INTEGRO-DIFFERENTIAL EQUATION
WITH TWO TIME LAGS

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Abstract. We consider an integro-differential equation with two time lags and we prove the existence, uniqueness and convergence of the sequence of the successive approximation by using contraction principle and step method with a weaker Lipschitz condition. Also, we propose a new algorithm of successive approximation sequence generated by the step method and we give an example to illustrate the applications of the abstract results.

Key Words and Phrases: Integro-differential equation, two time lags, step method, Picard operators, fibre contraction principle.

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1. Introduction

This paper is concerned with the following integro-differential equation

\[ x'(t) = g(t, x(t), x(t - \tau)) + \int_{t-h}^t K(s, x(s))ds, \quad t \in S, \quad (1.1) \]

where \( g \) and \( K \) are continuous functions on a Banach space and satisfy certain conditions to be specified later.

Regarding the earlier works on existence, uniqueness and convergence of the sequence of the successive approximation to integro-differential equations with delays and functional-differential equations with delays under different conditions, we refer to Guo et al [2], Kolmanowskii and Mishkis [5], Precup [7], Precup and Kirr [8], I.A. Rus [12] and the references therein. The related results for the existence and uniqueness, convergence of the sequence of the successive approximation, lower and upper solutions to the differential equations with delays can be found in Dobriţoiu et al [1], Ilea [3] and Otrocol [6].
The authors Rus, Şerban, Trif [13] have considered the following integral equation
\[ x(t) = g(t, x(t-\tau)) + \int_{t-\tau}^{t} K(t, s, x(s)) ds, \quad t \in [a, b], \quad \tau > 0 \]
\[ x(t) = \phi(t), \quad t \in [a - \tau, a], \]
in a Banach space and proved that the sequence of the successive approximation generated by the step method converges to the solution of this integral equation using the results of Rus [9].

Sakata and Hara [14] have considered the linear differential equation with two kinds of time lags
\[ x'(t) = ax(t-\tau) + b \int_{t-h}^{t} x(s) ds \]
where \( \tau > 0, \ h > 0 \) and \( a, b \) are both real and they have studied the dependence on delays towards stability regions.

In the present work we use the ideas of Rus, Şerban, Trif [13] to establish the convergence of the sequence of successive approximation to equation (1.1). Regarding the two delays we have the following cases: \( h > 0, \ \tau > h \) and \( h > 0, \ |\tau| > h \). Here, the authors study the first case, while the second case is studied in [4].

The aim of this paper is to obtain existence and uniqueness theorems using contraction principle and step method. Such kind of results have been proved in [13]. The approach proposed in the present paper is different to the ones in [13] and [1] and it is based on the different time lags. Also, we present here some lower and upper solution result, and a numerical example concerning equation (1.1).

We note that Sakata and Hara study in [14] the stability regions for similar integro-differential equation with two time lags.

2. Preliminaries

Let \( \tau > 0, \ h > 0, \ h < \tau \) and
\[ x'(t) = g(t, x(t), x(t-\tau)) + \int_{t-h}^{t} K(s, x(s)) ds, \quad t \in [0, T], \quad (2.1) \]
\[ x(t) = \varphi(t), \quad t \in [-\tau, 0]. \quad (2.2) \]
Relative to (2.1)–(2.2) we consider the following conditions:

\( (C_1) \) \((\mathbb{B}, \|\cdot\|)\) is a Banach space, \( g \in C([0, T] \times \mathbb{B}^2, \mathbb{B}), \ K \in C([0, T] \times \mathbb{B}, \mathbb{B}), \ \varphi \in C([-\tau, 0], \mathbb{B}); \)
\( (C_2) \) there exists \( L_g > 0 \) such that
\[ \|g(t, u_1, v_1) - g(t, u_2, v_2)\| \leq L_g (\|u_1 - u_2\| + \|v_1 - v_2\|), \quad u_i, v_i \in \mathbb{B}, t \in [0, T]; \]
\( (C'_2) \) there exists \( L'_g > 0 \) such that
\[ \|g(t, u_1, v) - g(t, u_2, v)\| \leq L'_g \|u_1 - u_2\|, \quad u_1, u_2, v \in \mathbb{B}, t \in [0, T]; \]
Let \( A \). Obviously on \( x \) (the sequence that:)

\[ \text{Lemma 3.3.} \]

\[ \text{Definition 3.2.} \]

\[ \text{Definition 3.1.} \]

\[ \text{Definition 3.2.} \]

\[ \text{Lemma 3.3.} \]
Theorem 3.4. (Fibre contraction principle, Rus [10]) Let \((X, d)\) be a metric space and \((Y, \rho)\) be a complete metric space. Let \(B : X \to X\) and \(C : X \times Y \to Y\) be two operators. We suppose that:

(i) \(B\) is a WPO;

(ii) \(C(x, \cdot) : Y \to Y\) is \(\alpha\)-contraction, for all \(x \in X\);

(iii) if \((x^*, y^*) \in F_A\), where \(A : X \times Y \to X \times Y\), \(A(x, y) = (B(x), C(x, y))\), then \(C(\cdot, y^*)\) is continuous in \(x^*\).

Then \(A\) is a WPO. Moreover, if \(B\) is PO then \(A\) is PO.

By induction, from the above result we have:

Theorem 3.5. (Rus [11]) Let \((X_i, d_i), i = 0, m, m \geq 1, i\) be some operator. We suppose that:

(i) \((X_i, d_i), i = 1, m,\) are complete metric spaces;

(ii) the operator \(A_0\) is WPO;

(iii) there exists \(\alpha_i \in (0, 1)\) such that:

\[A_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i, i = 0, m\]

are \(\alpha_i\)-contractions;

(iv) the operator \(A_i, i = 1, m\), are continuous.

Then the operator \(A : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m,\)

\[A(x_0, \ldots, x_m) = (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_0, \ldots, x_m))\]

is WPO. If \(A_0\) is PO, then \(A\) is PO.

4. Existence and uniqueness

In this section we give an existence theorem for the solution of the problem (2.1)–(2.2).

Theorem 4.1. In the condition \((C_1), (C_2), (C_3)\) and \((C_4)\), the problem (2.1)–(2.2) has in \(C([-\tau, T], \mathbb{B})\) a unique solution \(x^*\) and the sequence of successive approximations, \((x^n)_{n \in \mathbb{N}}\)

\[x^{n+1}(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\
\varphi(0) + \int_0^t g(s, x^n(s), x^n(s - \tau))dsd\xi + \int_0^t \int_{\xi - \tau}^{\xi} K(s, x^n(s))d\lambda dsd\xi, & t \in [0, T] \end{cases}\]

converges uniformly to \(x^*\), for every \(x^0 \in C([-\tau, T], \mathbb{B})\), with \(x^0|_{[-\tau, 0]} = \varphi\).

Proof. Let \(X_\varphi \subset X\), \(X_\varphi = \{ x \in X | x(t) = \varphi(t), t \in [-\tau, 0] \}\) and \(A : X_\varphi \to X_\varphi\) defined by

\[A(x)(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\
\varphi(0) + \int_0^t g(s, x(s), x(s - \tau))dsd\xi + \int_0^t \int_{\xi - \tau}^{\xi} K(s, x(s))d\lambda dsd\xi, & t \in [0, T] \end{cases}\]

Note that \(X_\varphi\) is a closed subset of \(X\), so \((X_\varphi, d_{\| \cdot \|_Y})\) is a complete metric space.
In a standard way we have
\[ \|A(x) - A(y)\|_{\lambda} \leq \frac{1}{\lambda} (L_y + L_R h) \|x - y\|_{\lambda}, \]
for all \(x, y \in X_{\lambda}\),

which proves that \(A\) is Lipschitz with \(L_A = \frac{1}{\lambda} (L_y + L_R h)\). We can choose \(\lambda\) sufficiently large such that \(L_A = \frac{1}{\lambda} (L_y + L_R h) < 1\), so \(A\) is contraction. Applying the contraction principle we get the conclusion. \(\square\)

**Remark 4.2.** From the proof of Theorem 4.1, it follows that the operator \(A\) is PO.

5. **Step method**

Using step method and contraction principle on each step for the problem (2.1)–(2.2), in this section we obtain a better result by replacing the condition (C2) from Theorem 4.1 with (C2').

