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ON A D.V. IONESCU'S PROBLEM FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. The paper is devoted to the study of the following D.V. Ionescu's problem

$$\begin{cases} -x_1''(t) = f_1(t, x_1(t), x_2(t), x_1'(t), x_2'(t)), \ t \in [a, b] \\ -x_2''(t) = f_2(t, x_1(t), x_2(t), x_1'(t), x_2'(t)) \end{cases}$$

with polylocal conditions

$$\begin{cases} x_1(a) = x_2(b) = 0 \\ x_1(c) = x_2(c) \\ x_1'(c) = x_2'(c). \end{cases}$$

Existence, uniqueness and data dependence (monotony, continuity, differentiability with respect to parameter) results of solution for the Cauchy problem are obtained using Perov fixed point theorem and weakly Picard operator theory.
Key Words and Phrases: Perov fixed point theorem, weakly Picard operators, polylocal problem, fixed points, data dependence.
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1. Introduction

Let there be given real numbers $a < c < b$ and two functions $f_1, f_2 : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$. We consider the boundary value problem for the system of functional-differential equations

$$\begin{cases} -x_1''(t) = f_1(t, x_1(t), x_2(t), x_1'(t), x_2'(t)), \ t \in [a, b] \\ -x_2''(t) = f_2(t, x_1(t), x_2(t), x_1'(t), x_2'(t)) \end{cases} \tag{1.1}$$

with polylocal conditions

$$\begin{cases} x_1(a) = x_2(b) = 0 \\ x_1(c) = x_2(c) \\ x_1'(c) = x_2'(c). \end{cases} \tag{1.2}$$

Boundary value problems that arise from different areas of applied mathematics and physics have received a lot of attention in the literature in the last decades (see for example [2], [3], [5], [6], [7] and references therein). In [9], D.V. Ionescu study the problem (1.1)-(1.2) using the successive approximation method. Several results of D.V. Ionescu have been cited and extended by: O Aramă [1], Gh. Coman [4], V. Ilea and D. Otrocol [8], G. Micula [10], A. Petruşel and I.A. Rus [12], etc. Our approach is based on the Perov fixed point theorem [11] and weakly Picard operator theory [15]-[17] in the following conditions

(C_1) $f_1, f_2 \in C^1([a,b] \times \mathbb{R}^4, \mathbb{R})$;
(C_2) there exists $L_i > 0$ such that

$$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \leq L_i \sum_{j=1}^{4} |u_j - v_j|,$$

for all $t \in [a,b], u_j, v_j \in \mathbb{R},\ i = 1, 2, j = \overline{1, 4}$.

2. Ionescu's problem in the linear case

In this section we study the existence and uniqueness theorem for the problem

$$\begin{cases} -x_1''(t) = \chi_1(t), t \in [a,b] \\ -x_2''(t) = \chi_2(t) \end{cases} \tag{2.1}$$

$$\begin{cases} x_1(a) = x_2(b) = 0 \\ x_1(c) = x_2(c) \\ x_1'(c) = x_2'(c) \end{cases} \tag{2.2}$$

where $\chi_i : [a,b] \to \mathbb{R},\ i = 1, 2$.

Theorem 2.1. *We suppose that $\chi_i \in C([a,b], \mathbb{R}),\ i = 1, 2$. Then the problem (2.1)-(2.2) has a unique solution $\overset{*}{x} = (\overset{*}{x}_1, \overset{*}{x}_2) \in C^1([a,b], \mathbb{R}^2)$ and*

$$\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \int_a^b \mathbf{G}(t,s) \begin{pmatrix} \chi_1(s) \\ \chi_2(s) \end{pmatrix} ds,$$

where $\mathbf{G}(t,s)$ *is the Green function of the problem*

$$\begin{cases} -x_2''(t) = \chi_1(t) \\ -x_2''(t) = \chi_2(t) \\ x_1(a) = x_2(b) = 0 \\ x_1(c) = x_2(c) \\ x_1'(c) = x_2'(c). \end{cases}$$

