITIES OF DIFFERENTIAL EQUATION WITH ABSTRACT VOLTERRA OPERATOR IN A BANACH SPACE

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**Abstract.** The paper is devoted to the study of Ulam–Hyers stability and Ulam–Hyers–Rassias stability for a class of abstract Volterra equations.

## 1. Introduction

The equations involving abstract Volterra operators have been investigated since 1928 by many authors: L. Tonelli (1928), S. Cinquini (1930), D. Graffi (1930), A.N. Tychonoff (1938). Such operators appear in many areas of investigation: control theory, continuum mechanics, engineering, dynamics of the nuclear reactors. Applications of such operators are contained in [1], [3], [6], [9], [7].

Equation stability is an important subject in the applications. Despite the large amount of works on Volterra integral equations, only the work [5] studies the conditions which ensure Ulam–Hyers–Rassias and Ulam–Hyers stability of a certain type of Volterra integral equations (see [4], [5], [8], [12]).

In the present paper we shall present Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability for a differential equation with abstract Volterra operator in a Banach space

$$x'(t) = f(t, x(t), V(x)(t)),\ t \in I \subset \mathbb{R},$$

where

(i) $I = [a, b]$ or $I = [a, \infty[$;
(ii) $(\mathbb{B}, |\cdot|)$ is a Banach space;
(iii) $f \in C([a, b] \times \mathbb{B}^2, \mathbb{B}),\ V \in C((C[a, b], \mathbb{B}), (C[a, b], \mathbb{B}))$.

## 2. Preliminaries

Let $(\mathbb{B}, |\cdot|)$ be a Banach space and $V : (C[a, b], \mathbb{B}) \to (C[a, b], \mathbb{B})$ an abstract Volterra operator.

[0]Received . Revised .
[0]2000 Mathematics Subject Classification: 34L05, 34K20, 47H10.
[0]Keywords: Ulam–Hyers stability, Ulam–Hyers–Rassias stability, abstract Volterra operators.

For $f \in C(I \times \mathbb{B}^2, \mathbb{B})$, $\varepsilon > 0$ and $\varphi \in C(I, \mathbb{R}_+)$ we consider the Cauchy problem

$$x'(t) = f(t, x(t), V(x)(t)),\ t \in I \tag{2.1}$$

$$x(a) = \alpha,\ \alpha \in \mathbb{R} \tag{2.1'}$$

and the following inequations

$$\left|y'(t) - f(t, y(t), V(y)(t))\right| \leq \varepsilon,\ t \in I, \tag{2.2}$$

$$\left|y'(t) - f(t, y(t), V(y)(t))\right| \leq \varphi(t),\ t \in I. \tag{2.3}$$

Following [11] we present some definitions and remarks.

**Definition 2.1.** *The equation* (2.1) *is Ulam–Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1(I, \mathbb{B})$ of* (2.2) *there exists a solution $x \in C^1(I, \mathbb{B})$ of* (2.1) *with*

$$|y(t) - x(t)| \leq c\varepsilon,\ \forall t \in I.$$

**Definition 2.2.** *The equation* (2.1) *is generalized Ulam–Hyers–Rassias stable with respect to $\varphi$, if there exists $c_\varphi > 0$, such that for each solution $y \in C^1(I, \mathbb{B})$ of the inequation* (2.3) *there exists a solution $x \in C^1(I, \mathbb{B})$ of* (2.1) *with*

$$|y(t) - x(t)| \leq c_\varphi \varphi(t),\ \forall t \in I.$$

**Remark 2.3.** *A function $y \in C^1(I, \mathbb{B})$ is a solution of* (2.2) *if and only if there exists a function $g \in C(I, \mathbb{B})$ (which depend on $y$) such that*

(i) $|g(t)| \leq \varepsilon,\ \forall t \in I;$
(ii) $y'(t) = f(t, y(t), V(y)(t)) + g(t),\ \forall t \in I.$

**Remark 2.4.** *A function $y \in C^1(I, \mathbb{B})$ is a solution of* (2.3) *if and only if there exists a function $\widetilde{g} \in C(I, \mathbb{B})$ (which depend on $y$) such that*

(i) $|\widetilde{g}(t)| \leq \varphi(t),\ \forall t \in I;$
(ii) $y'(t) = f(t, y(t), V(y)(t)) + \widetilde{g}(t),\ \forall t \in I.$