Let \(m \in \mathbb{N}^+\) such that \((m - 1)h \leq T, m h > T\). To simplify our presentation we suppose that \(h \leq \tau \leq 2h\). In the conditions (C1), (C2'), (C3) and (C4) the step method for (2.1)–(2.2) consist in the following:

\[ (p_0) \quad x_0(t) = \varphi(t), \quad t \in [-\tau, 0] \]

\[ (p_1) \quad x_1(t) = \varphi(0) + \int_0^t g(\xi, x_1(\xi), \varphi(\xi - \tau))d\xi + \int_0^t \int_{-h}^{t-h} K(s, \varphi(s))d\xi + \int_0^t \int_{-h}^{t-h} K(s, x_1(s))d\xi, \quad t \in [0, h] \]

\[ (p_2) \quad x_2(t) = x_1(h) + \int_0^h g(\xi, x_2(\xi), \varphi(\xi - \tau))d\xi + \int_0^t g(\xi, x_2(\xi), x_1(\xi - \tau))d\xi + \int_0^t \int_{-h}^{t-h} K(s, x_1(s))d\xi + \int_0^t \int_{-h}^{t-h} K(s, x_2(s))d\xi, \quad t \in [h, 2h] \]

\[ (p_3) \quad x_i(t) = x_{i-1}((i - 1)h) + \int_{(i-1)h}^{(i-1)h+\tau} g(\xi, x_i(\xi), x_{i-1}(\xi - \tau))d\xi + \int_0^t g(\xi, x_1(\xi), x_{i-1}(\xi - \tau))d\xi + \int_0^t \int_{(i-1)h}^{(i-1)h+\tau} K(s, x_{i-1}(s))d\xi d\xi + \int_0^t \int_{(i-1)h}^{(i-1)h+\tau} K(s, x_i(s))d\xi d\xi, \quad t \in [(i - 1) h, ih] \]

\[ (p_{m-1}) \quad x_{m-1}(t) = x_{m-2}((m - 2)h) + \int_{(m-2)h}^{(m-2)h+\tau} g(\xi, x_{m-1}(\xi), x_{m-2}(\xi - \tau))d\xi + \int_0^t g(\xi, x_1(\xi), x_{m-2}(\xi - \tau))d\xi + \int_0^t \int_{(m-2)h}^{(m-2)h+\tau} K(s, x_{m-2}(s))d\xi d\xi + \int_0^t \int_{(m-2)h}^{(m-2)h+\tau} K(s, x_{m-1}(s))d\xi d\xi, \quad t \in [(m - 2) h, (m - 1)h] \]

\[ (p_m) \quad x_m(t) = x_{m-1}((m - 1)h) + \int_{(m-1)h}^{(m-1)h+\tau} g(\xi, x_m(\xi), x_{m-2}(\xi - \tau))d\xi + \int_0^t g(\xi, x_1(\xi), x_{m-2}(\xi - \tau))d\xi + \int_0^t \int_{(m-1)h}^{(m-1)h+\tau} K(s, x_{m-2}(s))d\xi d\xi + \int_0^t \int_{(m-1)h}^{(m-1)h+\tau} K(s, x_{m-1}(s))d\xi d\xi, \quad t \in [(m - 1)h, T] \]

where \(x^*_i\) is the unique solution of \((p_i)\), \(i = 1, m\).

So, we have the following result:

**Theorem 5.1.** In the conditions (C1), (C2'), (C3) and (C4) we have:
a) the problem (2.1)–(2.2) has in $C([-\tau, T], \mathbb{B})$ a unique solution $x^*$,

$$x^*(t) = \begin{cases} 
\varphi(t), & t \in [-\tau, 0] \\
\frac{\partial}{\partial t} x^*_i(t), & t \in [0, h] \\
\vdots \\
x^*_m(t), & t \in [(m - 1)h, T] 
\end{cases}$$

b) for each $x^*_i \in C([(i - 1)h, ih], \mathbb{B})$, $i = 1, m - 1$,

$x^*_m \in C([(m - 1)h, T], \mathbb{B})$, the sequence defined by:

$$x^*_{i+1}(t) = \varphi(0) + \int_0^t g(s, x^*_i(s)) ds + \int_0^t \int_{\xi = -h}^h K(s, x^*_i(s)) d\xi ds + \int_0^t \int_{\xi = -h}^h K(s, x^*_i(s)) d\xi ds,$n

$$x^*_{i+1}(t) = x^*_i(h) + \int_h^t g(s, x^*_i(s), \varphi(s - \tau)) ds + \int_h^t \int_{\xi = -h}^h K(s, x^*_i(s)) d\xi ds,$n

