ULAM STABILITY FOR A DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper we study the Ulam-Hyers stability and generalized Ulam-Hyers-Rassias stability for a delay differential equation. Some examples are given.

MSC 2000: 34K20, 34L05, 47H10.
Keywords: Ulam-Hyers stability, Ulam-Hyers-Rassias stability, delay differential equation.

1. Introduction

In the last 30 years, the stability theory of functional equations was strongly developed. Very important contributions to this subject were brought by Ulam [15], Rassias [10], Hyers *et al.* [4], Jung [5], Guo *et al.* [3], Kolmanovskii and Myshkis [6] and Radu [9]. Our results are connected to some recent papers of Castro and Ramos [2] and Jung [5] (where integral and differential equations are considered), Bota-Boriceanu and Petruşel [1] and Petru *et al.* [8] (where the Ulam-Hyers stability for operatorial equations and inclusions are discussed). Following [13] and [7], in our paper we will investigate Ulam-Hyers stability, generalized Ulam-Hyers-Rassias stability for the following differential equation with modification of the argument

$$x'(t) = f(t, x(t), x(g(t))),\ t \in I \subset \overline{\mathbb{R}},$$

where

(i) $I = [a, b]$ or $I = [a, \infty[,\ a, b \in \mathbb{R}$;
(ii) $f \in C([a, b] \times \mathbb{R}^2, \mathbb{R}),\ g \in C([a, b], [a-h, b]),\ g(t) \leq t,\ h > 0$, respectively $f \in C([a, \infty[\times \mathbb{R}^2, \mathbb{R}),\ g \in C([a, \infty[, [a-h, \infty[,\ g(t) \leq t,\ h > 0$.

By a solution of the above equation we understand a function $x \in C([a-h, b], \mathbb{R}) \cap C^1([a, b], \mathbb{R})$, respectively $x \in C([a-h, \infty[, \mathbb{R}) \cap C^1([a, \infty[, \mathbb{R})$, that verifies the equation.

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2. Preliminaries

We begin our considerations with some notions and results from Ulam stability (see [13], [14]), for the case $I = [a, b]$.

For $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $\varepsilon > 0$, $\varphi \in C([a-h, b], \mathbb{R}_+)$ and $\psi \in C([a-h, a], \mathbb{R})$ we consider the following Cauchy problem

$$x'(t) = f(t, x(t), x(g(t))),\ t \in I \tag{2.1}$$

$$x(t) = \psi(t),\ t \in [a-h, a] \tag{2.2}$$

and the following inequations

$$|y'(t) - f(t, y(t), y(g(t)))| \le \varepsilon,\ t \in I \tag{2.3}$$

$$|y'(t) - f(t, y(t), y(g(t)))| \le \varphi(t),\ t \in I. \tag{2.4}$$

Definition 2.1. *The equation* (2.1) *is Ulam-Hyers stable if there exists a real number* $c > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $y \in C^1([a-h, b], \mathbb{R})$ *of* (2.3) *there exists a solution* $x \in C^1([a-h, b], \mathbb{R})$ *of* (2.1) *with*

$$|y(t) - x(t)| \le c\varepsilon, \quad \forall t \in [a-h, b].$$

Definition 2.2. *The equation* (2.1) *is generalized Ulam-Hyers-Rassias stable with respect to* φ*, if there exists* $c_\varphi > 0$*, such that for each solution* $y \in C^1([a-h, b], \mathbb{R})$ *of the inequation* (2.4) *there exists a solution* $x \in C^1([a-h, b], \mathbb{R})$ *of* (2.1) *with*

$$|y(t) - x(t)| \le c_\varphi \varphi(t),\ \forall t \in [a-h, b].$$

Remark 2.3. *A function* $y \in C^1(I, \mathbb{R})$ *is a solution of* (2.3) *if and only if there exists a function* $h \in C(I, \mathbb{R})$ *(which depends on* y*) such that*

(i) $|h(t)| \le \varepsilon,\ \forall t \in I$;
(ii) $y'(t) = f(t, y(t), y(g(t))) + h(t),\ \forall t \in I$.

Remark 2.4. *A function* $y \in C^1(I, \mathbb{R})$ *is a solution of* (2.4) *if and only if there exists a function* $\widetilde{h} \in C(I, \mathbb{R})$ *(which depends on* y*) such that*

(i) $\left|\widetilde{h}(t)\right| \le \varphi(t),\ \forall t \in I$;
(ii) $y'(t) = f(t, y(t), y(g(t))) + \widetilde{h}(t),\ \forall t \in I$.