$\mathbf{G}(t,s)$ *has the following form*

$$\mathbf{G}(t,s) = \begin{pmatrix} G_1(t,s) & 0 \\ 0 & G_2(t,s) \end{pmatrix}$$

where for $t \in (a,c)$

$$G_1(t,s) = \begin{pmatrix} \frac{(s-a)(b-t)}{b-a} & 0 \\ \frac{(t-a)(b-s)}{b-a} & 0 \\ 0 & \frac{(t-a)(b-s)}{b-a} \end{pmatrix},$$

$$G_2(t,s) = \begin{pmatrix} \frac{(s-a)(b-t)}{b-a} & 0 \\ \frac{(s-a)(b-t)}{b-a} & t-s \\ 0 & \frac{(t-a)(b-s)}{b-a} \end{pmatrix}$$

and for $t \in (c,b)$

$$G_1(t,s) = \begin{pmatrix} \frac{(s-a)(b-t)}{b-a} & 0 \\ s-t & \frac{(t-a)(b-s)}{b-a} \\ 0 & \frac{(t-a)(b-s)}{b-a} \end{pmatrix},$$

$$G_2(t,s) = \begin{pmatrix} \frac{(s-a)(b-t)}{b-a} & 0 \\ 0 & \frac{(s-a)(b-t)}{b-a} \\ 0 & \frac{(t-a)(b-s)}{b-a} \end{pmatrix}.$$

From Theorem 2.1 it follows that the problem (1.1)-(1.2) is equivalent with the system

$$\begin{pmatrix} -x_1(t) \\ -x_2(t) \end{pmatrix} = \int_a^b \mathbf{G}(t,s) \begin{pmatrix} f_1(s, x_1(s), x_2(s), x_1'(s), x_2'(s)) \\ f_2(s, x_1(s), x_2(s), x_1'(s), x_2'(s)) \end{pmatrix} ds. \tag{2.3}$$

In order to study the system (2.3), we shall use the weakly Picard operator technique. In the next section we present some notions and results from this theory.

3. Picard and weakly Picard operators

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper (see [15]-[17]). Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subset of A;

$A^{n+1} := A \circ A^n,\ A^0 = 1_X,\ A^1 = A,\ n \in \mathbb{N}.$

Definition 3.1. Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:

(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 3.2. Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.

Throughout this paper we denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m\times m$ matrices with positive elements and by I the identity $m \times m$ matrix. A square matrix Q with nonnegative elements is said to be convergent to zero if $Q^k \to 0$ as $k \to \infty$. It is known that the property of being convergent to zero is equivalent to each of the following three conditions (see [13], [14]):

(a) $I - Q$ is nonsingular and $(I - Q)^{-1} = I + Q + Q^2 + \cdots$ (where I stands for the unit matrix of the same order as Q);
(b) the eigenvalues of Q are located inside the unit disc of the complex plane;
(c) $I - Q$ is nonsingular and $(I - Q)^{-1}$ has nonnegative elements.

We finish this section by recalling the following fundamental result

Theorem 3.3. (Perov's fixed point theorem). *Let (X, d) with $d(x, y) \in \mathbb{R}^m$, be a complete generalized metric space and $A : X \to X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, such that*

(i) *$d(A(x), A(y)) \leq Qd(x, y)$, for all $x, y \in X$;*
(ii) *$Q^n \to 0$ as $n \to \infty$.*

Then

(a) *$F_A = \{x^*\}$,*
(b) *$A^n(x) = x^*$ as $n \to \infty$ and*

$$d(A^n(x), x^*) \leq (I - Q)^{-1}Q^n d(x_0, A(x_0)).$$

4. Existence and uniqueness

In this section we use Perov's fixed point theorem to obtain existence and uniqueness theorem for the solution of the problem (1.1)-(1.2).

