PROPERTIES OF THE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH MAXIMA, VIA WEAKLY PICARD OPERATOR THEORY

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Abstract. In this paper we present some properties of the solutions of a system of differential equation with maxima. Existence, uniqueness, inequalities of Čaplygin type and data dependence (monotony, continuity) results for the solution of the Cauchy problem of this system are obtained using weakly Picard operator technique.
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1. Introduction

In this work, we study the solutions of the nonlinear differential system with maxima of the type

$$y'(t) = A(t)y(t) + f(t, y(t), \max_{a \le \xi \le t} y(\xi)) \tag{1.1}$$

as a perturbed equations of

$$x'(t) = A(t)x(t). \tag{1.2}$$

Existence and uniqueness, inequalities of Čaplygin type and data dependence (monotony, continuity) results for the solution of the Cauchy problem of the system (1.1) shall be obtain using weakly Picard operator technique.

Differential equations with maxima arise naturally when solving practical phenomenon problems, in particular, in those which appear in the study of systems with automatic regulation and automatic control of various technical systems. In connections with many possible applications it is absolutely necessary to be developed qualitative theory of differential equations with maxima (see the monograph [1] and papers [2], [3], [4], [5], [9]).

We consider the following Cauchy problem

$$y'(t) = A(t)y(t) + f(t, y(t), \max_{a \le \xi \le t} y(\xi)),\ t \in [a, \infty[, \tag{1.3}$$

$$y(a) = y_0. \tag{1.4}$$

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Let $\mathbb{R}^n$ be the Euclidian n-space. For $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, let $\|u\| := \max\{|u_1|, \ldots, |u_n|\}$ be the norm of u. For a matrix $A \in M_{n\times n}(\mathbb{R})$, $A = (a_{ij})$, we define the norm $|A|$ of A by $|A| := \sup\limits_{\|u\|\leq 1} \|Au\|$. Then $|A| = \max\limits_{1\leq i\leq n} \sum\limits_{j=1}^{n} |a_{ij}|$.

Throughtout this paper we consider that $y_0 \in \mathbb{R}^n$, $f \in C([a, \infty[\times\mathbb{R}^n\times\mathbb{R}^n, \mathbb{R}^n)$ and $X(t)$ is the fundamental matrix of the system (1.2).

We remark that if $y \in C^1([a, \infty[, \mathbb{R}^n)$ is a solution of the problem (1.3)–(1.4), then y is a solution of

$$y(t) = X(t)X^{-1}(a)y_0+\int_a^t X(t)X^{-1}(s)f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))ds, t \in [a, \infty[\tag{1.5}$$

and if $y \in C([a, \infty[, \mathbb{R}^n)$ is a solution of (1.5), then $y \in C^1([a, \infty[, \mathbb{R}^n)$ and is a solution of (1.3)–(1.4).

Also, if $y \in C^1([a, \infty[, \mathbb{R}^n)$ is a solution of the problem (1.3), then y is a solution of

$$y(t) = X(t)X^{-1}(a)y(a)+\int_a^t X(t)X^{-1}(s)f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))ds, t \in [a, \infty[\tag{1.6}$$

and if $y \in C([a, \infty[, \mathbb{R}^n)$ is a solution of (1.6) then $y \in C^1([a, \infty[, \mathbb{R}^n)$ and is a solution of (1.3).

Let us consider the following operators:

$$B_f, E_f : C([a, \infty[, \mathbb{R}^n) \to C([a, \infty[, \mathbb{R}^n),$$

defined by

$$B_f(y)(t) := X(t)X^{-1}(a)y_0 + \int_a^t X(t)X^{-1}(s)f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))ds,$$

and

$$E_f(y)(t) := X(t)X^{-1}(a)y(a) + \int_a^t X(t)X^{-1}(s)f(s, y(s), \max_{a\leq\xi\leq s} y(\xi))ds.$$

For $y_0 \in \mathbb{R}^n$, we consider $X_{y_0} := \{y \in C([a, \infty[, \mathbb{R}^n)|\ y(a) = y_0\}$.

We remark that

$$C([a, \infty[, \mathbb{R}^n) = \bigcup_{y_0\in\mathbb{R}^n} X_{y_0}$$

is a partition of $C([a, \infty[, \mathbb{R}^n)$.

The following lemma is important for our further considerations.