Remark 2.5. *If* $y \in C^1(I, \mathbb{R})$ *is as solution of the inequation* (2.3)*, then* y *is a solution of the following integral inequation*

$$\left| y(t) - y(a) - \int_a^t f(s, y(s), y(g(s)))ds \right| \le (t-a)\varepsilon,\ \forall t \in I.$$

Remark 2.6. *If $y \in C^1(I,\mathbb{R})$ is as solution of the inequation* (2.4)*, then y is a solution of the following integral inequation*

$$\left| y(t) - y(a) - \int_a^t f(s, y(s), y(g(s)))ds \right| \leq \int_a^t \varphi(s)ds, \ \forall t \in I.$$

Analogously, one may have the above definitions and remarks for the case $I = [a, \infty[$, the interval $[a-h, b]$ would be replaced by $[a-h, \infty[$.

In the sequel we shall use the following Picard operator definition and the well-known Gronwall lemma and abstract Gronwall lemma (see, e.g. Rus [12]).

Definition 2.7. *(Rus* [11]*) Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*

(i) *$F_A = \{x^*\}$ where $F_A := \{x \in X \mid A(x) = x\}$ is the fixed point set of A;*

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Lemma 2.8. *(Gronwall Lemma) Let $g,\ h \in C([a, b], \mathbb{R}_+)$ be two functions. We suppose that g is increasing. If $x \in C([a, b], \mathbb{R}_+)$ is a solution of the inequation*

$$x(t) \leq g(t) + \int_a^b h(s)x(s)ds, \ t \in [a, b],$$

then

$$x(t) \leq g(t) \exp\left(\int_a^b h(s)ds\right), \ t \in [a, b].$$

Lemma 2.9. *(Abstract Gronwall Lemma) Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) *A is a Picard operator ($F_A = \{x^*_A\}$);*

(ii) *A is an increasing operator.*

Then we have: (a) *$x \in X,\ x \leq A(x) \Longrightarrow x \leq x^*_A$;*

(b) *$x \in X,\ x \geq A(x) \Longrightarrow x \geq x^*_A$.*

3. Ulam-Hyers stability on a compact interval $I = [a, b]$

In this section we present conditions for the equation (2.1) to admit the Ulam-Hyers stability on a compact interval $I = [a, b]$.

Theorem 3.1. *We suppose that*

(a) $f \in C([a, b] \times \mathbb{R}^2, \mathbb{R}),\ g \in C([a, b], [a-h, b]),\ g(t) \leq t,\ h > 0$;

(b) *there exists $L_f > 0$ such that $\forall t \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2,$ we have*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i| ;$$

(c) $2(b-a)L_f < 1$.

Then

(i) *the problem* (2.1)–(2.2) *has a unique solution in* $C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$;
(ii) *the equation* (2.1) *is Ulam-Hyers stable.*

Proof. (i) In the condition (a), the problem (2.1)–(2.2) is equivalent to the integral equation

$$x(t)=\begin{cases}\psi(t), & t\in[a-h,a],\\ \psi(t)+\displaystyle\int_a^t f(s,x(s),x(g(s)))ds, & t\in[a,b].\end{cases}$$

Let $X:=C([a-h,b],\mathbb{R})$ and $B_f:X\to X$ be given by

$$B_f(x)(t):=\begin{cases}\psi(t), & t\in[a-h,a],\\ \psi(t)+\displaystyle\int_a^t f(s,x(s),x(g(s)))ds, & t\in[a,b].\end{cases}$$

We show that B_f is a contraction on X with respect to the Chebyshev norm.

$$|B_f(x)(t)-B_f(y)(t)|=0,\ \ \forall x,y\in C([a-h,b],\mathbb{R}),\ t\in[a-h,a].$$

$$\begin{aligned}&|B_f(x)(t)-B_f(y)(t)|\\ &\le\left|\int_a^t f(s,x(s),x(g(s)))ds-\int_a^t f(s,y(s),y(g(s)))ds\right|\\ &\le L_f\left(\max_{a-h\le t\le b}|x(s)-y(s)|+\max_{a-h\le t\le b}|x(g(s))-y(g(s))|\right)(b-a)\\ &\le 2(b-a)L_f\|x-y\|,\ \forall x,y\in C([a-h,b],\mathbb{R}),\ t\in[a,b].\end{aligned}$$

So,

$$\|B_f(x)-B_f(y)\|\le 2(b-a)L_f\|x-y\|,\ \forall x,y\in C([a-h,b],\mathbb{R}),$$

i.e., B_f is a contraction w.r.t. the Chebyshev norm on X. The proof follows from the Banach contraction principle.