$$\cdots$$

$$x^*_{i+1}(t) = x^*_i((m - 1)h) + \int_{(m-1)h}^{(m+1)h} g(s, x^*_i(s), x^*_m(s)) ds + \int_{(m-1)h}^{(m+1)h} \int_{\xi = -h}^h K(s, x^*_i(s)) d\xi ds + \int_{(m-1)h}^{(m+1)h} \int_{\xi = -h}^h K(s, x^*_i(s)) d\xi ds,$n

$$\text{converge and } \lim_{n \to \infty} x^*_{i+1} = x^*_i, \quad i = 1, m.$$

Proof. In order to prove this theorem we apply the contraction principle for each step:

$$\|x\|_{\lambda_1} = \max_{t \in [0, h]} \{\|x(t)\| e^{-\lambda_1 t}\}$$

and the operator $A_1 : X_1 \to X_1$ defined by

$$A_1(x)(t) = \varphi(0) + \int_0^t g(s, x(s), \varphi(s - \tau)) ds + \int_0^t \int_{\xi = -h}^h K(s, x(s)) d\xi ds.$$n

For $x, y \in X_1$, we have

$$\|A_1(x) - A_1(y)\|_{\lambda_1} \leq \frac{1}{\lambda_1} (L_g + L_K h) \|x - y\|_{\lambda_1}.$$n

We can choose a $\lambda_1 > 0$ such that $\frac{1}{\lambda_1} (L_g + L_K h) < 1$, so $A_1$ is a contraction, therefore $F_{A_1} = \{x^*_1\}$.

For the next steps let us consider the following Banach spaces: for $i = \frac{m}{2}, m - 1$ given by

$$X_i := (C([(i - 1)h, ih]; \mathbb{B}), \|\cdot\|_{\lambda_i}), \text{ with } \|x\|_{\lambda_i} := \max_{t \in [(i - 1)h, ih]} \{\|x(t)\| e^{-\lambda_i (t - (i - 1)h)}\}$$

and

$$X_m := (C([(m - 1)h, T]; \mathbb{B}), \|\cdot\|_{\lambda_m}), \text{ with } \|x\|_{\lambda_m} := \max_{t \in [(m - 1)h, T]} \{\|x(t)\| e^{-\lambda_m (t - (m - 1)h)}\}$$

and
and the operators $A_i : X_i \to X_i$, $i = \overline{1,m}$ defined by

$$A_i(x)(t) := x^i_{-1}((i-1)h) + \int_{(i-1)h}^{(i-2)h+\tau} g(\xi, x^i(\xi), x^i_{-2}(\xi-\tau))d\xi +$$

$$+ \int_{(i-2)h+\tau}^{t} g(\xi, x^i(\xi), x^i_{-1}(\xi-\tau))d\xi + \int_{(i-1)h}^{t} \int_{\xi-h}^{\xi} K(s, x^i_{-1}(s))d\xi ds +$$

$$+ \int_{(i-1)h}^{t} \int_{(i-1)h}^{(i-1)h+1} K(s, x(s))d\xi ds,$$

For $x,y \in X_i$ we have $\|A_i(x) - A_i(y)\|_{\lambda_i} \leq \frac{1}{\lambda_i} (L^\prime_g + L_K) \|x - y\|_{\lambda_i}$, so $A_i$ is a contraction for a suitable choice of $\lambda_i$ such that $\frac{1}{\lambda_i} (L^\prime_g + L_K) < 1$. Therefore, we get that $F_{\lambda_i} = \{x^i_n\}, i = \overline{1,m}$.

From condition (C4) we get $\varphi(0) = x^i_n(0)$ and from definition of $A_i$, $i = \overline{1,m}$, we have

$$x^i_{-1}((i-1)h) = x^i((i-1)h), \quad i = \overline{1,m},$$

therefore

$$x^i(t) = \begin{cases} 
\varphi(t), & t \in [-\tau, 0] \\
x^i_1(t), & t \in [0, h) \\
\vdots \\
x^i_m(t), & t \in [(m-1)h, T]\end{cases}$$

is the unique solution in $C([-h, T], \mathbb{B})$. \hfill \Box

Now the question is: Can we put an approximation of $x^i_n$, $i = \overline{1,m}$ instead of $x^i$, $i = \overline{1,m}$?