Lemma 1.1. *(I.A. Rus, [8]) For $y_0 \in \mathbb{R}^n$ and $f \in C([a, \infty[\times\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, the following conditions hold:*

(a) $B_f(C([a, \infty[, \mathbb{R}^n)) \subset X_{y_0}$ *and* $E_f(X_{y_0}) \subset X_{y_0}$, $\forall y_0 \in \mathbb{R}^n$;

(b) $B_f|_{X_{y_0}} = E_f|_{X_{y_0}}$, $\forall y_0 \in \mathbb{R}^n$.

In this paper we prove that the operator E_f is weakly Picard operator (see [7]), and we study the equation (1.3) in the terms of the weakly Picard operators theory.

2. Weakly Picard operators

We start this section by presenting some notions and results from the weakly Picard operators theory.

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$ - the family of the nonempty invariant subsets of A;

We will denote by H the Pompeiu-Housdorff functional, $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined as

$$H(Y, Z) := \max\{\sup_{y\in Y} \inf_{z\in Z} d(y, z), \sup_{z\in Z} \inf_{y\in Y} d(y, z)\}$$

Definition 2.1. *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$;

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 2.2. *Let (X, d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$, and its limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is weakly Picard operator then we consider the operator A^∞ defined by*

$$A^\infty : X \to X, \ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Remark 2.4. *It is clear that $A^\infty(X) = F_A$.*

Definition 2.5. *Let A be a weakly Picard operator and $c > 0$. The operator A is c -weakly Picard operator if*

$$d(x, A^\infty(x)) \leq cd(x, A(x)), \ \forall x \in X.$$

For the theory of weakly Picard operator, see [6], [7], [8].

3. Cauchy problem

Let us consider the following Banach space $(BC([a, \infty[, \mathbb{R}^n), \|\cdot\|)$ where $BC([a, \infty[, \mathbb{R}^n) := \{y \in C([a, \infty[, \mathbb{R}^n) |\ y \text{ is bounded}\}$ with $\|y\| := \max_{t\in[a,\infty]} \{|y_1(t)|, \ldots, |y_n(t)|\}$. We have the following existence and uniqueness theorem

Theorem 3.1. *We suppose that:*

(i) *there exists* $L_f : [a, \infty[\to \mathbb{R}_+$ *with* $\int_a^\infty L_f(s)ds < \infty$ *such that*

$$\|f(t, u_1, u_2) - f(t, v_1, v_2)\| \le L_f(t) \max\{|u_1 - v_1|, |u_2 - v_2|\},$$
$$\forall t \in [a, \infty[\text{ and } u_i, v_i \in \mathbb{R}^n, i = 1, 2.$$

(ii) $\|X(t)X^{-1}(s)\| \le K$, *for* $a \le s \le t < \infty$;

(iii) $\int_a^t \|f(s, 0, 0)\| ds < \infty$;

(iv) $K \int_a^t L_f(s)ds < 1$.

Then the problem (1.3)–(1.4) has, in $BC([a, \infty[, \mathbb{R}^n)$, *a unique solution and this solution is the uniform limit of the successive approximations.*

Proof. The problem (1.3)–(1.4) is equivalent with the fixed point equation

$$B_f(y) = y, \; y \in BC([a, \infty[, \mathbb{R}^n).$$

We show that the space $BC([a, \infty[, \mathbb{R}^n)$ is invariant for the operator B_f.

If $x \in BC([a, \infty[, \mathbb{R}^n)$, then

$$\begin{aligned}
&|B_f(x)(t)| \\
&\le \left|X(t)X^{-1}(a)y_0\right| + \int_a^t \left|X(t)X^{-1}(a)\right| \left|f(s, y(s), \max_{a \le \xi \le s} y(\xi)) - f(s, 0, 0)\right| ds \\
&\quad + \int_a^t \left|X(t)X^{-1}(a)\right| |f(s, 0, 0)| \, ds \\
&\le K y_0 + \int_a^t K L_f(s) \max\left\{|y(s)|, \left|\max_{a \le \xi \le s} y(\xi)\right|\right\} ds + \int_a^t K |f(s, 0, 0)| \, ds < \infty.
\end{aligned}$$

So $B_f(BC([a, \infty[, \mathbb{R}^n)) \subseteq BC([a, \infty[, \mathbb{R}^n)$.