We consider the Banach space $X = (C^1([a,b],\mathbb{R}^2), \|\cdot\|_{C^1})$ where $\|\cdot\|_{C^1}$, is the Chebyshev norm defined by

$$\|(x_1,x_2)-(y_1,y_2)\|_{C^1} := \|(x_1,x_2)-(y_1,y_2)\|_\infty + \left\|(x_1',x_2')-(y_1',y_2')\right\|_\infty$$

and the operator $A : X \to X$ defined by

$$A(x_1,x_2)(t) = \begin{pmatrix} A_1(x_1,x_2)(t) \\ A_2(x_1,x_2)(t) \end{pmatrix}$$

$$:= \int_a^b \mathbf{G}(t,s) \begin{pmatrix} f_1(s,x_1(s),x_2(s),x_1'(s),x_2'(s)) \\ f_2(s,x_1(s),x_2(s),x_1'(s),x_2'(s)) \end{pmatrix} ds.$$

We consider the problem (1.1)-(1.2) in the conditions (C_1) and (C_2). The problem (1.1)-(1.2) is equivalent with the fixed point equation

$$A(x_1,x_2)(t) = (x_1,x_2),\ x_i \in C^1([a,b],\mathbb{R}),\ i = 1,2.$$

For $t \in [a,b]$ we have

$$|A_1(x_1,x_2)(t) - A_1(y_1,y_2)(t)|$$

$$= \left| \int_a^b \mathbf{G}(t,s)[f_1(s,x_1(s),x_2(s),x_1'(s),x_2'(s)) - f_1(s,y_1(s),y_2(s),y_1'(s),y_2'(s))]ds \right|$$

$$\leq \max_{x\in[a,b]} \left| \int_a^b G_1(t,s)ds \right| L_1(\|(x_1,x_2)-(y_1,y_2)\|_\infty + \left\|(x_1',x_2')-(y_1',y_2')\right\|_\infty)$$

$$\leq \frac{3}{2}(b-a)^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} L_1 \|(x_1,x_2)-(y_1,y_2)\|_{C^1} .$$

At the same time we have

$$\|A_2(x_1,x_2)(t) - A_2(y_1,y_2)(t)\|_\infty \leq \frac{3}{2}(b-a)^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} L_2 \|(x_1,x_2)-(y_1,y_2)\|_{C^1} .$$

For $t \in [a,b]$ we have

$$\left| \frac{d}{dt}A_1(x_1,x_2)(t) - \frac{d}{dt}A_1(y_1,y_2)(t) \right|$$

$$\leq \left| \int_a^b \frac{\partial}{\partial t}\mathbf{G}(t,s)[f_1(s,x_1(s),x_2(s),x_1'(s),x_2'(s)) - f_1(s,y_1(s),y_2(s),y_1'(s),y_2'(s))]ds \right|$$

$$\leq \left| \int_a^b \frac{\partial}{\partial t}\mathbf{G}(t,s)ds \right| L_1 \cdot (\|(x_1,x_2)-(y_1,y_2)\|_\infty + \left\|(x_1',x_2')-(y_1',y_2')\right\|_\infty)$$

$$\leq \left| \int_a^b \frac{\partial}{\partial t}G_1(t,s)ds \right| L_1 \|(x_1,x_2)-(y_1,y_2)\|_{C^1}$$

$$\leq \frac{3}{2}(b-a)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} L_1 \left\| (x_1,x_2)-(y_1,y_2) \right\|_{C^1}.$$

Analogous we have

$$\left\| \frac{d}{dt}A_2(x_1,x_2)(t) - \frac{d}{dt}A_2(y_1,y_2)(t) \right\|_\infty$$

$$\leq \frac{3}{2}(b-a)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} L_2 \left\| (x_1,x_2)-(y_1,y_2) \right\|_{C^1}.$$

