INTEGRO-DIFFERENTIAL EQUATION WITH TWO TIME MODIFICATIONS

VERONICA-ANA ILEA, DIANA OTROCOL

Abstract. We consider an integro-differential equation with two time modifications. Existence, uniqueness and monotony results of solution for the Cauchy problem are obtained using weakly Picard operator theory. In the last section we present a step method for this type of equation.

1. Introduction

This paper is concerned with the following integro-differential equation

\( x'(t) = g(t, x(t), x(t - \tau)) + \int_{t-h}^{t} K(s, x(s))ds, \ t \in I. \)  

The aim of this paper is to obtain existence and uniqueness theorems using contraction principle, step method and monotony results for the Cauchy problem, see \[7\] and \[11\]. Such kind of results have been proved for an integro delay equation in \[17\]. The approach proposed in the present paper is different to the ones in \[4\], \[17\] and \[18\] and it is based on the different time modifications.

In our paper we consider \( I = [0, \infty) \).

Regarding the two delays we have the following cases: \( h > 0, \ \tau > 0, \ \tau > h \), discussed in \[8\], and here we take the case: \( \tau < 0, \ h > 0, \ h = |\tau| \).

The equation becomes

\( x'(t) = g(t, x(t), x(t + h)) + \int_{t-h}^{t} K(s, x(s))ds, \ t \in [0, \infty], \)  

with the condition

\( x(t) = \varphi(t), \ t \in [-h, h]. \)

Relative to (1.2)--(1.3) we consider the following conditions:

(\( C_1 \)) \( (\mathbb{B}, |.|) \) is a Banach space, \( g \in C([0, \infty[ \times \mathbb{B}^2, \mathbb{B}), K \in C([0, \infty[ \times \mathbb{B}, \mathbb{B}), \varphi \in C([-h, h], \mathbb{B}); \)

(\( C_1' \)) \( (\mathbb{B}, |.|) \) is a Banach space, \( g \in C^\infty([0, \infty[ \times \mathbb{B}^2, \mathbb{B}), K \in C^\infty([0, \infty[ \times \mathbb{B}, \mathbb{B}), \varphi \in C^\infty([-h, h], \mathbb{B}); \)

(\( C_2 \)) there exists \( L_1, L_2 > 0 \) such that

\[ |g(t, u_1, v_1) - g(t, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \ u_i, v_i \in \mathbb{B}, t \in [0, \infty[; \]

(\( C_3 \)) there exists \( L_3 > 0 \) such that

\[ |K(s, u) - K(s, v)| \leq L_3 |u - v|, \ u, v \in \mathbb{B}, t \in [0, \infty[; \]

(\( C_4 \)) \( (L_1 + L_2 + 2L_3 h)h < 1; \)

(\( C_5 \)) \( \varphi''(0) = g(h, \varphi(0), \varphi(h)) + \int_{-h}^{0} K(s, \varphi(s))ds. \)

2010 Mathematics Subject Classification. 47H10, 47N20.

Key words and phrases. Integro-differential equation, two time modifications, step method, Picard operators.
In what follow we shall present some notions that will help us obtaining the results bellow.

Let $X$ be a nonempty set,
\[
s(X) := \{ (x_n)_{n \in \mathbb{N}^*} \mid x_n \in X, n \in \mathbb{N}^* \}
\]
and
\[
M(X) := \{ (x_{ij})_{i,j \in \mathbb{N}^*} \}
\]
where
\[
(x_{ij})_{i,j} := \begin{pmatrix}
x_{11} & x_{12} & x_{13} & \cdots \\
x_{21} & x_{22} & x_{23} & \cdots \\
x_{31} & x_{32} & x_{33} & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{pmatrix}
\]
is a infinite matrix.

For $A \in M(\mathcal{B})$ we denote
\[
|A| := \sup_{1 \leq i \leq \infty} \sum_{j \in \mathbb{N}^*} |a_{ij}|.
\]

Let $d : X \times X \to s(\mathcal{B})$ be the generalized metric.

**Remark 1.1.** [13] A functional $d : X \times X \to s(\mathcal{B})$, $(x, y) \mapsto (d_k(x, y))_{k \in \mathbb{N}^*}$ is a generalized metric of $X$ iff:

(a) $d_k$ is a pseudometric, $\forall k \in \mathbb{N}^*$;
(b) $\forall x, y \in X, x \neq y$, there exist $k \in \mathbb{N}^*$ such as $d_k(x, y) \neq 0$.