On the other hand we have that (see [4])

$$\begin{aligned}
&|B_f(y_1)(t) - B_f(y_2)(t)| \\
&= \Big|X(t)X^{-1}(a)y_0 + \int_a^t X(t)X^{-1}(s) f(s, y_1(s), \max_{a \le \xi \le s} y_1(\xi))ds \\
&\quad - X(t)X^{-1}(a)y_0 - \int_a^t X(t)X^{-1}(s) f(s, y_2(s), \max_{a \le \xi \le s} y_2(\xi))ds\Big| \\
&\le \int_a^t L_f(s) \left|X(t)X^{-1}(s)\right| \max\left\{|y_1(s) - y_2(s)|, \left|\max_{a \le \xi \le s} y_1(\xi) - \max_{a \le \xi \le s} y_2(\xi)\right|\right\} ds \\
&\le K \int_a^t L_f(s) \max\left\{|y_1(s) - y_2(s)|, \max_{a \le \xi \le s} |y_1(\xi) - y_2(\xi)|\right\} ds \\
&\le \left(K \int_a^t L_f(s)ds\right) \|y_1 - y_2\|, \forall t \in [a, \infty[, \; \forall y_1, y_2 \in BC([a, \infty[, \mathbb{R}^n).
\end{aligned}$$

So,

$$\|B_f(y_1) - B_f(y_2)\| \leq \left(K \int_a^t L_f(s)ds\right) \|y_1 - y_2\|,$$

$\forall t \in [a, \infty[, \forall y_1, y_2 \in BC([a, \infty[, \mathbb{R}^n)$, i.e., B_f is a contraction w.r.t. the norm $\|\cdot\|$ on $BC([a, \infty[, \mathbb{R}^n)$. The proof follows from the Banach fixed point theorem. □

Remark 3.2. *In the conditions of Theorem 3.1, the operator B_f is Picard operator. But*

$$B_f|_{X_{y_0}} = E_f|_{X_{y_0}}, \ \forall y_0 \in \mathbb{R}^n.$$

Hence, the operator E_f is weakly Picard operator and $F_{E_f} \cap X_{y_0} = \{y^*_{y_0}\}, \forall y_0 \in \mathbb{R}^n$, where $F_{E_f} = \{y^*_{y_0} \in X_{y_0} | E_f(y^*_{y_0}) = y^*_{y_0}\}$ and $y^*_{y_0}$ is the unique solution of the problem (1.3)–(1.4).

From the WPO theory, we present in the following sections inequalities of Čaplygin type and data dependence results for the solution of the system of differential equations.

4. Inequalities of Čaplygin type

From the weakly Picard operator theory we have

Theorem 4.1. *(Theorem of Čaplygin type) We suppose that:*

(a) *the hypothesis of Theorem 3.1 are satisfied;*
(b) *$f(t, \cdot, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is increasing, i.e., $u_i \leq v_i \Rightarrow f(t, u_1, u_2) \leq f(t, v_1, v_2), \ \forall t \in [a, \infty[$ and $u_i, v_i \in \mathbb{R}^n, i = 1, 2$.*

Let y be a solution of equation (1.3) and x a solution of the inequality

$$x'(t) \leq A(t)x(t) + f(t, x(t), \max_{a \leq \xi \leq t} x(\xi)), \ t \in [a, \infty[.$$

Then

$$x(a) \leq y(a) \text{ implies that } x \leq y.$$

Proof. In the terms of the operator E_f, we have

$$y = E_f(y) \text{ and } x \leq E_f(x),$$

and $y(a) \leq x(a)$.

From Remark 3.2, E_f is weakly Picard operator. From the condition (b), E_f^∞ is increasing ([7]). If $y_0 \in \mathbb{R}^n$, then we denote by $\widetilde{y}_0$ the following function

$$\widetilde{y}_0 : [a, \infty[\to \mathbb{R}^n, \ \widetilde{y}_0(t) = y_0, \ \forall t \in [a, \infty[.$$

We have

$$x \leq E_f(x) \leq \ldots \leq E_f^\infty(x) = E_f^\infty(\widetilde{x}(a)) \leq E_f^\infty(\widetilde{y}(a)) = y.$$

□

5. Data dependence: monotony

The following concept is important for our further considerations.