The answer of this question is given by the following theorem:

**Theorem 5.2.** In the condition of Theorem 5.1, for each $x^i_n \in C([((i-1)h, ih] \cup \mathbb{B}), i = \overline{1,m-1}$, $x^i_0 \in C([((m-1)h, T], \mathbb{B})$, the sequences defined by:

$$x^i_{n+1}(t) = \varphi(0) + \int_{0}^{h} g(\xi, x^i_n(\xi), \varphi(\xi-\tau))d\xi + \int_{0}^{h} \int_{\xi-h}^{h} K(s, x^i_n(s))d\xi ds +$$

$$+ \int_{0}^{h} \int_{0}^{h} K(s, x^i_0(s))d\xi ds, \quad t \in [0, h],$$

$$x^i_{2n+1}(t) = x^i_n(h) + \int_{h}^{2h} g(\xi, x^i_2(\xi), \varphi(\xi-\tau))d\xi + \int_{h}^{2h} g(\xi, x^i_n(\xi), x^i_n(n-1)(\xi-\tau))d\xi +$$

$$+ \int_{h}^{2h} \int_{\xi-h}^{2h} K(s, x^i_0(s))d\xi ds + \int_{h}^{2h} \int_{h}^{2h} K(s, x^i_2(s))d\xi ds, \quad t \in [h, 2h],$$

$$\cdots$$

$$x^i_{mn+1}(t) = x^i_{(m-1)n}(h) + \int_{(m-1)n}^{(m-2)n+\tau} g(\xi, x^i_n(\xi), x^i_{m-2}(\xi-\tau))d\xi +$$

$$+ \int_{(m-2)n+\tau}^{(m-1)n+\tau} g(\xi, x^i_n(\xi), x^i_{m-1}(\xi-\tau))d\xi +$$

$$+ \int_{(m-1)n+\tau}^{(m-1)n+\tau} K(s, x^i_{m-1}(s))d\xi ds +$$

$$+ \int_{(m-1)n+\tau}^{(m-1)n+\tau} \int_{(m-1)n+\tau}^{(m-1)n+\tau} K(s, x^i_n(s))d\xi ds, \quad t \in [(m-1)n, T]\]

converge and $\lim_{n \to \infty} x^i_n = x^i$, $i = \overline{1,m}$. 

Proof. We consider the following Banach spaces (with \( \lambda > 0 \)):

\[
X_0 = \left\{ C([0, \mathbb{T}], \mathbb{B}), \| \cdot \|_{\lambda_0}, \| \cdot \|_{\lambda_0} \right\} = \max_{\tau \in [-\tau, 0]} \left\{ \| x(t) \| e^{-\lambda_0(t+\tau)} \right\},
\]

\[
X_i = \left( C([0, 1), \mathbb{B}), \| \cdot \|_{\lambda_i}, \| \cdot \|_{\lambda_i} \right) = \max_{\tau \in [0, 1)} \left\{ \| x(t) \| e^{-\lambda_i(t+\tau)} \right\}, \quad i = 1, m-1,
\]

\[
X_m = \left( C([0, 1), \mathbb{T}], \mathbb{B}), \| \cdot \|_{\lambda_m}, \| \cdot \|_{\lambda_m} \right) = \max_{\tau \in [0, 1)} \left\{ \| x(t) \| e^{-\lambda_m(t+\tau)} \right\},
\]

and the operators:

\[
A_0 : X_0 \rightarrow X_0, \quad A_0(x_0)(t) = \varphi(t), \quad t \in [-\tau, 0],
\]

\[
A_1 : X_0 \times X_1 \rightarrow X_1,
\]

\[
A_1(x_0, x_1)(t) = \varphi(0) + \int_0^t g(x_1(\xi), x_0(\xi - \tau))d\xi + \int_0^t \int_{\xi-h}^\xi K(s, x_0(s))dsd\xi + \int_0^t \int_{\xi-h}^\xi K(s, x_1(s))dsd\xi, \quad t \in [0, h],
\]

\[
A_i : X_{i-2} \times X_{i-1} \times X_i \rightarrow X_i, \quad i = 2, m-1
\]