(ii) Let $y\in C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ be a solution of the inequation (2.3). We denote by $x\in C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$ the unique solution of the Cauchy problem

$$\begin{aligned}x'(t)&=f(t,x(t),x(g(t))),\ t\in[a,b],\\ x(t)&=y(t),\ t\in[a-h,a].\end{aligned}$$

From condition (a) we have

$$x(t)=\begin{cases}y(t), & t\in[a-h,a],\\ y(a)+\displaystyle\int_a^t f(s,x(s),x(g(s)))ds, & t\in[a,b].\end{cases}$$

Remark 2.5 gives

$$\left|y(t)-y(a)-\int_a^t f(s,y(s),y(g(s)))ds\right|\le(t-a)\varepsilon,\ t\in[a,b].$$

It follows that $|y(t)-x(t)|=0$, for $t\in[a-h,a]$ and for $t\in[a,b]$ we have

$$
\begin{aligned}
(3.1)\quad & |y(t)-x(t)| \leq \\
& \leq \left| y(t)-y(a)-\int_a^t f(s,y(s),y(g(s)))ds \right| + \\
& + \int_a^t |f(s,y(s),y(g(s)))-f(s,x(s),x(g(s)))|\, ds \\
& \leq (t-a)\varepsilon + L_f\Big(\int_a^t |y(s)-x(s)|\, ds + \int_a^t |y(g(s))-x(g(s))|\, ds \Big).
\end{aligned}
$$

According to the last inequality, for $u\in C([a-h,b],\mathbb{R}_+)$ we consider the following operator $A: C([a-h,b],\mathbb{R}_+)\to C([a-h,b],\mathbb{R}_+)$ defined by

$$
A(u)(t) := \begin{cases} 0, & t\in[a-h,a], \\ (t-a)\varepsilon + L_f\displaystyle\int_a^t u(s)ds + L_f\int_a^t u(g(s))ds, & t\in[a,b]. \end{cases}
$$

In order to verify that A is a Picard operator (Definition 2.7) we prove that A is a contraction.

For $t\in[a,b]$:

$$
\begin{aligned}
& |A(u)(t)-A(v)(t)| \leq \\
& \leq L_f\left(\int_a^t |u(s)-v(s)|\, ds + \int_a^t |u(g(s))-v(g(s))|\, ds \right) \\
& \leq L_f\left(\max_{a-h\leq t\leq b} |u(s)-v(s)| + \max_{a-h\leq t\leq b} |u(g(s))-v(g(s))| \right)(b-a) \\
& \leq 2(b-a)L_f\,\|u-v\|\,,\ \forall u,v\in C([a-h,b],\mathbb{R}_+).
\end{aligned}
$$

So, $\|A(u)-A(v)\| \leq 2(b-a)L_f\,\|u-v\|\,,\ \forall u,v\in C([a-h,b],\mathbb{R}_+)$, i.e., A is a contraction w.r.t. the Chebyshev norm on $C([a-h,b],\mathbb{R}_+)$. Applying the Banach contraction principle, we have that A is Picard operator and $F_A=\{u^*\}$. Then

$$
u^*(t) = (t-a)\varepsilon + L_f\int_a^t u^*(s)ds + L_f\int_a^t u^*(g(s))ds,\ t\in[a,b].
$$

The solution u^* is increasing and $(u^*)'\geq 0$. So, $u^*(g(t))\leq u^*(t)$ and

$$
u^*(t) \leq (t-a)\varepsilon + 2L_f\int_a^t u^*(s)ds.
$$

From the Gronwall Lemma we obtain

$$
u^*(t)\leq c\varepsilon,\ t\in[a-h,b],\ \text{where } c:=(b-a)\exp(2L_f(b-a)).
$$

In particular, if $u:=|y-x|$, from (3.1), $u(t)\leq A(u)(t)$ and applying the abstract Gronwall lemma we obtain $u(t)\leq u^*(t)$ (A is a Picard and an increasing operator). It follows that

$$|y(t) - x(t)| \leq c\varepsilon, \ t \in [a-h, b],$$

i.e., the equation (2.1) is Ulam-Hyers stable. □

4. Generalized Ulam-Hyers-Rassias stability on $I = [a, \infty[$

In this section we present conditions for the equation (2.1) to admit the generalized Ulam-Hyers-Rassias stability on the interval $I = [a, \infty[$.