Lemma 5.1. *(Comparison principle, [6]) Let $(X, d, \leq)$ an ordered metric space and $A, B, C : X \to X$ be such that*

(a) $A \leq B \leq C$;
(b) *the operator A, B, C, are WPOs;*
(c) *the operator B is increasing.*

Then $x \leq y \leq z$ imply that $A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

From this abstract result we have

Theorem 5.2. *Let $f_i \in C([a, \infty[\times\mathbb{R}^{2n}, \mathbb{R}^n), i = 1, 2,$ be as in Theorem 3.1. We suppose that:*

(i) $f_1 \leq f_2 \leq f_3$;
(ii) $f_2(t, \cdot, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^n$ *is increasing;*

Let $y_i \in BC^1([a, \infty[, \mathbb{R}^n)$ be a solution of the equation

$$y_i'(t) = A(t)y_i(t) + f_i(t, y(t), \max_{a\leq\xi\leq t} y(\xi)),\ t \in [a, \infty[\text{ and } i = 1, 2, 3.$$

If $y_1(a) \leq y_2(a) \leq y_3(a)$, then $y_1 \leq y_2 \leq y_3$.

Proof. From Theorem 3.1 we have that the operator $E_{f_i}, i = 1, 2, 3,$ are WPOs. From the condition (ii) the operator E_{f_2} is monotone increasing. From the condition (i) it follows that

$$E_{f_1} \leq E_{f_2} \leq E_{f_3}.$$

Let $\widetilde{y}_i(a) \in BC([a, \infty[, \mathbb{R}^n)$ be defined by $\widetilde{y}_i(a)(t) = y_i(a),\ \forall t \in [a, \infty[$. It is clear that

$$\widetilde{y}_1(a)(t) \leq \widetilde{y}_2(a)(t) \leq \widetilde{y}_3(a)(t),\ \forall t \in [a, \infty[.$$

From Lemma 5.1 we have that

$$E^\infty_{f_1}(\widetilde{y}_1(a)) \leq E^\infty_{f_2}(\widetilde{y}_2(a)) \leq E^\infty_{f_3}(\widetilde{y}_3(a)).$$

But $y_i = E^\infty_{fi}(\widetilde{y}_i(a))$, and $y_1 \leq y_2 \leq y_3$. □

6. Data dependence: continuity

Consider the Cauchy problem (1.3)–(1.4) and suppose the conditions of Theorem 3.1 are satisfied. Denote by $y^*(\cdot; y_0, f)$ the solution of this problem.

In order to study the continuous dependence of the fixed points we will use the following result:

Theorem 6.1. *(I.A. Rus, [7]) Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

(i) *the operator A is a α-contraction;*
(ii) $F_B \neq \emptyset$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(A(x), B(x)) \leq \eta, \ \forall x \in X.$$

Then, if $F_A = \{x_A^*\}$ *and* $x_B^* \in F_B$, *we have*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

Then, accordingly to Theorem 6.1 we have the result as follows.

Theorem 6.2. *Let* $y_0^i, f_i, i = 1, 2$ *be as in Theorem 3.1. Furthermore, we suppose that there exists* $\eta_i \in \mathbb{R}_+^n, i = 1, 2$ *with* $\int_a^t \eta_2(s)ds < \infty$ *such that*

(i) $\|y_0^1 - y_0^2\| \leq \eta_1$;
(ii) $\|X(t)X^{-1}(s)\| \leq K$ *for* $a \leq s \leq t < \infty$;
(iii) $\|f_1(t, u, v) - f_2(t, u, v)\| \leq \eta_2(t), \forall t \in [a, \infty[, u \in \mathbb{R}^n$.

Then

$$\left\| y_1^*(t; y_0^1, f_1) - y_2^*(t; y_0^2, f_2) \right\| \leq \frac{K\eta_1 + K \int_a^t \eta_2(s)ds}{1 - K \int_a^t L_f(s)ds},$$

where $y_i^*(t; x_0^i, f_i), i = 1, 2$ *are the solution of the problem (1.3)–(1.4) with respect to* y_0^i, f_i *and* $L_f = \max\{L_{f_1}, L_{f_2}\}$.

Proof. Consider the operators $B_{y_0^i, f_i}, i = 1, 2$. From Theorem 3.1 these operators are contractions.