**Remark 2.5.** *If $y \in C^1(I, \mathbb{B})$ is a solution of the inequation* (2.2)*, then $y$ is a solution of the following integral equation*

$$\left|y(t) - y(a) - \int_a^t f(s, y(s), V(y)(s))ds\right| \leq (t-a)\varepsilon,\ \forall t \in I.$$

**Remark 2.6.** *If $y \in C^1(I, \mathbb{B})$ is a solution of the inequation* (2.3)*, then $y$ is a solution of the following integral equation*

$$\left|y(t) - y(a) - \int_a^t f(s, y(s), V(y)(s))ds\right| \leq \int_a^t \varphi(s)ds,\ \forall t \in I.$$

## 3. Ulam–Hyers stability on a compact interval $I = [a, b]$

This section is totaly devoted to find out conditions under which the Volterra equation (2.1) admits the Ulam–Hyers stability on a compact interval $I = [a, b]$. This is assembled in the next theorem.

**Theorem 3.1.** *We suppose that*

(a) $f \in C([a, b] \times \mathbb{B}^2, \mathbb{B}),\ V \in C((C[a, b], \mathbb{B}), (C[a, b], \mathbb{B}))$;

(b) *there exists* $L_f > 0$ *such that*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|, \forall t \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2;$$

(c) *there exits* $L_V > 0$ *such that*

$$|V(x)(t) - V(y)(t)| \leq L_V |x(t) - y(t)|,\ \forall x, y \in C[a, b], t \in [a, b].$$

*Then*

(i) *the problem* (2.1)–(2.1') *has a unique solution in* $C([a, b], \mathbb{B})$;

(ii) *the equation* (2.1) *is Ulam–Hyers stable.*

*Proof.* Let $y \in C^1([a, b], \mathbb{B})$ be a solution of the inequation (2.2).

From [7], the problem (2.1)–(2.1') has a unique solution in $C^1([a, b], \mathbb{B})$. We denote by $x \in C^1([a, b], \mathbb{B})$ the unique solution of the Cauchy problem

$$\begin{aligned} &x'(t) = f(t, x(t), V(x)(t)),\ t \in [a, b], \\ &x(a) = y(a). \end{aligned}$$

From condition (a) we have

$$x(t) = y(a) + \int_a^t f(s, x(s), V(x)(s))ds,\ t \in [a, b].$$

From Remark 2.5 we have

$$\left| y(t) - y(a) - \int_a^t f(s, y(s), V(y)(s))ds \right| \leq (t - a)\varepsilon,\ t \in [a, b].$$

From above relations we have

$$\begin{aligned} |y(t) - x(t)| \leq &\left| y(t) - y(a) - \int_a^t f(s, y(s), V(y)(s))ds \right| + \\ &+ \int_a^t |f(s, y(s), V(y)(s)) - f(s, x(s), V(x)(s))| \, ds \\ &\leq (t - a)\varepsilon + L_f\left(\int_a^t |y(s) - x(s)| \, ds + \int_a^t |V(y)(s) - V(x)(s)| \, ds\right) \\ &\leq (t - a)\varepsilon + L_f(1 + L_V) \int_a^t |y(s) - x(s)| \, ds. \end{aligned}$$

From the Gronwall Lemma (see [10], Example 6.2) we have that

$$|y(t) - x(t)| \leq (t - a)\varepsilon e^{L_f(1+L_V)(b-a)} = c\varepsilon,\ t \in [a, b],$$

i.e. the equation (2.1) is Ulam–Hyers stable. □

## 4. Generalized Ulam–Hyers–Rassias stability on $I = [a, \infty[$

This section is devoted to the analysis of the generalized Ulam–Hyers–Rassias stability of the Volterra equation (2.1) but when considering infinite intervals. Such stability is here obtained for this case under the conditions of the next result.

**Theorem 4.1.** *We suppose that*

(a) $f \in C([a, \infty[\times \mathbb{B}^2, \mathbb{B}),\ V \in C((C[a, b], \mathbb{B}), (C[a, b], \mathbb{B}))$;

(b) *there exists* $l_f \in L^1([a, \infty[, \mathbb{R}_+)$ *such that*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le l_f(t)(|u_1 - v_1|+|u_2 - v_2|),\ \forall t \in [a, \infty[, u_i, v_i \in \mathbb{B};$$

(c) *there exists* $l_V \in L^1([a, \infty[, \mathbb{R}_+)$ *such that*

$$|V(x)(t) - V(y)(t)| \le l_V(t)\,|x(t) - y(t)|\,,\ \forall x, y \in C[a, \infty[, t \in [a, \infty[;$$

(d) *the function* $\varphi \in C[a, \infty[$ *is increasing;*

(e) *there exists* $\lambda > 0$ *such that*

$$\int_a^t \varphi(s)ds \le \lambda\varphi(t),\ t \in [a, \infty[.$$

*Then*

(i) *the problem* (2.1)–(2.1') *has a unique solution in* $C([a, \infty[, \mathbb{B})$;

(ii) *the equation* (2.1) *is generalized Ulam–Hyers–Rassias stable with respect to* $\varphi$.