Theorem 4.1. *We suppose that*

(a) $f \in C([a, \infty[\times \mathbb{R}^2, \mathbb{R}), \ g \in C([a, \infty[, [a-h, \infty[), \ g(t) \leq t, \ h > 0$;

(b) *there exists* $l_f \in L^1([a, \infty[, \mathbb{R}_+)$ *such that* $\forall t \in [a, \infty[, u_i, v_i \in \mathbb{R}, i = 1, 2,$ *we have*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq l_f(t)(|u_1 - v_1| + |u_2 - v_2|);$$

(c) *the function* $\varphi \in C[a, \infty[$ *is increasing;*

(d) *there exists* $\lambda > 0$ *such that*

$$\int_a^t \varphi(s)ds \leq \lambda \varphi(t), \ t \in [a, \infty[.$$

Then

(i) *the problem* (2.1)–(2.2) *has a unique solution in* $C([a-h, \infty[, \mathbb{R}) \cap C^1([a, \infty[, \mathbb{R})$;

(ii) *the equation* (2.1) *is generalized Ulam-Hyers-Rassias stable with respect to* φ.

Proof. The proof follows the same steps as in Theorem 3.1. Let $y \in C([a-h, \infty[, \mathbb{R}) \cap C^1([a, \infty[, \mathbb{R})$ be a solution of the inequation (2.4). The equation (2.1) has a unique solution in $C([a-h, \infty[, \mathbb{R}) \cap C^1([a, \infty[, \mathbb{R})$. We denote by $x \in C([a-h, \infty[, \mathbb{R}) \cap C^1([a, \infty[, \mathbb{R})$ the unique solution of the Cauchy problem

$$x'(t) = f(t, x(t), x(g(t))), \ t \in [a, \infty[,$$
$$x(a) = y(t), \ t \in [a-h, a].$$

So

$$x(t) = \begin{cases} y(t), & t \in [a-h, a] \\ y(a) + \int_a^t f(s, x(s), x(g(s)))ds, & t \in [a, \infty[. \end{cases}$$

Remark 2.6 gives

$$\left| y(t) - y(a) - \int_a^t f(s, y(s), y(g(s)))ds \right| \leq$$
$$\leq \int_a^t \varphi(s)ds \leq \lambda \varphi(t), \ t \in [a, \infty[.$$

From the above relations, for $t \in [a-h,a]$ we have $|y(t)-x(t)| = 0$ and for $t \in [a,\infty[$, we obtain

$$\begin{aligned}|y(t)-x(t)| \leq &\left| y(t) - y(a) - \int_a^t f(s,y(s),y(g(s)))ds \right| + \\ &+ \int_a^t |f(s,y(s),y(g(s))) - f(s,x(s),x(g(s)))|\, ds \\ &\leq \lambda\varphi(t) + \int_a^t l_f(s)|y(s)-x(s)|\, ds + \int_a^t l_f(s)|y(g(s))-x(g(s))|\, ds.\end{aligned}$$

As in the proof of Theorem 3.1 (ii), it follows that

$$|y(t)-x(t)| \leq \lambda\varphi(t)\exp\left(\int_a^t 2l_f(s)ds\right) = c_\varphi\varphi(t),\ t \in [a,\infty[,$$

where $c_\varphi := \lambda \exp\left(\int_a^t 2l_f(s)ds\right)$, i.e., the equation (2.1) is generalized Ulam-Hyers-Rassias stable. □

5. Applications

Here we present some consequences of the above theory.