Then

$$\begin{pmatrix} \|A_1(x_1,x_2)(t)-A_1(y_1,y_2(t)\|_{C^1} \\ \|A_2(x_1,x_2)(t)-A_2(y_1,y_2(t)\|_{C^1} \end{pmatrix}$$

$$\leq \left(\frac{3}{2}(b-a)^2 \begin{pmatrix} L_1 & L_1 \\ L_2 & L_2 \end{pmatrix} + \frac{3}{2}(b-a)\begin{pmatrix} L_1 & L_1 \\ 0 & L_2 \end{pmatrix} \right) \begin{pmatrix} \|(x_1,x_2)-(y_1,y_2)\|_{C^1} \\ \|(x_1,x_2)-(y_1,y_2)\|_{C^1} \end{pmatrix}$$

with $Q := \left(\frac{3}{2}(b-a)^2 \begin{pmatrix} L_1 & L_1 \\ L_2 & L_2 \end{pmatrix} + \frac{3}{2}(b-a)\begin{pmatrix} L_1 & L_1 \\ 0 & L_2 \end{pmatrix} \right)$.

So, we have the following existence and uniqueness theorem

Theorem 4.1. *We suppose that:*

(i) *the conditions* (C_1) *and* (C_2) *are satisfied;*

(ii) $Q^n \to 0$ *as* $n \to \infty$.

Then

(a) *the problem (1.1)-(1.2) has a unique solution*

$$\overset{*}{x} = (\overset{*}{x}_1, \overset{*}{x}_2) \in C^1([a,b],\mathbb{R}^2);$$

(b) *for all* $\left(x_1^0, x_2^0\right) \in C^1([a,b],\mathbb{R}^2)$, *the sequence* $(x_1^n, x_2^n)_{n\in\mathbb{N}}$ *defined by*

$$\left(x_1^{n+1}, x_2^{n+1}\right) = A(x_1^n, x_2^n),$$

converges uniformly to $(\overset{*}{x}_1, \overset{*}{x}_2)$, *for all* $t \in [a,b]$, *and*

$$\left\| (x_1^n, x_2^n) - (\overset{*}{x}_1, \overset{*}{x}_2) \right\|_{C^1} \leq (I-Q)^{-1} Q^n \left\| \left(x_1^0, x_2^0\right) - (x_1^1, x_2^1) \right\|_{C^1}.$$

5. Data dependence: continuity

Consider the problem (1.1)-(1.2) with the dates $f = (f_1, f_2), g = (g_1, g_2)$ and suppose that the conditions from Theorem 4.1 are satisfied.

Let $f, g \in C^2([a,b]\times\mathbb{R}^4, \mathbb{R}^2)$. For simplicity we denote

$$Q_f := \left(\frac{3}{2}(b-a)^2 \begin{pmatrix} L_1^f & L_1^f \\ L_2^f & L_2^f \end{pmatrix} + \frac{3}{2}(b-a)\begin{pmatrix} L_1^f & L_1^f \\ 0 & L_2^f \end{pmatrix} \right), \tag{5.1}$$

$$Q_g := \left(\frac{3}{2}(b-a)^2 \begin{pmatrix} L_1^g & L_1^g \\ L_2^g & L_2^g \end{pmatrix} + \frac{3}{2}(b-a)\begin{pmatrix} L_1^g & L_1^g \\ 0 & L_2^g \end{pmatrix} \right) \tag{5.2}$$

and

$$Q := max\{Q_f, Q_g\}.$$

where *max* is taken w.r.t. the ordered relation of $M_{22}(\mathbb{R})$.

Theorem 5.1. *Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ satisfy the condition (C_1). Furthermore, we suppose that there exist $\eta := (\eta_1, \eta_2) \in \mathbb{R}^2_+$, such that*

$$|f_i(t, u_1, u_2, u_3, u_4) - g_i(t, u_1, u_2, u_3, u_4)| \le \eta_i,$$
$$i = 1, 2, \forall t \in [a, b],\ u_j \in \mathbb{R},\ j = \overline{1, 4}.$$

Then

$$\| \overset{*}{x}(t; f) - \overset{*}{x}(t; g) \|_{C^1} \le (I - Q_f)^{-1}\eta_1,$$

where $\overset{*}{x}(t; f)$ and $\overset{*}{x}(t; g)$ are the solution of the problem (1.1)-(1.2) with respect to f and g, with $f_i = (f_1, f_2)$, $g_i = (g_1, g_2)$.