**Definition 2.1.** [13] Let $(X, d)$ be a complete generalized metric space, $A : X \to X$ and $S \in M(\mathcal{B})$. The operator $A$ is a $S$-contraction iff:

(i) $S$ is row and column finite (meaning that there are only a finite number of nonzero elements in each row and each column);
(ii) $S$ is a Neumann matrix (meaning that if $S^n$ is definite for all $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} S^n$ converges for all $x, y \in X$);
(iii) $\sum_{n \in \mathbb{N}} S^n d(x, y)$ converges $\forall x, y \in X$;
(iv) $d(A(x), A(y)) \leq S d(x, y)$ $\forall x, y \in X$.

We consider the space $X = C([-h, \infty[, \mathcal{B})$ endowed with the norm
\[
\| \cdot \| : X \to s(\mathbb{R}^+), \quad \| x \| := \begin{pmatrix}
\| x \|_0 \\
\| x \|_1 \\
\| x \|_2 \\
\| x \|_m \\
\| x \|_n \\
\| x \|_\infty 
\end{pmatrix},
\]
where $\| x \|_0 = \max_{-h \leq t \leq h} |x(t)|$ and $\| x \|_m = \max_{m h \leq t \leq (m+1)h} |x(t)|, m \geq 1$.

This generalized norm induces a generalized metric, $d(x, y) := \| x - y \|$.

2. Preliminaries

Let $(X, d)$ be a generalized metric space and $A : X \to X$ an operator. In this paper we shall use the terminologies and notations from [13]–[15]. For the convenience of the reader we shall recall some of them.

We denote by $A_0 := 1_X$, $A^1 := A$, $A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$, the iterate operators of the operator $A$. Also we shall use the following notations:

$F_A := \{ x \in X \mid A(x) = x \}$ - the fixed point set of $A$;
$I(A) := \{ Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset \}$ - the family of the nonempty invariant subset of $A$.

**Definition 2.1.** $A : X \to X$ is called a Picard operator (briefly PO) if:
(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \to x^*$ as $n \to \infty$, $\forall x \in X$.

**Definition 2.2.** $A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n \in \mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on $x$) is a fixed point of $A$.

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n \to \infty} A^n(x).$$

Obviously $A^\infty(X) = F_A$. Moreover, if $A$ is a PO and we denote by $x^*$ its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

**Lemma 2.3.** Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:

(i) $A$ is WPO;
(ii) $A$ is increasing.

Then, the operator $A^\infty$ is increasing.

**Lemma 2.4.** Let $(X, d, \leq)$ an ordered metric space and $A, B, C : X \to X$ be such that:

(i) the operator $A, B, C$ are WPOs;
(ii) $A \leq B \leq C$;
(iii) the operator $B$ is increasing.

Then $x \leq y \leq z$ implies that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

**Theorem 2.5.** [13] Let $(X, d)$ be a complete metric space and $A : X \to X$ a $S$-contraction. Then we have

(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \xrightarrow{d} x^*$, as $n \to \infty$, $\forall x \in X$;
(iii) $d(A^n(x), x^*) \leq (E - S)^{-1}S^n d(x, A(x))$;
(iv) $d(x, x^*) \leq (E - S)^{-1}d(x, A(x))$.

In what follow we shall apply the above results to the problem (1.2)–(1.3). For other applications of these abstract results, see [2], [3], [8], [9], [12], [16].

### 3. Existence and uniqueness

From Theorem 2.5 we have

**Theorem 3.1.** In the condition $(C_1), (C_2)$ $(C_3)$, and $(C_4)$ the problem (1.2)–(1.3) has in $C([-h, \infty[, B)$ a unique solution $\hat{x}$ which is the limit of the sequence of successive approximation.