\[
A_i(x_{i-2}, x_{i-1}, x_i)(t) = x_{i-1}(0) + \int_{(i-1)h}^{(i-2)h+\tau} g(x_i(\xi), x_{i-2}(\xi - \tau))d\xi + \int_{(i-1)h}^{(i-2)h+\tau} g(x_i(\xi), x_{i-1}(\xi - \tau))d\xi + \int_0^{(i-1)h} \int_{\xi-h}^\xi K(s, x_{i-1}(s))dsd\xi + \int_0^{(i-1)h} \int_{\xi-h}^\xi K(s, x_i(s))dsd\xi, \quad t \in [(i-1)h, ih]
\]

\[
A_m : X_{m-2} \times X_{m-1} \times X_m \rightarrow X_m
\]

\[
A_m(x_{m-2}, x_{m-1}, x_m)(t) = x_{m-1}(0) + \int_{(m-1)h}^{(m-2)h+\tau} g(x_m(\xi), x_{m-2}(\xi - \tau))d\xi + \int_{(m-1)h}^{(m-2)h+\tau} g(x_m(\xi), x_{m-1}(\xi - \tau))d\xi + \int_0^{(m-1)h} \int_{\xi-h}^\xi K(s, x_{m-1}(s))dsd\xi + \int_0^{(m-1)h} \int_{\xi-h}^\xi K(s, x_m(s))dsd\xi, \quad t \in [(m-1)h, T]
\]
and

\[ A : X_0 \times \ldots \times X_m \rightarrow X_0 \times \ldots \times X_m, \]

\[ A(x_0, \ldots, x_m) = (A_0(x_0), A_1(x_0, x_1), A_2(x_0, x_1, x_2), \ldots, A_m(x_{m-2}, x_{m-1}, x_m)). \]

It is easy to see that for fixed \((x_0, \ldots, x_m) \in X_0 \times \ldots \times X_m\) the sequence defined by (5.1) means

\[ (x_0^n, \ldots, x_m^n) = A^n(x_0, \ldots, x_m). \]

To prove the conclusion we need to prove that the operator \(A\) is PO and for this we apply Theorem 3.5.

Since \(A_0 : X_0 \rightarrow X_0\) is a constant operator then \(A_0\) is \(\alpha_0\)-contraction with \(\alpha_0 = 0\), so \(A_0\) is PO and \(F_{A_0} = \{x_0^0\}\), where \(x_0^0 = \varphi\). We have the inequalities:

\[ \|A_1(x_0, x_1) - A_1(x_0, y_1)\|_{\lambda_1} \leq \frac{1}{\lambda_1} (L'_g + L_K h) \|x_1 - y_1\|_{\lambda_1} \]

for all \(x_0 \in X_0, x_1, y_1 \in X_1\), and

\[ \|A_i(x_{i-2}, x_{i-1}, x_i) - A_i(x_{i-2}, x_{i-1}, y_i)\|_{\lambda_i} \leq \frac{1}{\lambda_i} (L'_g + L_K h) \|x_i - y_i\|_{\lambda_i} \]

for all \(x_{i-2} \in X_{i-2}, x_{i-1} \in X_{i-1}, x_i, y_i \in X_i, i = 2, m\). For \(\lambda_i\) sufficiently large, \((\lambda_i > L'_g + L_K h)\), we get that \(A_i(x_{i-2}, o) : X_1 \rightarrow X_1\) is \(\alpha_i\)-contraction and \(A_i(x_{i-2}, x_{i-1}, o) : X_i \rightarrow X_i\) are \(\alpha_i\)-contractions with \(\alpha_i = \frac{1}{\lambda} (L'_g + L_K h), i = 1, m\), so we are in the conditions of Theorem 3.5, therefore \(A\) is PO and \(F_A = \{(x_0^\ast, \ldots, x_m^\ast)\}\), thus

\[ (x_0^n, \ldots, x_m^n) = A^n(x_0, \ldots, x_m) \rightarrow (x_0^\ast, \ldots, x_m^\ast) \]

with \(x_0^0 = \varphi\) and \(x_1^\ast, \ldots, x_m^\ast\) for all \(n \in \mathbb{N}\), are defined by (5.1). From condition \((C_4)\) and from the definitions of \(A_i, i = 1, m\), we have

\[ x_i^\ast((i - 2)h) = x_i^\ast((i - 1)h), i = 1, m \]

therefore

\[ x^\ast(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ x_1^\ast(t), & t \in [0, h] \\ \ldots \\ x_m^\ast(t), & t \in [(m - 1)h, T] \end{cases} \]

is the unique solution in \(C([-\tau, T], \mathbb{B})\).

\[ \square \]

6. LOWER SOLUTIONS, UPPER SOLUTIONS AND THE SOLUTION

In this section we shall prove that the solution of the equation (1.1) is an upper bound of the lower solutions set and a lower bound of the upper solutions set.