Proof. Consider the operators A_f and A_g. From Theorem 4.1 it follows that

$$\| A_f(x_1, x_2) - A_g(y_1, y_2) \|_{C^1} \le Q \| (x_1, x_2) - (y_1, y_2) \|_{C^1},\ (x_1, x_2), (y_1, y_2) \in X.$$

We have now

$$\| \overset{*}{x}_i(t; f_i) - \overset{*}{x}_i(t; g_i) \|_{C^1} = \| A_{f_i}(\overset{*}{x}_i(t; f_i)) - A_{g_i}(\overset{*}{x}_i(t; g_i)) \|_{C^1} \le$$
$$\le \| A_{f_i}(\overset{*}{x}_i(t; f_i)) - A_{f_i}(\overset{*}{x}_i(t; g_i)) \|_{C^1} + \| A_{f_i}(\overset{*}{x}_i(t; g_i)) - A_{g_i}(\overset{*}{x}_i(t; g_i)) \|_{C^1} \le$$
$$\le Q_f \| \overset{*}{x}_i(t; f_i) - \overset{*}{x}_i(t; g_i) \|_{C^1} + \eta_i, i = 1, 2.$$

Because $Q^n \to 0$ as $n \to \infty$ imply that

$$(I - Q)^{-1} \in M_{22}(\mathbb{R}^2),$$

so we have

$$\| \overset{*}{x}_i(t; f_i) - \overset{*}{x}_i(t; g_i) \|_{C^1} \le (I - Q_f)^{-1}\eta_i.$$

□

6. Data dependence: differentiability

In this section we present the dependence by parameter λ of the solution of the problem (1.1)-(1.2).

We shall use the following theorem

Theorem 6.1. (Fibre contraction principle). *Let (X, d) and (Y, ρ) be two metric spaces and $A : X \times Y \to X \times Y$, $A = (B, C)$, ($B : X \to X$, $C : X \times Y \to Y$) a triangular operator. We suppose that*

(i) *(Y, ρ) is a complete metric space;*
(ii) *the operator B is Picard operator;*
(iii) *there exists $l \in [0, 1)$ such that $C(x, \cdot) : Y \to Y$ is a l-contraction, for all $x \in X$;*
(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is Picard operator.

Consider the following differential system with parameter:

$$-x''(t) = f(t, x_1(t), x_2(t), x_1'(t), x_2'(t); \lambda), \ t \in [a, b], \tag{6.1}$$

$$\begin{cases} x_1(a) = x_2(b) = 0 \\ x_1(c) = x_2(c) \\ x_1'(c) = x_2'(c). \end{cases} \tag{6.2}$$

where $x = (x_1, x_2)$ and $f = (f_1, f_2)$.

We suppose that

(C$_1$) $a, b \in \mathbb{R}, c \in (a, b)$ given, $\lambda \in J \subset \mathbb{R}$ a compact interval;

(C$_2$) $f = (f_1, f_2) \in C^2([a, b] \times \mathbb{R}^4 \times J, \mathbb{R}^2)$;

(C$_3$) there exists $L_i > 0$ such that

$$\left(\left| \frac{\partial f_i(t, u_1, u_2, u_3, u_4; \lambda)}{\partial u_j} \right| \right)_{\substack{i=1,2 \\ j=\overline{1,4}}} \leq L_i$$

for all $t \in [a, b], \ u_j \in \mathbb{R}, \ i = 1, 2, \ j = \overline{1, 4}$;

(C$_4$) for $Q = \max\{Q_f, Q_g\}$ with

$$Q_f := \left(\frac{3}{2}(b-a)^2 \begin{pmatrix} L_1^f & L_1^f \\ L_2^f & L_2^f \end{pmatrix} + \frac{3}{2}(b-a) \begin{pmatrix} L_1^f & L_1^f \\ 0 & L_2^f \end{pmatrix} \right)$$

and Q_g analogous, we have $Q^n \to 0$ as $n \to \infty$.