**Proof.** We consider the operator $A : X \to X$ defined by

$$A(x)(t) = \begin{cases} 2 \varphi(t), & t \in [-h, h] \\ \varphi(h) + \int_{-h}^{h} g(\xi, x(\xi + h)) \, d\xi + K(s, x(s)) \, ds, & t \in [h, \infty[. \end{cases}$$

$(X, d)$ is a complete metric space with $d = (\|\cdot\|_m)_{m \in \{-1, 0, 1, \ldots\}}$ where

$$d(x, y) = \begin{pmatrix} d_0(x, y) \\ \vdots \\ d_m(x, y) \\ \vdots \end{pmatrix}.$$
For \( t \in [-h, h] \) we have
\[
\|A(x)(t) - A(y)(t)\|_0 = 0, \; \forall x, y \in X.
\]
For \( t \in [h, 2h] \) we have
\[
|A(x)(t) - A(y)(t)|_1 \leq \frac{L_1}{h} \int_h^t |x(\xi) - y(\xi)| \, d\xi + L_2 \int_h^t |x(\xi + h) - y(\xi + h)| \, d\xi + L_3 \int_{\xi = h}^{t} |x(s) - y(s)| \, dsd\xi
\]
\[
\leq L_1 h \|x - y\|_1 + L_2 h \|x - y\|_2 + L_3 \int_h^t (h \|x - y\|_0 + h \|x - y\|_1) \, d\xi
\]
\[
\leq L_3 h^2 \|x - y\|_0 + (L_1 h + L_3 h^2) \|x - y\|_1 + L_2 h \|x - y\|_2.
\]
So, \( \|A(x)(t) - A(y)(t)\|_1 \leq L_3 h^2 \|x - y\|_0 + (L_1 h + L_3 h^2) \|x - y\|_1 + L_2 h \|x - y\|_2 \).

For \( t \in [2h, 3h] \) we have
\[
\|A(x)(t) - A(y)(t)\|_2 \leq L_3 h^2 \|x - y\|_1 + (L_1 h + L_3 h^2) \|x - y\|_2 + L_2 h \|x - y\|_3.
\]
By induction, for \( t \in [mh, (m+1)h] \) we have that
\[
\|A(x)(t) - A(y)(t)\|_m \leq L_3 h^2 \|x - y\|_{m-1} + (L_1 h + L_3 h^2) \|x - y\|_m + L_2 h \|x - y\|_{m+1}.
\]
Then
\[
\left(\begin{array}{c}
|A(x)(t) - A(y)(t)|_0 \\
|A(x)(t) - A(y)(t)|_1 \\
|A(x)(t) - A(y)(t)|_2 \\
\vdots \\
|A(x)(t) - A(y)(t)|_m
\end{array}\right) \leq
\left(\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
L_3 h^2 & L_3 h + L_3 h^2 & L_3 h & \cdots & 0 & 0 \\
0 & L_3 h^2 & L_3 h + L_3 h^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & L_3 h + L_3 h^2 & L_3 h \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots
\end{array}\right)
\left(\begin{array}{c}
\|x - y\|_0 \\
\|x - y\|_1 \\
\|x - y\|_2 \\
\vdots \\
\|x - y\|_m
\end{array}\right)
\]
So \( d(A(x), A(y)) \leq Sd(x, y) \), where \( S : s(\mathbb{R}) \rightarrow s(\mathbb{R}), \; \|S\| := \sup_{i \in \mathbb{N}} \sum_{j=0}^{\infty} |L_{ij}| = (L_1 + L_2 + 2L_3 h)h \), which proves that \( A \) is Lipschitz with
\[
S = \left(\begin{array}{cccccc}
0 & 0 & 0 & \cdots & 0 & 0 \\
L_3 h^2 & L_3 h + L_3 h^2 & L_3 h & \cdots & 0 & 0 \\
0 & L_3 h^2 & L_3 h + L_3 h^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & L_3 h + L_3 h^2 & L_3 h \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots
\end{array}\right).
\]
From condition (C4) we have that \( A \) is \( S \)-contraction. Applying Theorem 2.5 we have the conclusion.

**Remark 3.2.** From the proof of Theorem 3.1, it follows that the operator \( A \) is PO in \( ([C][-h, \infty], \mathcal{B}, d) \).
In this section we shall study the relation between the solution of the problem (1.2)–(1.3) and the subsolution of the same problem.

