ITERATES OF MULTIVARIATE CHENEY-SHARMA OPERATORS

TEODORA CĂTINAŞ AND DIANA OTROCOL

Abstract. Using the weakly Picard operators technique, we study the convergence of the iterates of some bivariate and trivariate Cheney-Sharma operators. Also, we generalize the procedure for the multivariate case.
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1. Preliminaries

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [17], [20]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by

$F_A := \{x \in X \mid A(x) = x\}$-the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subset of A

$A^0 := 1_X,\ A^1 := A,\ \ldots,\ A^{n+1} := A \circ A^n, \quad n \in \mathbb{N}.$

Definition 1.1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 1.2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 1.3. *We define the operator A^∞, $A^\infty : X \to X$, by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Theorem 1.4. [17] *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

2. Cheney-Sharma operator

In [21] there was given an extension to two variables of the second univariate operator of Cheney-Sharma introduced in [5].

Let f be a real-valued function defined on $D = [0,1] \times [0,1]$. The bivariate Cheney-Sharma operator is defined by

$$(S_{m,n}f)(x,y;\beta,b) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x;\beta)\, q_{n,j}(y;b) f\left(\tfrac{i}{m},\tfrac{j}{n}\right), \tag{1}$$

with

$$p_{m,i}(x;\beta) = \frac{\binom{m}{i} x(x+i\beta)^{i-1}(1-x)\left[1-x+(m-i)\beta\right]^{m-i-1}}{(1+m\beta)^{m-1}},$$

and

$$q_{n,j}(y;b) = \frac{\binom{n}{j} y(y+jb)^{j-1}(1-y)\left[1-y+(n-j)b\right]^{n-j-1}}{(1+nb)^{n-1}},$$

where β and b are nonnegative parameters.

For a function f defined on $D_1 = [0,1] \times [0,1] \times [0,1]$, the trivariate operator Cheney-Sharma is defined by [22]

$$(S_{m,n,l}f)(x,y,z;\beta,\gamma,\delta) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l} p_{m,i}(x;\beta)\, q_{n,j}(y;\gamma) r_{l,k}(z;\delta) f\left(\tfrac{i}{m},\tfrac{j}{n},\tfrac{k}{r}\right), \tag{2}$$

with

$$p_{m,i}(x;\beta) = \frac{\binom{m}{i} x(x+i\beta)^{i-1}(1-x)\left[1-x+(m-i)\beta\right]^{m-i-1}}{(1+m\beta)^{m-1}},$$

$$q_{n,j}(y;\gamma) = \frac{\binom{n}{j} y(y+j\gamma)^{j-1}(1-y)\left[1-y+(n-j)\gamma\right]^{n-j-1}}{(1+n\gamma)^{n-1}},$$

and

$$r_{l,k}(z;\delta) = \frac{\binom{l}{k} z(z+k\delta)^{k-1}(1-z)\left[1-z+(l-k)\delta\right]^{l-k-1}}{(1+l\delta)^{l-1}}$$

where β, γ and δ are nonnegative parameters. This operator represents an extension to three variables of the second univariate operator of Cheney-Sharma [5].

Theorem 2.1. [21] *If f is a real-valued function defined on D then we have*

$$\left(S_{m,n}e_{ij}\right)(x,y) = x^i y^j, \quad i,j = 0,1,$$

and therefore, $\mathrm{span}\{e_{00}, e_{10}, e_{01}, e_{11}\} \subset F_{S_{m,n}}$*, where* $F_{S_{m,n}}$ *denotes the fixed points set of* $S_{m,n}$*.*

Theorem 2.2. [22] *If f is a real-valued function defined on D_1 then we have*

$$\left(S_{m,n,l}e_{ijk}\right)(x,y,z) = x^i y^j z^k, \quad i,j,k \in \{0,1\},$$

and therefore, $\mathrm{span}\{e_{000}, e_{100}, e_{001}, e_{001}, e_{110}, e_{011}, e_{101}, e_{111}\} \subset F_{S_{m,n,l}}$*, where* $F_{S_{m,n,l}}$ *denotes the fixed points set of* $S_{m,n,l}$*.*

3. Iterates of Cheney-Sharma operator

Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of the bivariate Cheney-Sharma operator given in (1).