In the above conditions, from Theorem 4.1 we have that the problem (1.1)-(1.2) has a unique solution, $\overset{*}{x}(\cdot; \lambda) = (\overset{*}{x}_1(\cdot; \lambda), \overset{*}{x}_2(\cdot; \lambda))$, for any $\lambda \in \mathbb{R}$. In what follows we shall prove that $\overset{*}{x}(t; \cdot) \in C^2(J, \mathbb{R}^2)$, for all $t \in [a, b]$.

For this we consider the system

$$-x''(t) = f(t, x_1(t; \lambda), x_2(t; \lambda), x_1'(t; \lambda), x_2'(t; \lambda); \lambda), \tag{6.3}$$

for all $t \in [a, b], \ \lambda \in J, x = (x_1, x_2) \in C^1([a, b] \times J, \mathbb{R}^2), \ f = (f_1, f_2)$.

The system (6.3) is equivalent with

$$\begin{pmatrix} -x_1(t) \\ -x_2(t) \end{pmatrix} = \int_a^b \mathbf{G}(t, s) \begin{pmatrix} f_1(s, x_1(s; \lambda), x_2(s; \lambda), x_1'(s; \lambda), x_2'(s; \lambda); \lambda) \\ f_2(s, x_1(s; \lambda), x_2(s; \lambda), x_1'(s; \lambda), x_2'(s; \lambda); \lambda) \end{pmatrix} ds. \tag{6.4}$$

Let $X := (C^1([a, b] \times J, \mathbb{R}^2), \|\cdot\|_{C^1})$ with the Chebyshev norm

$$\|(x_1, x_2) - (y_1, y_2)\|_{C^1} := \|(x_1, x_2) - (y_1, y_2)\|_\infty + \left\|(x_1', x_2') - (y_1', y_2')\right\|_\infty.$$

Now we consider the operator $B : C^1([a, b] \times J, \mathbb{R}^2) \to C^1([a, b] \times J, \mathbb{R}^2)$ where

$$B(x_1, x_2)(t; \lambda) = \begin{pmatrix} B_1(x_1, x_2)(t; \lambda) \\ B_2(x_1, x_2)(t; \lambda) \end{pmatrix} := \text{second part of (6.4)}.$$

It is clear, from the proof of the Theorem 4.1, that in the conditions (C_1)-(C_4), the operator B is Picard operator, since

$$\|B(y_1, y_2) - B(z_1, z_2)\|_{C^1} \leq Q \, \|(y_1, y_2) - (z_1, z_2)\|_{C^1},$$

$$\forall (y_1, y_2), (z_1, z_2) \in C^1([a, b] \times J, \mathbb{R}^2).$$

Let $\overset{*}{x} = (\overset{*}{x}_1, \overset{*}{x}_2)$ be the unique fixed point of B.

We suppose that there exists $\dfrac{\partial \overset{*}{x}_i}{\partial \lambda}, i = 1, 2$. From condition ($C_3$) we have

$$\frac{\partial \overset{*}{x}_i(t;\lambda)}{\partial \lambda} = \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_1} \cdot \frac{\partial \overset{*}{x}_1(s;\lambda)}{\partial \lambda} ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_2} \cdot \frac{\partial \overset{*}{x}_2(s;\lambda)}{\partial \lambda} ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_3} \cdot \frac{\partial \overset{*}{x}{}_1'(s;\lambda)}{\partial \lambda} ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_4} \cdot \frac{\partial \overset{*}{x}{}_2'(s;\lambda)}{\partial \lambda} ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial \lambda} ds$$

for $t \in [a, b], \ \lambda \in J, \ i = 1, 2$.