Additionally

$$\left\| B_{y_0^1, f_1}(y) - B_{y_0^2, f_2}(y) \right\| \leq K\eta_1 + K \int_a^t \eta_2(s)ds, \ \forall y \in BC([a, \infty[, \mathbb{R}^n).$$

Now the proof follows from the Theorem 6.1, with $A := B_{y_0^1, f_1}$, $B = B_{y_0^2, f_2}$, $\eta = K\eta_1 + K \int_a^t \eta_2(s)ds$ and $\alpha := K \int_a^t L_f(s)ds$, where $L_f = \max\{L_{f_1}, L_{f_2}\}$. □

We shall use the c-WPOs techniques to give some data dependence results.

Theorem 6.3. *(I.A. Rus, [7]) Let* (X, d) *be a metric space and* $A_i : X \to X, \ i = 1, 2$. *Suppose that*

(i) *the operator* A_i *is* c_i*-weakly Picard operator,* $i = 1, 2$;
(ii) *there exists* $\eta > 0$ *such that*

$$d(A_1(x), A_2(x)) \leq \eta, \ \forall x \in X.$$

Then $H(F_{A_1}, F_{A_2}) \leq \eta \max\{c_1, c_2\}$.

Based upon Theorem 6.3 we have the next result.

Theorem 6.4. *Let* f_1 *and* f_2 *be as in Theorem 3.1. Let* $S_{E_{f_1}}, S_{E_{f_2}}$ *be the solution sets of system (1.3) corresponding to* f_1 *and* f_2. *Suppose that*

(i) $\|X(t)X^{-1}(s)\| \leq K$ *for* $a \leq s \leq t < \infty$;

(ii) *there exists* $\eta \in \mathbb{R}^n_+$, $\int_a^t \eta(s)ds < \infty$ *such that*

$$\|f_1(t,u,v) - f_2(t,u,v)\| \leq \eta(t) \text{ for all } t \in [a,\infty[, u \in \mathbb{R}^n. \tag{6.1}$$

Then

$$H_{\|\cdot\|_C}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{K\int_a^t \eta(s)ds}{1 - K\int_a^t L_f(s)ds},$$

where $L_f = \max\{L_{f_1}, L_{f_2}\}$ *and* $H_{\|\cdot\|}$ *denotes the Pompeiu-Housdorff functional with respect to* $\|\cdot\|$ *on* $BC([a,\infty[,\mathbb{R}^n)$.

Proof. In the condition of Theorem 3.1, the operators E_{f_1} and E_{f_2} are c_i-weakly Picard operators, $i = 1,2$. Let $X_{y_0} := \{y \in BC([a,\infty[,\mathbb{R}^n) | \, y(a) = y_0\}$. It is clear that $E_{f_1}|_{X_{y_0}} = B_{f_1},\ E_{f_2}|_{X_{y_0}} = B_{f_2}$. Therefore,

$$\left\|E^2_{f_1}(y) - E_{f_1}(y)\right\| \leq \left(K\int_a^t L_{f_1}(s)ds\right)\|E_{f_1}(y) - y\|,\ \forall y \in BC([a,\infty[,\mathbb{R}^n),$$

$$\left\|E^2_{f_2}(y) - E_{f_2}(y)\right\| \leq \left(K\int_a^t L_{f_2}(s)ds\right)\|E_{f_2}(y) - y\|,\ \forall y \in BC([a,\infty[,\mathbb{R}^n).$$

Now, choosing $y_0^1 = K\int_a^t L_{f_1}(s)ds$ and $y_0^2 = K\int_a^t L_{f_2}(s)ds$, we get that E_{f_1} and E_{f_2} are c_i-weakly Picard operators, $i = 1,2$ with $c_1 = (1 - y_0^1)^{-1}$ and $c_2 = (1 - y_0^2)^{-1}$. From (6.1) we obtain that

$$\|E_{f_1}(y) - E_{f_2}(y)\| \leq K\int_a^t \eta(s)ds,\ \forall y \in BC([a,\infty[,\mathbb{R}^n).$$

Applying Theorem 6.3 we have that

$$H_{\|\cdot\|}(S_{E_{f_1}}, S_{E_{f_2}}) \leq \frac{K\int_a^t \eta(s)ds}{1 - K\int_a^t L_f(s)ds},$$

where $L_f = \max\{L_{f_1}, L_{f_2}\}$ and $H_{\|\cdot\|}$ is the Pompeiu-Housdorff functional with respect to $\|\cdot\|$ on $BC([a,\infty[,\mathbb{R}^n)$. □

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