A similar approach for the univariate case was given in [4]. Some other linear and positive operators lead to similar results in [1], [2], [7], [18] and [19]. The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [3], [8]-[16].

Let f be a real-valued function defined on D.

Theorem 3.1. *The operator $S_{m,n}$ is a weakly Picard operator and*

$$\begin{aligned}\left(S^{\infty}_{m,n}f\right)(x,y;\beta,b) =& (1-x)(1-y)f(0,0) + (1-x)yf(1,0) \\ &+ x(1-y)f(0,1) + xyf(1,1).\end{aligned} \tag{3}$$

Proof. Taking into account the interpolation properties (Theorem 2.1), of $S_{m,n}$, consider

$$X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \{f \in C(D) \mid f(0,0) = \alpha_1, f(1,0) = \alpha_2, f(0,1) = \alpha_3, f(1,1) = \alpha_4\}, \tag{4}$$

and denote by

$$f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,y) := (1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4,$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$.

We have the following properties:

(i) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is closed subset of $C(D)$;

(ii) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is an invariant subset of $S_{m,n}$, for $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, $m, n \in \mathbb{N}_+$;

(iii) $C(D) = \bigcup\limits_{\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{R}} X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a partition of $C(D)$;

(iv) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \cap F_{S_{m,n}} = \{f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\}$.

The statements (i) and (iii) are obvious.

(ii) By interpolation properties of $S_{m,n}$ we have that $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is an invariant subset of $S_{m,n}$, for any $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{R},\ m,n \in \mathbb{N}_+$;

(iv) We prove that

$$S_{m,n}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}} : X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \to X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$$

is a contraction for $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{R},\ m,n \in \mathbb{N}_+$.

Let $f,g \in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$. From (1) and (4) we obtain

$$\begin{aligned}
&|S_{m,n}(f)(x,y) - S_{m,n}(g)(x,y)| = \\
&= |S_{m,n}(f-g)(x,y)| \leq \\
&\leq |p_{m,0}(x;\beta)\, q_{n,0}(y;b)\, [f(0,0) - g(0,0)]| \\
&\quad + \left| \sum_{i=1}^{m} \sum_{j=1}^{n} p_{m,i}(x;\beta)\, q_{n,j}(y;b) \left[f\left(\tfrac{i}{m},\tfrac{j}{n}\right) - g\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right] \right| \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} p_{m,i}(x;\beta)\, q_{n,j}(y;b) \left| f\left(\tfrac{i}{m},\tfrac{j}{n}\right) - g\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right| \\
&\leq \sum_{i=1}^{m} p_{m,i}(x;\beta) \sum_{j=1}^{n} q_{n,j}(y;b)\, \|f-g\|_\infty \\
&= \left[\sum_{i=0}^{m} p_{m,i}(x;\beta) - p_{m,0}(x;\beta) \right] \left[\sum_{j=0}^{n} q_{n,j}(y;b) - q_{n,0}(y;b) \right] \|f-g\|_\infty \\
&= \left[1 - \left(1 - \tfrac{x}{1+m\beta}\right)^{m-1} \right] \left[1 - \left(1 - \tfrac{y}{1+nb}\right)^{n-1} \right] \|f-g\|_\infty \\
&\leq \left[1 - \left(1 - \tfrac{1}{1+m\beta}\right)^{m-1} \right] \left[1 - \left(1 - \tfrac{1}{1+nb}\right)^{n-1} \right] \|f-g\|_\infty .
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm.