This relation suggest us to consider the following operator

$$C : X \times X \to X, \ (x_1, x_2, y_1, y_2) \to C(x_1, x_2, y_1, y_2),$$

defined by

$$C(x_1, x_2, y_1, y_2)(t; \lambda) :=$$

$$= \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_1} \cdot y_1(s;\lambda) ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_2} \cdot y_2(s;\lambda) ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_3} \cdot y_1'(s;\lambda) ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial u_4} \cdot y_2'(s;\lambda) ds$$

$$+ \int_a^b \mathbf{G}(t,s) \frac{\partial f_i(s, x_1(s;\lambda), x_2(s;\lambda), x_1'(s;\lambda), x_2'(s;\lambda); \lambda)}{\partial \lambda} ds$$

where $y_i(t;\lambda) := \frac{\partial \overset{*}{x}_i(t;\lambda)}{\partial \lambda}$ and $y_i'(t;\lambda) := \frac{\partial \overset{*}{x}{}_i'(t;\lambda)}{\partial \lambda}$ for $t \in [a,b], \lambda \in J, i = 1,2$. In this way we have the triangular operator

$$A : X \times X \to X \times X,$$
$$A(x_1,x_2,y_1,y_2) = (B(x_1,x_2)(t;\lambda), C(x_1,x_2,y_1,y_2)(t;\lambda))$$

where B is a Picard operator and $C(x_1,x_2,\cdot,\cdot) : X \to X$ is Q-contraction. Indeed we have

$$\left\| C(\overset{*}{x}_1,\overset{*}{x}_2,u_1,u_2)(t;\lambda) - C(\overset{*}{x}_1,\overset{*}{x}_2,v_1,v_2)(t;\lambda) \right\|_{C^1} \le Q \left\| (u_1,u_2) - (v_1,v_2) \right\|_{C^1},$$
$$\forall u = (u_1,u_2), v = (v_1,v_2) \in X,\ t \in [a,b], \lambda \in J.$$

Since $Q^n \to 0$ as $n \to \infty$, from the theorem of fibre contraction (see [15]) follows that the operator A is Picard operator and has a unique fixed point $(\overset{*}{x}_1,\overset{*}{x}_2,\overset{*}{y}_1,\overset{*}{y}_2) \in X \times X$. So the sequences

$$(x_1^{n+1},x_2^{n+1},y_1^{n+1},y_2^{n+1}) = (B(x_1^n,x_2^n), C(x_1^n,x_2^n,y_1^n,y_2^n)), n \in \mathbb{N}$$

converges uniformly (with respect to $t \in [a,b],\ \lambda \in J$) to $(\overset{*}{x}_1,\overset{*}{x}_2,\overset{*}{y}_1,\overset{*}{y}_2) \in F_A$, for any $(x_1^0,x_2^0),\ (y_1^0,y_2^0) \in X$. If we take

$$x_i^0 = 0, y_i^0 = \frac{\partial x_i^0}{\partial \lambda} = 0, \text{ then } y_i^1 = \frac{\partial x_i^1}{\partial \lambda}, i = 1,2.$$

By induction we prove that

$$y_i^n = \frac{\partial x_i^n}{\partial \lambda},\ \forall n \in \mathbb{N}, i = \overline{1,2}.$$

Thus

$$x_i^n \overset{unif}{\to} \overset{*}{x}_i, \text{ as } n \to \infty, i = 1,2$$
$$\frac{\partial x_i^n}{\partial \lambda} \overset{unif}{\to} \overset{*}{y}_i, \text{ as } n \to \infty, i = \overline{1,2}.$$

These imply that there exists $\dfrac{\partial \overset{*}{x}_i}{\partial \lambda}$ and

$$\frac{\partial x_i(t;\lambda)}{\partial \lambda} = \overset{*}{y}_i(t;\lambda),\ i = \overline{1,2}.$$

So, we have

Theorem 6.2. *Suppose that conditions* (C$_1$)-(C$_4$) *hold. Then,*

(i) *the problem (1.1)-(1.2) has a unique solution*

$$\overset{*}{x} = (\overset{*}{x}_1,\overset{*}{x}_2) \in C^2([a,b] \times J, \mathbb{R}^2);$$

(ii) $\overset{*}{x}(t;\cdot) \in C^1(J,\mathbb{R}^2), \forall t \in [a,b]$.

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