Let the integro-differential equation (1.1) under the conditions \((C_1), (C_2), (C_3), (C_4)\) and we denote by \(x^\ast \in (C[0, T], \mathbb{B})\) the unique fixed point of the operator \(A\). In addition, we suppose that:

\((C_5)\) \(g(t, \cdot, \cdot) : \mathbb{B}^2 \rightarrow \mathbb{B}\) is increasing, for every \(t \in [0, T]\);

\((C_6)\) \(K(t, \cdot, \cdot) : \mathbb{B} \rightarrow \mathbb{B}\) is increasing, for every \(t \in [0, T]\).

We have
Theorem 6.1. We suppose that the conditions \((C_1)-(C_6)\) are satisfied. The following implications hold:

(a) If \(x(t) \leq g(t, x(t), x(t - \tau)) + \int_{t-k}^{t} K(s, x(s)) ds, x \in \mathbb{B}\) then \(x \leq x_A\).

(b) If \(x(t) \geq g(t, x(t), x(t - \tau)) + \int_{t-k}^{t} K(s, x(s)) ds, x \in \mathbb{B}\) then \(x \geq x_A^*\).

Proof. (a) We consider the operator \(A\) defined by

\[
A(x)(t) = \int_{0}^{t} g(\xi, x(\xi), x(\xi - \tau)) d\xi + \int_{0}^{\xi} \int_{\xi-h}^{\xi} K(s, x(s)) ds d\xi.
\]

Under the conditions \((C_1)-(C_4)\) the operator \(A\) is PO and by \((C_5)-(C_6)\) we have that the operator \(A\) is increasing. Since all the conditions of the Abstract Gronwall Lemma 3.3 are satisfied, we obtain \(x \leq x_A^*\) and the proof is complete.

For (b) the proof is similar. \(\square\)

7. Numerical example

In this section we give a numerical example to illustrate the convergence of the solution defined in theorem 5.1 to the solution. We consider the following integro-differential equation:

\[
x'(t) = (-6 + \sin(t)) x(t) + x(t - \frac{\pi}{2}) - \int_{t-\frac{\pi}{2}}^{t} \sin(s) x(s) ds + 5 e^{\cos(t)} + e^{\cos(t - \frac{\pi}{2})} - e^{\cos(t - \frac{\pi}{2})}, \quad t \in [\frac{\pi}{4}; \pi]
\]

\[
x(t) = e^{\cos(t)}, \quad t \in [0; \frac{\pi}{4}]
\]

which has the exact solution \(x(t) = e^{\cos(t)}\).

Numerical method. (For more details see N.L. Trefethen [15], D. Trif [16]) We divide the working interval by the points \(P_k = k, k = 0, 1, ..., M\), \(M = 4\) and represents the number of subintervals). On each subinterval \(I_k = [P_{k-1}; P_k]\), \(k = 1, ..., M\), we find the numerical solution by the form

\[
x_k(t) = c_{0,k} \frac{T_0}{2} + c_{1,k} T_1(\xi) + c_{2,k} T_2(\xi) + ... + c_{n-1,k} T_{n-1}(\xi),
\]

where \(T_i(\xi) = \cos(i \arccos(\xi))\) are Chebyshev polynomials of \(i\) degree, \(i = 0, ..., n-1\), \(n = 8\), and \(t = \alpha \xi + \beta\) with \(\alpha = (P_k - P_{k-1}) / 2\), respectively \(\beta = (P_k + P_{k-1}) / 2\).