From [2, Lemma 8] it follows that

$$\begin{aligned}
&|S_{m,n}(f)(x,y) - S_{m,n}(g)(x,y)| = \\
&\leq \left[1 - \left(1 - \tfrac{1}{1+m\beta}\right)^{m-1} \left(1 - \tfrac{1}{1+nb}\right)^{n-1} \right] \|f-g\|_\infty .
\end{aligned}$$

So,

$$\begin{aligned}
&\|S_{m,n}(f)(x,y) - S_{m,n}(g)(x,y)\|_\infty \\
&\leq \left[1 - \left(1 - \tfrac{1}{1+m\beta}\right)^{m-1} \left(1 - \tfrac{1}{1+nb}\right)^{n-1} \right] \|f-g\|_\infty ,\ \forall f,g \in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4},
\end{aligned}$$

i.e., $S_{mn}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}$ is a contraction for $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{R}$.

On the other hand, we have that

$$f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,y) := (1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4$$

and

$$S_{m,n}\left((1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4\right) = \\ = (1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4.$$

From the contraction principle we have that $f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is the unique fixed point of $S_{m,n}$ in $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ and $S_{m,n}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}$ is a Picard operator, so (3) holds. Consequently, taking into account (ii), by Theorem 1.4 it follows that the operator $S_{m,n}$ is a weakly Picard operator. We remark that $F_{S_{m,n}} = \text{span}\{e_{00}, e_{10}, e_{01}, e_{11}\}$. □

Next, we study the convergence of the iterates of the trivariate Cheney-Sharma operator given in (2).

Let f be a real-valued function defined on D_1.

Theorem 3.2. *The operator $S_{m,n,l}$ is a weakly Picard operator and*

$$\left(S^{\infty}_{m,n,l}f\right)(x,y,z;\beta,\gamma,\delta) = \tag{5}$$
$$= (1-x)(1-y)(1-z)f(0,0,0) + x(1-y)(1-z)f(1,0,0) \\ + (1-x)y(1-z)f(0,1,0) + (1-x)(1-y)zf(0,0,1) + xy(1-z)f(1,1,0) \\ + x(1-y)zf(1,0,1) + (1-x)yzf(0,1,1) + xyzf(1,1,1).$$

Proof. The proof follows the same steps as in Theorem 3.1. Using the following inequality

$$|S_{m,n,l}(f)(x,y,z) - S_{m,n,l}(g)(x,y,z)| \leq \\ \leq \left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\left[1-\left(1-\tfrac{1}{1+n\gamma}\right)^{n-1}\right]\left[1-\left(1-\tfrac{1}{1+l\delta}\right)^{l-1}\right]\|f-g\|_\infty,$$

and further [2, Lemma 8]

$$\|S_{m,n,l}(f)(x,y,z) - S_{m,n,l}(g)(x,y,z)\|_\infty \leq \\ \leq \left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\left(1-\tfrac{1}{1+n\gamma}\right)^{n-1}\left(1-\tfrac{1}{1+l\delta}\right)^{l-1}\right]\|f-g\|_\infty,$$

$\forall f,g \in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$, we prove that $S_{m,n,l}$ is a contraction. □

We generalize these results to multivariate case.

Theorem 3.3. *Consider a function $f \in C(D_p)$, with $D_p = [0,1] \times \underbrace{\ldots}_{p\ times} \times [0,1]$. The p-variate Cheney-Sharma operator, denoted by $S_{i_1,\ldots,i_p}$,* *is a weakly Picard operator and*

$$\left(S^{\infty}_{i_1,...,i_p} f\right)(x_1, ..., x_p) = \sum_{\alpha_i \in \{0,1\}, i=\overline{1,p}} s^{\infty}_{i_1,...,i_p}(x_1, ..., x_p)\, f(\alpha_1, ..., \alpha_p), \tag{6}$$

where $\alpha_i \in \{0, 1\}$, $i = 1, ..., p$ *and*

$$s^{\infty}_{i_1,...,i_p}(x_1, ..., x_p) = x_1^{\alpha_1} \cdot ... \cdot x_p^{\alpha_p}(1 - x_1)^{(1-\alpha_1)} \cdot ... \cdot (1 - x_p)^{(1-\alpha_p)}.$$

Proof. The proof follows the same steps as in Theorem 3.1. □

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