Let \( \hat{x} \) be the unique solution of the problem (1.2)–(1.3) and \( y \) a subsolution of the same problem, i.e.

\[
y'(t) \leq g(t, y(t), y(t + h)) + \int_{t-h}^{t} K(s, y(s))ds, \ t \in [0, \infty[,
\]

where \( g \) and \( K \) satisfy the conditions (\( C_1 \))–(\( C_4 \)) and

\[
y(t) = \varphi(t), \ t \in [-h, h].
\]

In this section we consider an ordered Banach space \((\mathbb{B}, || \cdot ||, \leq)\) and the operator \( A \) defined by (3.1) on the ordered Banach space \( X = ((C[a, b], \mathbb{B}), || \cdot ||, \leq) \). We have the following theorem

**Theorem 4.1.** We suppose that:

(a) the conditions \( (C_1) \)–(\( C_4 \)) are satisfied;

(b) \( g(t, \cdot, \cdot) : \mathbb{B}^2 \rightarrow \mathbb{B} \) and \( K(t, \cdot) : \mathbb{B} \rightarrow \mathbb{B} \) are increasing, \( \forall t \in [0, \infty[ \).

Then \( y \leq \hat{x} \) for all \( t \in [0, \infty[ \).

**Proof.** In terms of the operator \( A \) defined by the relation (3.1), we have \( \hat{x} = A(\hat{x}) \) and \( y \leq A(y) \). On the other hand from condition (b) and Lemma 2.3, we have that the operator \( A^\infty \) is increasing. Hence \( y \leq A(y) \leq A^2(y) \leq \cdots \leq A^\infty(y) \leq A^\infty(\hat{x}) = \hat{x} \). So, \( y \leq \hat{x} \).

\( \square \)

5. **Data dependence: monotony**

In this section we study the monotony of the system (1.2)–(1.3) with respect to \( g \) and \( K \). For this we use the abstract comparison Lemma from Section 2.

Consider the following equations

\[
x_i'(t) = g_i(t, x_i(t), x_i(t + h)) + \int_{t-h}^{t} K_i(s, x_i(s))ds, \ t \in [0, \infty[, \ i = 1, 3
\]

with the conditions (1.3) for each problem and let \( \hat{x}_i, \ i = 1, 3 \) the unique solutions of these problems. Then we need the operators \( A_i : X \rightarrow X \) defined by

\[
A_i(x)(t) = \begin{cases} 
\varphi(t), & t \in [-h, h] \\
\varphi(h) + \int_{h}^{t} g(\xi, x_i(\xi), x_i(\xi + h))d\xi + \\
+ \int_{h}^{t} K(\xi, x_i(\xi))d\xi, & t \in [h, \infty[.
\end{cases}
\]

**Theorem 5.1.** Let \( g_i, K_i, i = 1, 3 \), that satisfy the conditions \( (C_1) \)–(\( C_4 \)).

We suppose that we have

(i) \( g_1 \leq g_2 \leq g_3 \); 

(ii) \( g(\cdot, \cdot, \cdot) : \mathbb{B}^2 \rightarrow \mathbb{B} \) and \( K(t, \cdot) : \mathbb{B} \rightarrow \mathbb{B} \) are increasing.

Let \( \hat{x}_i \) the solutions of the equations (5.1), \( i = 1, 3 \).

Then \( \hat{x}_1(t) \leq \hat{x}_2(t) \leq \hat{x}_3(t), \forall t \in [0, \infty[. \)

**Proof.** From Theorem 3.1 the operators \( A_i \) are POs. From the condition (ii) it follows that the operator \( A_2 \) is monotone increasing and from condition (i) we have \( A_1 \leq A_2 \leq A_3 \). But \( \hat{x}_1 = A_1^\infty(\hat{x}_1) \), \( \hat{x}_2 = A_2^\infty(\hat{x}_2) \) and \( \hat{x}_3 = A_3^\infty(\hat{x}_3) \).

By applying the abstract comparison Lemma 2.4 follows that the unique solution of the problem (1.2)–(1.3) is increasing with respect to \( A \).

\( \square \)

**Remark 5.2.** The conclusion of the Theorem 5.1. means that the unique solution of (1.2)–(1.3) is increasing with respect to the right hand.
6. Step method

Next we apply the step method for (1.2)–(1.3). Let the conditions \((C_1'), (C_2), (C_3)\) and \((C_5)\) and we suppose also the condition

\((C_6)\) For all \(t \in [-h, \infty)\), \(u_1, u_2, u_3 \in \mathbb{B}\) there exists a unique \(u_2 \in \mathbb{B}\), \(u_2 = f(t, u_1, u_3)\) such as \(u_3 = g(t, u_1, u_2) + \int_{t-h}^{t} K(s, u_1)ds\).