Example 5.1. *We consider the following Cauchy problem*

$$x'(t) = f(t,x(t),x(t-h)),\ t \in [a,b) \tag{5.1}$$

$$x(a) = x_0 \tag{5.2}$$

and the following inequations

$$|y'(t) - f(t,y(t),y(t-h))| \leq \varepsilon,\ t \in [a,b)$$
$$|y'(t) - f(t,y(t),y(t-h))| \leq \varphi(t),\ t \in [a,b).$$

In this case, from Theorem 3.1 we have:

Theorem 5.2. *We suppose that*

(a) $f \in C([a,b]\times\mathbb{R}^2,\mathbb{R})$;
(b) *there exists* $L_f > 0$ *such that* $\forall t \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2$ *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|\,;$$

Then

(i) *the problem* (5.1)–(5.2) *has a unique solution in* $C([a-h,b],\mathbb{R})\cap C^1([a,b],\mathbb{R})$;
(ii) *the equation* (5.1) *is Ulam-Hyers stable.*

Let $b = +\infty$. The conditions (a)-(d) from Theorem 4.1 are the same, so the problem (5.1)–(5.2) has a unique solution in $C([a-h,\infty[,\mathbb{R})\cap C^1([a,\infty[,\mathbb{R})$ and the equation (5.1) is generalized Ulam-Hyers-Rassias stable on $[a,\infty[$.

Theorem 5.3. *We suppose that*

(a) $f \in C([a,\infty[\times\mathbb{R}^2,\mathbb{R})$;

(b) *there exists* $l_f \in L^1([a,\infty[,\mathbb{R}_+)$ *such that* $\forall t \in [a,\infty[, u_i, v_i \in \mathbb{R}, i = 1,2,$ *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq l_f(t)(|u_1 - v_1| + |u_2 - v_2|);$$

(c) *the function* $\varphi \in C[a,\infty[$ *is increasing;*

(d) *there exists* $\lambda > 0$ *such that*

$$\int_0^t \varphi(s)ds \leq \lambda\varphi(t),\ t \in [a,\infty[.$$

Then

(i) *the problem* (5.1)–(5.2) *has a unique solution in* $C([a-h,\infty[,\mathbb{R}) \cap C^1([a,\infty[,\mathbb{R})$;

(ii) *the equation* (5.1) *is generalized Ulam-Hyers-Rassias stable with respect to* φ.

Example 5.4. *We consider the following Cauchy problem*

$$x'(t) = f(t, x(t), x(t^2)),\ t \in [0,1] \tag{5.3}$$

$$x(0) = x_0 \tag{5.4}$$

and the following inequations

$$\left|y'(t) - f(t, y(t), y(t^2))\right| \leq \varepsilon,\ t \in [0,1)$$

$$\left|y'(t) - f(t, y(t), y(t^2))\right| \leq \varphi(t),\ t \in [0,1).$$

For this example, Theorem 3.1 and Theorem 4.1 become

Theorem 5.5. *We suppose that*

(a) $f \in C([0,1] \times \mathbb{R}^2,\mathbb{R})$;

(b) *there exists* $L_f > 0$ *such that* $\forall t \in [0,1], u_i, v_i \in \mathbb{R}, i = 1,2,$ *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i|\,;$$

Then

(i) *the problem* (5.3)–(5.4) *has a unique solution in* $C([0,1],\mathbb{R}) \cap C^1([0,1],\mathbb{R})$;

(ii) *the equation* (5.3) *is Ulam-Hyers stable.*

Theorem 5.6. *We suppose that*

(a) $f \in C([0,\infty[\times\mathbb{R}^2,\mathbb{R})$;

(b) *there exists* $l_f \in L^1([0,\infty[,\mathbb{R}_+)$ *such that* $\forall t \in [0,\infty[, u_i, v_i \in \mathbb{R}, i = 1,2,$ *we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq l_f(t)(|u_1 - v_1| + |u_2 - v_2|);$$

(c) *the function* $\varphi \in C[0,\infty[$ *is increasing;*

(d) *there exists* $\lambda > 0$ *such that*

$$\int_0^t \varphi(s)ds \leq \lambda\varphi(t),\ t \in [0, \infty[.$$

Then

(i) *the problem* (5.3)–(5.4) *has a unique solution in* $C([0, \infty[, \mathbb{R}) \cap C^1([0, \infty[, \mathbb{R})$;

(ii) *the equation* (5.3) *is generalized Ulam-Hyers-Rassias stable with respect to* φ.

Acknowledgement The authors would like to thank the referees for their useful and valuable suggestions.

The work of the second author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

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