Choosing a mesh \(\xi, j = 1, ..., n\), on interval \([-1; 1]\) consisting by the knots of Gauss quadrature formula generated by Matlab subprogram \([\text{csi}, \text{w}] = \text{pd}(\text{n})\), the transformation \(t = \alpha \xi + \beta\) corresponding to each interval \(I_k = [P_{k-1}; P_k]\) construct a local mesh on that subinterval. The coefficients \(c_{i,k}\) of \(x_k\) expansion after the Chebyshev polynomials \(T_i\) are obtained from \(x_k\) values on the local mesh using Fast Fourier Transforms (if \(n\) is large) or using a matrix \(T\) generated by the subprogram \(\text{T} = \text{x2t}(\text{n}, \text{csi})\)
INTEGRO-DIFFERENTIAL EQUATION WITH TWO TIME LAGS

(for \( n \) small)

\[
\begin{pmatrix}
c_{0,k} \\
c_{1,k} \\
\vdots \\
c_{n-2,k} \\
c_{n-1,k}
\end{pmatrix} = (T')^{-1} \cdot
\begin{pmatrix}
x_k(t_1) \\
x_k(t_2) \\
\vdots \\
x_k(t_{n-1}) \\
x_k(t_n)
\end{pmatrix}.
\]

The same formula allows the quick pass from the local coefficients to the values on local mesh.

The formulae

\[
\int_{\xi-h}^{\xi} T_i(s) \, ds = \frac{T_{i+1}(\xi)}{2(i+1)} - \frac{T_{i-1}(\xi)}{2(i-1)}
\]

allow to obtain the coefficients \( C_i \) of a primitive \( F \) for a function \( f \) given by its coefficients \( c_i \), from multiplication of them with a sparse matrix \( J \) generated by the subprogram \( J=tchebj(n) \)

\[
\begin{pmatrix}
C_0 \\
C_1 \\
\vdots \\
C_{n-2} \\
C_{n-1}
\end{pmatrix} = J \cdot
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_{n-2} \\
c_{n-1}
\end{pmatrix}.
\]

Of course, if the primitive is calculated for other interval \([P_{k-1}; P_k]\) instead of \([-1; 1]\), the matrix \( J \) is replaced by \( \alpha J \), where \( \alpha = (P_k - P_{k-1})/2 \).

The algorithm from Theorem 5.1 is implemented in the following way in program \([X,sol]=step\_meth2\), which can be obtained from the authors \( mserban@math.ubbcluj.ro \):

Step 0. We generate a global mesh \( X \) on \([0; \pi]\) by the union of all local meshes on which we also add the points \( P_k \) of subintervals. We calculate the values of \( x(0) \) on the global mesh from the values of the function \( \varphi \) on the local mesh of the first interval \([0; \pi/4]\) and from the constant value \( \varphi(\pi/4) \) on the other knots.

Step \( k \). Taking the values of \( x^{(k)} \) on the global mesh, we obtain the values of \( \sin(X) \cdot x^{(k)} \) on the local mesh, we calculate the coefficients of \( \sin(X) \cdot x^{(k)} \) on each subinterval, then we get the coefficients of a primitive for \( \sin(X) \cdot x^{(k)} \) on each subinterval and finally we obtain the values of that primitive on the local mesh. We add the contribution of nonintegrated part (where it is used the history from the previous intervals with two steps). The implementation of the formulae from Theorem 5.1 is now immediately, getting the values of the new iteration \( x^{(k+1)} \) on the global mesh by a new integration: we pass from the values on the mesh to coefficients, then we use the integration matrix \( J \) and finally we return to the values in order to find \( x^{(k+1)} \).

Stoping test. We evaluate the difference in norm between the values of \( x^{(k)} \) and \( x^{(k+1)} \) and iterations stop when this is below than a chosen value (concretely \( 10^{-9} \)). We represent the graph of solution and the norm of difference for different \( k \).
For the efficiency estimation of this algorithm, the integro-differential equation is written in the form of delay differential equation system and we use the Matlab command \texttt{dde23} to solve it. We impose the relative error to $10^{-9}$ and the absolute error to $10^{-12}$ to obtain an accuracy comparable with the step method. We display the graph of solution.

**Results.** Running the program we get the following results:

```
>> [X,sol] = step_meth2;
Step method solution
Elapsed time is 0.054235 seconds.
Matlab dde23 solution 1253 successful steps
0 failed attempts
3760 functions evaluations
Elapsed time is 0.671015 seconds.
```

The graph of solutions and the evolution of the differences between two successive iterations are given below:

![Graph of solutions](image1)

![Evolution of differences](image2)
Conclusions. For the chosen example, the step method obtains the solution in 68 iterations with an error of $10^{-10}$ in 0.054 CPU seconds. The Matlab program dde23 needs 0.671 CPU seconds (12 times bigger) for a similar precision. The above comparisons validate the step method from the accuracy and efficiency point of view.

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References


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