Note that if \(x \in C^1(\mathbb{B})\) is a solution for (1.2)–(1.3) then, by mathematical induction, follows that \(x \in C^\infty(\mathbb{B})\).

**Theorem 6.1.** Suppose that we have \((C_1'), (C_2), (C_3), (C_5)\) and \((C_6)\). Then the problem (1.2)–(1.3) has a solution if and only if

\[\varphi^{(n+1)}(0) = g^{(n)}(0, \varphi(0), \varphi(h)) + \left[\int_{t-h}^{t} K(s, \varphi(s))ds\right]^{(n)}_{t=0}, \ n \in \mathbb{N}.\]

More, the solution is unique.

**Proof.** By the step method we have

\((p_0) \ x_0(t) = \varphi(t), \ t \in [0, h].\)

Also we have

\[x_0'(t) = g(t, x_0(t), x(t+h)) + \int_{t-h}^{t} K(s, x_0(s))ds\]

or

\[\varphi' = g(t, \varphi(t), x(t+h)) + \int_{t-h}^{t} K(s, \varphi(s))ds.\]

From condition \((C_6)\) we have that

\[x(t) := x_1(t) = f(t-h, \varphi(t-h), \varphi'(t-h)), \ \forall t \in [h, 2h].\]

From the regularity condition we have that \(x(t) \in C^\infty[-h, 2h]\) where

\[(6.1) \quad x(t) = \begin{cases} \varphi(t), & t \in [-h, h] \\ f(t-h, \varphi(t-h), \varphi'(t-h)), & t \in [h, 2h]. \end{cases}\]

The next step is

\((p_1) \ x_1'(t) = g(t, x_1(t), x(t+h)) + \int_{t-h}^{t} K(s, x_1(s))ds.\)

From condition \((C_5)\) we have that

\[x(t) := x_2(t) = f(t-h, x_1(t-h), x_1'(t-h)), \ \forall t \in [2h, 3h].\]

From the regularity condition we have that \(x(t) \in C^\infty[-h, 3h]\) where

\[(6.2) \quad x(t) = \begin{cases} \varphi(t), & t \in [-h, h] \\ f(t-h, \varphi(t-h), \varphi'(t-h)), & t \in [h, 2h] \\ f(t-h, x_1(t-h), x_1'(t-h)), & t \in [2h, 3h]. \end{cases}\]

By induction we can obtain the solution on \([-h, \infty]\) of the form

\[(6.3) \quad x(t) = \begin{cases} \varphi(t), & t \in [-h, h] \\ x_1, & t \in [h, 2h] \\ x_2, & t \in [2h, 3h] \\ \vdots \\ x_n, & t \in [nh, (n+1)h]. \end{cases}\]

In order to prove the necessity of the regularity condition we have \(x \in C^\infty[-h, \infty]\) a solution of the problem (1.2)–(1.3). By successive derivations we have

\[x^{(n+1)}(t) = g^{(n)}(t, \varphi(t), \varphi(t+h)) + \left[\int_{t-h}^{t} K(s, \varphi(s))ds\right]^{(n)}_{t=0}, \ n \in \mathbb{N}.\]

For \(t = 0\) follows that
\[ \varphi^{(n+1)}(0) = g^{(n)}(0, \varphi(0), \varphi(h)) + \left[ \int_{-h}^{0} K(s, \varphi(s)) \, ds \right]^{(n)}. \]

**Remark 6.2.** If \( B = \mathbb{R}^n \), then (1.2) is a finite system of equations, see [6], [10].

**Remark 6.3.** If \( B = \mathcal{P} \), then (1.2) is an infinite system of equations, see [1], [5], [19].

**References**


Babeș-Bolyai University
Department of Applied Mathematics
Kogălniceanu Str., No. 1, Cluj-Napoca, Romania
E-mail address: vdarzu@math.ubbcluj.ro

T. Popoviciu Institute of Numerical Analysis
Romanian Academy
Cluj-Napoca, Romania
E-mail address: dotrocol@ictp.acad.ro