*Proof.* Let $y \in C^1([a, \infty[, \mathbb{B})$ be a solution of the inequation (2.3).

From [7], the problem (2.1)–(2.1') has a unique solution in $C^1([a, \infty[, \mathbb{B})$. We denote by $x \in C^1([a, \infty[, \mathbb{B})$ the unique solution of the Cauchy problem

$$x'(t) = f(t, x(t), V(x)(t)),\ t \in [a, \infty[,$$
$$x(a) = y(a).$$

We have that

$$x(t) = y(a) + \int_a^t f(s, x(s), V(x)(s))ds,\ t \in [a, \infty[.$$

From (2.3) we have

$$\left| y(t) - y(a) - \int_a^t f(s, y(s), V(y)(s))ds \right| \le$$
$$\le \int_a^t \varphi(s)ds \le \lambda\varphi(t),\ t \in [a, \infty[.$$

From the above relations it follows

$$|y(t)-x(t)| \le \left| y(t)-y(a)-\int_a^t f(s,y(s),V(y)(s))ds \right| +$$
$$+\int_a^t |f(s,y(s),V(y)(s))-f(s,x(s),V(x)(s))|\, ds$$
$$\le \lambda\varphi(t)+\int_a^t l_f(s)(1+l_V(s))\,|y(s)-x(s)|\, ds.$$

From the Gronwall Lemma (see [10], Example 6.2) we have that

$$|y(t)-x(t)| \le \lambda\varphi(t)e^{\int_a^t l_f(s)(1+l_V(s))ds}$$
$$=\left[\lambda e^{\int_a^t l_f(s)(1+l_V(s))ds}\right]\varphi(t)=c_\varphi\varphi(t),\ t\in[a,\infty[,$$

i.e. the equation (2.1) is generalized Ulam–Hyers–Rassias stable. □

## 5. Applications

**Example 5.1.**

$$x'(t)=\int_a^t x(t)dt,\ t\in I. \tag{5.1}$$

*For this example we have* $\mathbb{B}=\mathbb{R}$ *and* $V(x)(t)=\int_a^t x(t)dt$*, see* [2].

*So, the equation* (5.1) *is Ulam–Hyers stable on* $I=[a,b]$ *and is generalized Ulam–Hyers–Rassias stable on* $I=[a,\infty[$.

**Example 5.2.**

$$x'(t)=f(t,x(t)),\ t\in I. \tag{5.2}$$

*For this example we have* $\mathbb{B}=\mathbb{R}$ *and* $V(x)(t)=0$.

*The equation* (5.2) *is Ulam–Hyers stable on* $I=[a,b]$ *and is generalized Ulam–Hyers–Rassias stable on* $I=[a,\infty[$*, see* [11].

**Example 5.3.**

$$x'(t)=f(t,x(t),\int_a^t k(t,s,x(s))ds,\ t\in I. \tag{5.3}$$

*For this example we have* $\mathbb{B}=\mathbb{R}$, $V(x)(t)=\int_a^t k(t,s,x(s))ds$ *and conditions* (a)*-*(c) *from Theorem 3.1 become:*

(a) $f\in C([a,b]\times\mathbb{R}^2,\mathbb{R}),\ k\in C([a,b]\times[a,b]\times\mathbb{R},\mathbb{R})$ *are given;*

(b) *there exists* $L_f>0$ *such that*

$$|f(t,u_1,u_2)-f(t,v_1,v_2)|\le L_f\sum_{i=1}^{2}|u_i-v_i|\,,\ \forall t\in[a,b],u_i,v_i\in\mathbb{R},i=1,2;$$

(c) *there exists* $L_k>0$ *such that*

$$|k(t,s,u)-k(t,s,v)|\le L_k\,|u-v|\,,\ \forall t,s\in[a,b],\ u,v\in\mathbb{R}.$$

*In this case, the equation* (5.2) *has a unique solution in* $C([a, b], \mathbb{R})$*, see* [7]*, is Ulam–Hyers stable on* $I = [a, b]$ *and is generalized Ulam–Hyers–Rassias stable on* $I = [a, \infty[$.

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